Decompositions of the group $G(2)$ and related integrable hierarchies

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Outline of the talk

- The group $G(2)$
- Hierarchies
- The Lie algebra Psd
- The Lie algebra $\mathrm{Ps}\Delta$
- Decompositions in $\mathrm{Ps}\Delta$
- Decompositions in Psd
- The infinite Toda chain
- Compatible Lax equations
- Linearizations
- The geometric construction of solutions

The group $G(2)$

- $\mathcal{H}$ Hilbert space with Hilbert basis $\{e_i \mid i \in \mathbb{Z}\}$.
- For each bounded operator $b : \mathcal{H} \to \mathcal{H}$, a $\mathbb{Z} \times \mathbb{Z}$-matrix $[b] = (b_{ij})$ by the formula

$$b(e_j) = \sum_{i \in \mathbb{Z}} b_{ij} e_i.$$

- $S_2(\mathcal{H})$ ideal of Hilbert Schmidt operators, i.e. $A : \mathcal{H} \mapsto \mathcal{H}$ s.t.

$$||A||_2^2 := \mathrm{trace}(A^* A) = \mathrm{trace}(|A|^2) = \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} |A_{ij}|^2 < \infty.$$

- The relevant group in all cases is

$$G(2) = \left\{ g = (g_{ij}) \in \mathrm{GL}(\mathcal{H}) \;\middle|\; g - \mathrm{Id} \in S_2(\mathcal{H}) \right\}.$$

- $O(G(2)) = \{g \in G(2) \mid [g]^T [g] = \mathrm{Id}\}$
- If $\mathcal{H}$ is complex, $U(G(2)) = \{g \in G(2) \mid [g]^* [g] = \mathrm{Id}\}$.

The group $G(2)$ 2

- LU-**decomposition** in $G(2)$: on dense, open subset $g = LU$

$$[L] = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \mathbf{1} & 0 & 0 & \ddots \\ \ddots & l_{n\,n-1} & \mathbf{1} & 0 & \ddots \\ \ddots & l_{n+1\,n-1} & l_{n+1\,n} & \mathbf{1} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix},$$

$$[U] = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \boldsymbol{u_{n-1\,n-1}} & u_{n-1\,n} & u_{n-1\,n+1} & \ddots \\ \ddots & 0 & \boldsymbol{u_{n\,n}} & u_{n\,n+1} & \ddots \\ \ddots & 0 & 0 & \boldsymbol{u_{n+1\,n+1}} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

The group $G(2)$ 3

- **Gauss- or Iwasawa-decomposition**: each $g \in G(2)$

$$g = o(g)b^+(g) \text{ real case, or } g = u(g)b^+(g) \text{ complex case,}$$

where $o(g) \in O(G(2))$, $u(g) \in U(G(2))$ and

$$[b^+(g)] = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \boldsymbol{b_{n-1\ n-1}} & b_{n-1\ n} & b_{n-1\ n+1} & \ddots \\ \ddots & 0 & \boldsymbol{b_{n\ n}} & b_{n\ n+1} & \ddots \\ \ddots & 0 & 0 & \boldsymbol{b_{n+1\ n+1}} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

, with all $b_{ii} > 0, i \in \mathbb{Z}$.

Hierarchies 1

- General set-up for hierarchies: Lie algebra $\mathfrak{g}$
- $\mathfrak{g}_i$, $i = 1, 2$, Lie subalgebras of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

- π_i the projection of $\mathfrak{g}$ onto $\mathfrak{g}_i$ induced by this decomposition
- $\mathfrak{g}_2$ Lie algebra of the Lie subgroup G_2
- Set linear independent, commuting elements:

$$\{F_j \mid j \geq 1\} \in \mathfrak{g}_1$$

- t_j flow parameter w.r.t. F_j, $\partial_j = \frac{\partial}{\partial t_j}$, $t = \{t_j\}$.

Hierarchies 2

- Search for $g_2(t) \in G_2$ such that the deformations

$$\mathcal{F}_j := g_2(t)^{-1} F_j g_2(t), j \geq 1$$

satisfy for all $j_1 \geqslant 1$ and $j_2 \geqslant 1$:

$$\frac{\partial}{\partial t_{j_1}}(\mathcal{F}_{j_2}) = [\mathcal{F}_{j_2}, \pi_2(\mathcal{F}_{j_1})] = [\pi_1(\mathcal{F}_{j_1}), \mathcal{F}_{j_2}] \tag{1}$$

- The last equality in (1) follows from $[\mathcal{F}_{j_1}, \mathcal{F}_{j_2}] = 0$.
- (1): **compatible Lax equations**, for in practice it implies

$$\frac{\partial}{\partial t_{j_1}}(\pi_1(\mathcal{F}_{j_2})) - \frac{\partial}{\partial t_{j_2}}(\pi_1(\mathcal{F}_{j_1})) - [\pi_1(\mathcal{F}_{j_1}), \pi_1(\mathcal{F}_{j_2})] = 0,$$

a set of **zero curvature relations**.

Pseudo differential operators 1

- R k-algebra, $k = \mathbb{R}$ or $\mathbb{C}$, ∂ k-linear derivation of R.
- $R[\partial] = \{\sum_{i=0}^{n} a_i \partial^i, a_i \in R$ for all $i \geq 0\}$
- Assume $\{\partial^n \mid n \geqslant 0\}$ R-linear independent. Then

$$R[\partial] \subset R[\partial, \partial^{-1}) = \mathrm{Psd}, \text{ the pseudo differential operators .}$$

- Psd: extension of $R[\partial]$ with integral operators $\{\partial^m \mid m < 0\}$.
- For all m and $n \in \mathbb{Z}$

$$\partial^n \partial^m = \partial^{n+m} \text{ and } \partial^0 \text{ is the unit element.}$$

Pseudo differential operators 2

- Pseudo differential operators

$$\mathrm{Psd} = R[\partial, \partial^{-1}) = \{p = \sum_{j=-\infty}^{N} p_j \partial^j, p_j \in R\},$$

- Significant class of invertible elements in $R[\partial, \partial^{-1})$:

Lemma

Every scalar pseudo differential operator $P = \sum_{j \leqslant m} p_j \partial^j$, with $p_m \in R^*$, has an inverse P^{-1} of the form

$$P^{-1} = \sum_{i \leqslant -m} q_i \partial^i, \text{ with } q_{-m} = p_m^{-1}.$$

- **Dressing** $P \in R[\partial, \partial^{-1})$ with $B \in R[\partial, \partial^{-1})^*$: BPB^{-1}.

Pseudo differential operators 3

- Taking roots in Psd:

Lemma

Consider any monic pseudo differential operator

$$U = \partial^m + \sum_{i<m} u_{m-i}\partial^i$$

of order $m \geqslant 1$. There is a unique monic pseudo differential operator of order one

$$U^{\frac{1}{m}} = L = \partial + \sum_{i=0}^{\infty} \ell_{1+i}\partial^{-i},$$

with $U = (U^{\frac{1}{m}})^m$. We call $U^{\frac{1}{m}}$ the m-th root of U.

Pseudo difference operators 1

- Commutative k-algebra R, $k = \mathbb{R}$ or $\mathbb{C}$.
- $M_{\mathbb{Z}}(R)$: $\mathbb{Z} \times \mathbb{Z}$-matrices, coefficients from R
- $A = (a_{ij}) \in M_{\mathbb{Z}}(R)$:

$$A = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \boldsymbol{a_{n-1\ n-1}} & a_{n-1\ n} & a_{n-1\ n+1} & \ddots \\ \ddots & a_{n\ n-1} & \boldsymbol{a_{n\ n}} & a_{n\ n+1} & \ddots \\ \ddots & a_{n+1\ n-1} & a_{n+1\ n} & \boldsymbol{a_{n+1\ n+1}} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Pseudo difference operators 2

- To $\{d(s)|s \in \mathbb{Z}\}$ in R is associated $\mathrm{diag}(d(s))$:

$$\begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \mathbf{d(n-1)} & 0 & 0 & \ddots \\ \ddots & 0 & \mathbf{d(n)} & 0 & \ddots \\ \ddots & 0 & 0 & \mathbf{d(n+1)} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

- Diagonal matrices:

$$\mathcal{D}_1(R) = \{d = \mathrm{diag}(d(s))|d(s) \in R \text{ for all } s \in \mathbb{Z}\}.$$

Pseudo difference operators 3

- Shift matrix Λ

$$\Lambda = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \mathbf{0} & 1 & 0 & \ddots \\ \ddots & 0 & \mathbf{0} & 1 & \ddots \\ \ddots & 0 & 0 & \mathbf{0} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

- Action of the $\{\Lambda^m \mid m \in \mathbb{Z}\}$ on $\mathcal{D}_1(R)$:

$$\Lambda^m \mathrm{diag}(d(s))\Lambda^{-m} = \mathrm{diag}(d(s+m)).$$

- Each $A = (a_{ij}) \in M_{\mathbb{Z}}(R)$: decomposes uniquely

$$A = \sum_{i \in \mathbb{Z}} d_i \Lambda^i, d_i \in \mathcal{D}_1(R)$$

Pseudo difference operators 4

- Lower triangular matrices

$$LT(R) = \{L \mid L = \sum_{i \leq N} \ell_i \Lambda^i, \ell_i \in \mathcal{D}_1(R)\}$$

- Each $L = \sum_{i \leq N} \ell_i \Lambda^i, \ell_N \in \mathcal{D}_1(R)^*$, is invertible.
- Consider a $L_0 = \sum_{i \leq 1} \ell_i \Lambda^i, \ell_1 \in \mathcal{D}_1(R)^*$. Then:

$$L_0 = K_0 \Lambda K_0^{-1},$$

with $K_0 = \sum_{i \leq 0} k_i \Lambda^i, k_i \in \mathcal{D}_1(R), k_0 \in \mathcal{D}_1(R)^*$ and

$$LT(R) = \{P \mid P = \sum_{i \leq N} p_i L_0^i, p_i \in \mathcal{D}_1(R)\}$$

Pseudo difference operators 5

- Consider the invertible operator $\Delta := \Lambda - \mathrm{Id}$:

$$\Delta\left(\begin{pmatrix} \vdots \\ x_{n-1} \\ x_n \\ x_{n+1} \\ \vdots \end{pmatrix}\right) = \begin{pmatrix} \vdots \\ x_n - x_{n-1} \\ x_{n+1} - x_n \\ x_{n+2} - x_{n+1} \\ \vdots \end{pmatrix}$$

- For the difference operator Δ we have

$$\mathrm{Ps}\Delta = LT(R) = \{L \mid L = \sum_{i \leq N} \ell_i \Delta^i, \ell_i \in \mathcal{D}_1(R)\}$$

Elements of $\mathrm{Ps}\Delta$ also called: *pseudo difference operators*.

Infinite Toda chain 1

- Particles on a straight line with nearest neighbour interaction:

- q_n is the displacement of the n-th particle, $n \in \mathbb{Z}$.
- Equations of motion in dimensionless form are described by

$$\frac{dq_n}{dt} = p_n \text{ and } \frac{dp_n}{dt} = e^{-(q_n - q_{n-1})} - e^{-(q_{n+1} - q_n)}, \quad n \in \mathbb{Z}.$$

- Put

$$a_n := \frac{1}{2} e^{-(q_n - q_{n-1})} \text{ and } b_n := \frac{1}{2} p_n.$$

Infinite Toda chain 2

- Introduce the $\mathbb{Z} \times \mathbb{Z}$-matrices L resp. B by

$$\begin{pmatrix} \ddots & \ddots & \ddots & & 0 \\ \ddots & \mathbf{b_{n-1}} & a_n & 0 & \ddots \\ \ddots & a_n & \mathbf{b_n} & a_{n+1} & \ddots \\ & 0 & a_{n+1} & \mathbf{b_{n+1}} & \ddots \\ 0 & & \ddots & \ddots & \ddots \end{pmatrix}, \begin{pmatrix} \ddots & \ddots & \ddots & & 0 \\ \ddots & \mathbf{0} & a_n & 0 & \ddots \\ \ddots & -a_n & \mathbf{0} & a_{n+1} & \ddots \\ & 0 & -a_{n+1} & \mathbf{0} & \ddots \\ 0 & & \ddots & \ddots & \ddots \end{pmatrix}$$

- Equations of motion equivalent to:

$$\frac{dL}{dt} = -BL + LB = [L, B].$$

Decompositions in PsΔ 1

- Consider in LT the Lie subalgebra

$$LT_{\geqslant 0} := \{A = \sum_{0 \leqslant j \leqslant N} a_j \Lambda^j \mid \text{ all } a_j \in \mathcal{D}_1(R)\}$$

- We write $\pi_{\geqslant 0}$ for the projection of LT onto $LT_{\geqslant 0}$,

$$\pi_{\geqslant 0}(\sum_{-\infty \leqslant j \leqslant N} a_j \Lambda^j) = \sum_{0 \leqslant j \leqslant N} a_j \Lambda^j.$$

- Similarly, we have the Lie subalgebras $LT_{<0}$, $LT_{\leqslant 0}$, $LT_{>0}$ and the respective projections $\pi_{<0}$, $\pi_{\leqslant 0}$ and $\pi_{>0}$.
- A $\mathbb{Z} \times \mathbb{Z}$-matrix A for which there is an $N \geqslant 0$ such that

$$A = \sum_{-N \leqslant j \leqslant N} a_j \Lambda^j, a_j \in \mathcal{D}_1(R) \tag{2}$$

  is called a **finite band** matrix in $M_{\mathbb{Z}}(R)$.
- This set of matrices is a Lie subalgebra and is denoted by $\mathcal{FB}$.

Decompositions in PsΔ 2

- Inside $\mathfrak{FB}$ we have the antisymmetric matrices

$$\mathfrak{FB}_{as}(R) = \mathfrak{FB}_{as} = \{X \in \mathfrak{FB} \mid X^T = -X\}$$

- There is a natural projection π_{as} from LT to $\mathfrak{FB}_{as}$

$$\pi_{as}(\sum_{j \leqslant N} a_j \Lambda^j) = \sum_{j \geqslant 1} (a_j \Lambda^j - \Lambda^{-j} a_j),$$

  with $LT_{\leqslant 0}$ as a kernel.
- Note that at the infinite Toda chain, we had $\pi_{as}(L) = B$.
- This gives the following 3 decompositions of LT:

$$\begin{aligned} LT &= LT_{\geqslant 0} \oplus LT_{<0}, \\ LT &= LT_{>0} \oplus LT_{\leqslant 0}, \\ LT &= \mathfrak{FB}_{as} \oplus LT_{\leqslant 0}. \end{aligned}$$

Decompositions in Psd 1

- First decomposition in Psd:

$$P = \sum_{j} P_j \partial^j = \sum_{j<0} P_j \partial^j + \sum_{j\geqslant 0} P_j \partial^j = P_{<0} + P_{\geqslant 0}$$

- Lie algebra $\mathrm{Psd} = \mathrm{Psd}_{<0} \oplus \mathrm{Psd}_{\geqslant 0} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$
- Group corresponding to $\mathfrak{g}_1$

$$G_1 = \{g = 1 + \sum_{j<0} g_j \partial^j, g_j \in R\}$$

Decompositions in Psd 2

- Second decomposition in Psd:

$$P = \sum_j P_j \partial^j = \sum_{j \leqslant 0} P_j \partial^j + \sum_{j > 0} P_j \partial^j = P_{\leqslant 0} + P_{>0}$$

- Lie algebra decomposition $\mathrm{Psd} = \mathrm{Psd}_{\leqslant 0} \oplus \mathrm{Psd}_{>0}$
- Group corresponding to $\mathfrak{g}_1$

$$G_1 = \{g = \sum_{j \leqslant 0} g_j \partial^j, g_j \in R, g_0 \in R^*\}$$

Compatible Lax equations in $\mathrm{Ps}\Delta$ 1

- Each decomposition starting point of a compatible set of Lax equations
- Given R , set $\{\partial_i \mid i \geqslant 1\}$ of commuting derivations of R
- Example: $R = k[t_i \mid i \geqslant 1]$ or $R = k[[t_i \mid i \geqslant 1]]$ and

$$\partial_i := \partial_{t_i} := \frac{\partial}{\partial t_i}.$$

- Consider the first decomposition in $\mathrm{Ps}\Delta$:

$$LT_{\geqslant 0}(\Lambda) \oplus LT_{<0}(\Lambda) = \mathrm{Ps}\Delta_{\geqslant 0} \oplus \mathrm{Ps}\Delta_{<0} = \mathfrak{g}_1 \oplus \mathfrak{g}_2.$$

Compatible Lax equations in $\mathrm{Ps}\Delta$ 2

- Group corresponding to $\mathfrak{g}_2 = \mathrm{Ps}\Delta_{<0}$:

$$U_- = \{\mathrm{Id} + B \mid B \in \mathrm{Ps}\Delta_{<0}\}$$

- Basic commuting directions : the $\{\Lambda^k \mid k \geqslant 1\}$
- Deformation of Λ:

$$\mathcal{L} = \Lambda + \sum_{i=1}^{\infty} d_i \Lambda^{1-i}, d_i \in \mathcal{D}_1(R).$$

- Examples: $\mathcal{L} = U \Lambda U^{-1}$, with $U \in U_-$.

Compatible Lax equations in $\mathrm{Ps}\Delta$ 3

- Let $\mathcal{B}_r := (\mathcal{L}^r)_{\geqslant 0}, r \geqslant 1$.
- Search for deformations $\mathcal{L}$ that satisfy:

$$\partial_{k_1}(\mathcal{L}^{k_2}) = [\mathcal{B}_{k_1}, \mathcal{L}^{k_2}] = [\mathcal{L}^{k_2}, \mathcal{L}^{k_1}_{\leqslant 0}], k_1 \text{ and } k_2 \geqslant 1.$$

- Sufficient the Lax equations for $\mathcal{L}$

$$\partial_{k_1}(\mathcal{L}) = [\mathcal{B}_{k_1}, \mathcal{L}] = [\mathcal{L}, \mathcal{L}^{k_1}_{<0}], k_1 \geqslant 1,$$

  the **Lower Triangular Toda (LTT)-hierarchy**.
- For each solution $\mathcal{L}$ the *zero curvature relations* hold:

$$\partial_{k_1}(\mathcal{B}_{k_2}) - \partial_{k_1}(\mathcal{B}_{k_1}) - [\mathcal{B}_{k_1}, \mathcal{B}_{k_2}] = 0, k_1 \text{ and } k_2 \geqslant 1.$$

Compatible Lax equations in $\mathrm{Ps}\Delta$ 4

- Next relevant decomposition in $\mathrm{Ps}\Delta$:

$$LT_{>0}(\Lambda) \oplus LT_{\leqslant 0}(\Lambda) = \mathrm{Ps}\Delta_{>0} \oplus \mathrm{Ps}\Delta_{\leqslant 0} = \mathfrak{g}_1 \oplus \mathfrak{g}_2.$$

- Group corresponding to $\mathfrak{g}_2 = \mathrm{Ps}\Delta_{\leqslant 0}$:

$$P_- = \{d \,\mathrm{Id} + B \mid, d \in \mathcal{D}_1(R)^*, B \in \mathrm{Ps}\Delta_{<0}\}.$$

- Basic commuting directions : the $\{\Lambda^k \mid k \geqslant 1\}$.
- Deformation of Λ:

$$\mathcal{M} = d_0\Lambda + \sum_{i=1}^{\infty} d_i \Lambda^{1-i}, d_i \in \mathcal{D}_1(R) \text{ and } d_0 \in \mathcal{D}_1(R)^*.$$

- Examples: $\mathcal{M} = P\Lambda P^{-1}$, with $P \in P_-$.

Compatible Lax equations in $\mathrm{Ps}\Delta$ 5

- Consider the cut-off's $\mathcal{C}_r := (\mathcal{M}^r)_{>0}, r \geqslant 1$.
- Search for deformations $\mathcal{M}$ that satisfy:

$$\partial_{r_1}(\mathcal{M}^{r_2}) = [\mathcal{C}_{r_1}, \mathcal{M}^{r_2}] = [\mathcal{M}^{r_2}, \mathcal{M}^{r_1}_{\leqslant 0}], r_1 \text{ and } r_2 \geqslant 1.$$

- Sufficient Lax equations for $\mathcal{M}$ the

$$\partial_{r_1}(\mathcal{M}) = [\mathcal{C}_{r_1}, \mathcal{M}] = [\mathcal{M}, \mathcal{M}^{r_1}_{\leqslant 0}], r_1 \geqslant 1,$$

  the **Strict Lower Triangular Toda (SLTT)-hierarchy**.
- Consequence: *zero curvature relations*

$$\partial_{r_1}(\mathcal{C}_{r_2}) - \partial_{r_2}(\mathcal{C}_{r_1}) - [\mathcal{C}_{r_1}, \mathcal{C}_{r_2}] = 0, r_1 \text{ and } r_2 \geqslant 1.$$

Compatible Lax equations in $\mathrm{Ps}\Delta$ 6

- The last relevant decomposition in $\mathrm{Ps}\Delta$:

$$\mathcal{FB}_{as} \oplus LT_{\leqslant 0} = \mathrm{Ps}\Delta_{as} \oplus \mathrm{Ps}\Delta_{\leqslant 0} = \mathfrak{g}_1 \oplus \mathfrak{g}_2.$$

- Group corresponding to $\mathfrak{g}_2 = \mathrm{Ps}\Delta_{\leqslant 0}$:

$$P_- = \{d \operatorname{Id} + B \mid, d \in \mathcal{D}_1(R)^*, B \in \mathrm{Ps}\Delta_{<0}\}.$$

- Basic commuting directions : the $\{\Lambda^r - \Lambda^{-r} \mid r \geqslant 1\}$.
- Commuting deformations of the $\{\Lambda^r - \Lambda^{-r}\}$:

$$\mathcal{F}_r = m_0\Lambda^r + \sum_{i=1}^{\infty} m_i\Lambda^{r-i}, m_i \in \mathcal{D}_1(R) \text{ and } m_0 \in \mathcal{D}_1(R)^*.$$

- Examples: $\mathcal{F}_r = P(\Lambda^r - \Lambda^{-r})P^{-1}$, with $P \in P_-$.

Compatible Lax equations in $\mathrm{Ps}\Delta$ 7

- Consider the cut-off's $\mathcal{E}_r := \pi_{as}(\mathcal{M}_r), r \geqslant 1$.
- Search for deformations $\mathcal{F}_r$ that satisfy:

$$\partial_{r_1}(\mathcal{F}_{r_2}) = [\mathcal{M}_{r_2}, \mathcal{E}_{r_1}] = [\pi^c_{as}(\mathcal{M}_{r_1}), \mathcal{M}_{r_2}], r_1 \text{ and } r_2 \geqslant 1,$$

where $\pi^c_{as} = \mathrm{Id} - \pi_{as}$ is a projection on $LT_{\leqslant 0}$.
- This is called the **Infinite Toda Chain (ITC)-hierarchy**.
- These $\{-\mathcal{E}_r\}$ satisfy the *zero curvature relations*

$$\partial_{r_1}(\mathcal{E}_{r_2}) - \partial_{r_2}(\mathcal{E}_{r_1}) - [\mathcal{E}_{r_2}, \mathcal{E}_{r_1}] = 0, r_1 \text{ and } r_2 \geq 1.$$

Compatible Lax equations in $\mathrm{Ps}\Delta$ 8

- $\mathcal{L}$ solution of the LTT-hierarchy, $\mathcal{A}_k := -(\mathcal{L}^k)_{<0}, k \geq 1$.
- Zero curvature relations for the $\{\mathcal{A}_k \mid k \geqslant 1\}$:

$$\partial_{k_1}(\mathcal{A}_{k_2}) - \partial_{k_2}(\mathcal{A}_{k_1}) - [\mathcal{A}_{k_1}, \mathcal{A}_{k_2}] = 0, k_1 \text{ and } k_2 \geqslant 1.$$

- $\mathcal{M}$ solution of the SLTT-hierarchy, $\mathcal{D}_r := -(\mathcal{M}^r)_{\leqslant 0}, r \geqslant 1$.
- Zero curvature for the $\{\mathcal{D}_r \mid r \geqslant 1\}$:

$$\partial_{r_1}(\mathcal{D}_{r_2}) - \partial_{r_2}(\mathcal{D}_{r_1}) - [\mathcal{D}_{r_1}, \mathcal{D}_{r_2}] = 0, r_1 \text{ and } r_2 \geqslant 1.$$

- $\{\mathcal{F}_r\}$ solutions of ITC-hierarchy, $\mathcal{G}_r := \pi^c_{as}(\mathcal{F}_r), r \geqslant 1$.
- Zero curvature for the $\{\mathcal{G}_r \mid r \geqslant 1\}$:

$$\partial_{r_1}(\mathcal{G}_{r_2}) - \partial_{r_2}(\mathcal{G}_{r_1}) - [\mathcal{G}_{r_1}, \mathcal{G}_{r_2}] = 0, r_1 \text{ and } r_2 \geqslant 1.$$

Compatible Lax equations in $\mathrm{Ps}\Delta$ 9

- $\mathcal{L}$ potential solution LTT-hierarchy, $\mathcal{A}_k := -(\mathcal{L}^k)_{<0}, k \geqslant 1$.
- Related Cauchy problem: find a $u \in U_-(R)$ s.t. for all $k \geqslant 1$

$$\partial_k(u) = \mathcal{A}_k u, \tag{3}$$

- $\mathcal{M}$ potential solution SLTT-hierarchy, $\mathcal{D}_r := -(\mathcal{M}^r)_{\leqslant 0}, r \geqslant 1$.
- Related Cauchy problem: find a $p \in P_-(R)$ s.t. for all $r \geqslant 1$

$$\partial_r(p) = \mathcal{D}_r p, \tag{4}$$

- $\{\mathcal{F}_r\}$ potential solutions ITC-hierarchy
- Related Cauchy problem: find a $g \in P_-(R)$ s.t. for all $j \geqslant 1$

$$\partial_j(g) = \pi^c_{LT,as}(\mathcal{F}_j)g, \tag{5}$$

Compatible Lax equations in Psd

- Decomposition $\mathrm{Psd} = \mathrm{Psd}_{<0} \oplus \mathrm{Psd}_{\geqslant 0} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$
- Deformation $L = \partial + \sum_{i \geqslant 1} \ell_{i+1} \partial^{-i}$, $B_k = (L^k)_{\geqslant 0}$
- Examples: $L = P \partial P^{-1}$, $P \in G_1, P = \mathrm{Id} + \sum_{i \geqslant 1} p_i \partial^{-i}$
- Assume R has a collection of k-linear derivations $\{\partial_k \mid k \geqslant 1\}$, all commuting with ∂
- Lax equations of the **KP hierarchy**

$$\partial_{k_1}(L^{k_2}) = [B_{k_1}, L^{k_2}] = [L^{k_2}, L^{k_1}_{<0}], k_1 \text{ and } k_2 \geqslant 1.$$

Compatible Lax equations in Psd 2

- Decomposition $\mathrm{Psd} = \mathrm{Psd}_{\leqslant 0} \oplus \mathrm{Psd}_{>0} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$
- Consider deformations

$$M = \partial + m_1 + m_2\partial^{-1} + \cdots$$

- Examples: $M = P\partial P^{-1}$,
  $P \in G_1, P = p_0 + \sum_{i\geqslant 1} p_i\partial^{-i}, p_0 \in R^*$
- R and $\{\partial_r \mid r \geqslant 1\}$ as above
- Let $C_r = (M^r)_{>0}, r \geq 1$.
- **Strict KP hierarchy** for M and its powers:

$$\partial_{r_1}(M^{r_2}) = [C_{r_1}, M^{r_2}] = [M^{r_2}, M^{r_1}_{\leqslant 0}], r_1 \text{ and } r_2 \geqslant 1$$

Linearizations 1

- Linearization of LTT-hierarchy:

$$\mathcal{L}\varphi = \varphi\Lambda,$$
$$\partial_k(\varphi) = \pi_{\geqslant 0}(\mathcal{L}^k)\varphi \text{ for all } k \geqslant 1.$$

- Linearization of SLTT-hierarchy:

$$\mathcal{M}\psi = \psi\Lambda$$
$$\partial_r(\psi) = \pi_{>0}(\mathcal{M}^r)\psi \text{ for all } r \geqslant 1.$$

- Linearization of ITC-hierarchy:

$$\mathcal{F}_j\phi = \phi(\Lambda^j - \Lambda^{-j}) \text{ for all } j \geqslant 1,$$
$$\partial_j(\phi) = -\pi_{as}(\mathcal{F}_j)\phi \text{ for all } j \geqslant 1.$$

Linearizations 2

- For suitable φ, ψ, ϕ the linearization implies the Lax equations

$$\begin{aligned}
&\partial_{j_1}(\mathcal{F}_{j_2}\phi - \phi(\Lambda^{j_2} - \Lambda^{-j_2})) \\
&= \partial_{j_1}(\mathcal{F}_{j_2})\phi + \mathcal{F}_{j_2}(\partial_{j_1}(\phi)) - (\partial_{j_1}(\phi))(\Lambda^{j_2} - \Lambda^{-j_2}) \\
&= \partial_{j_1}(\mathcal{F}_{j_2})\phi - \mathcal{F}_{j_2}\pi_{as}(\mathcal{F}_{j_1})\phi + \pi_{as}(\mathcal{F}_{j_1})\phi(\Lambda^{j_2} - \Lambda^{-j_2}) \\
&= \{\partial_{j_1}(\mathcal{F}_{j_2}) - [\mathcal{F}_{j_2}, \pi_{as}(\mathcal{F}_{j_1})\,]\}\phi = 0.
\end{aligned}$$

- φ, ψ, ϕ belong to a PsΔ-module of perturbations of the solution of the linearization corresponding to the trivial solutions of the hierarchies:

$$\mathcal{L} = \Lambda, \mathcal{M} = \Lambda, \mathcal{F}_j = \Lambda^j - \Lambda^{-j}.$$

- For *LTT*- and *SLTT*-hierarchy:

$$\varphi_0 = \psi_0 = \exp(\sum_{k=1}^{\infty} t_k \Lambda^k)$$

Linearizations 3

- For ITC-hierarchy:

$$\phi_0 = \exp(\sum_{k=1}^{\infty} -t_k(\Lambda^k - \Lambda^{-k}))$$

- Appropriate PsΔ-module for ITC-hierarchy: $M(ITC)$.
- $M(ITC)$ consists of formal products:

$$\{\ell\}\phi_0 = \{\sum_{j=-\infty}^{N} d_j \Lambda^j\} \exp(\sum_{j=1}^{\infty} -t_j(\Lambda^j - \Lambda^{-j})), \text{ where } \ell \in \text{Ps}\Delta.$$

- Elements of $M(ITC)$ are called *oscillating matrices*.

Linearizations 4

- PsΔ-action on $M(ITC)$:

$$\ell_1\{\ell_2\}\phi_0 = \{\ell_1\ell_2\}\phi_0.$$

- Right multiplication with $\{\Lambda^j - \Lambda^{-j}\}$

$$\{\ell\}\phi_0(\Lambda^j - \Lambda^{-j}) := \{\ell(\Lambda^j - \Lambda^{-j})\}\phi_0.$$

- Action of the derivations ∂_j on $M(ITC)$:

$$\partial_j(\{\sum_{j=-\infty}^{N} d_j\Lambda^j\}\phi_0) = \{\sum_{j=-\infty}^{N} \partial_j(d_j)\Lambda^j\} - \sum_{j=-\infty}^{N} d_j\Lambda^j(\Lambda^j - \Lambda^{-j})\}\phi_0.$$

- $M(ITC)$ is a free PsΔ-module with generator ϕ_0

Linearizations 5

- An oscillating matrix $\phi = \hat{\phi}\phi_0$, with $\hat{\phi} = \sum_{i=-\infty}^{m} d_i \Lambda^i$, with d_m invertible, is called **a wave matrix** for the matrices $\{\mathfrak{F}_j\}$, if it satisfies satisfies the linearization.
- The $\{\mathfrak{F}_j\}$ form then a solution of the ITC-hierarchy
- It even suffices to show:

Proposition

Let $\phi = \hat{\phi}\phi_0$, with $\hat{\phi} = \sum_{i=-\infty}^{m} d_i \Lambda^i$ and $d_m \in \mathcal{D}_1(R)$ invertible, be an oscillating matrix. If it satisfies for all $j \geqslant 1$

$$\partial_j(\phi) = G_j \phi, \text{ with } G_j \in \mathfrak{F}\mathcal{B}_{as},$$

then $G_j = -\pi_{as}(\mathfrak{F}_j)$, where $\mathfrak{F}_j := \hat{\phi}(\Lambda^j - \Lambda^{-j})\hat{\phi}^{-1}$. In particular the $\{\mathfrak{F}_j\}$ form a solution to the ITC-hierarchy and ϕ is a wave matrix for this solution.

Linearizations 6

- To get the oscillating matrices of the LTT- resp. $SLTT$-hierarchy, replace ϕ_0 by φ_0, resp. ψ_0.
- **Wave matrices** at the LTT-hierarchy have the form

$$\hat{\varphi}\varphi_0 = \{\mathrm{Id}\,\Lambda^N + \sum_{j<N} d_j \Lambda^j\}\varphi_0$$

  and lead to a solution $\mathcal{L} = \hat{\varphi}\Lambda\hat{\varphi}^{-1}$ of the LTT-hierarchy.
- **Wave matrices** at the $SLTT$-hierarchy have the form

$$\hat{\psi}\psi_0 = \{\sum_{j\leqslant N} d_j \Lambda^j\}\psi_0, \text{ with } d_N \text{ invertible}$$

  and lead to a solution $\mathcal{M} = \hat{\psi}\Lambda\hat{\psi}^{-1}$ of the $SLTT$-hierarchy.
- Similar Propositions hold in the LTT- resp. $SLTT$-case.

Linearizations 7

- Linearization of the KP hierarchy:

$$
\begin{aligned}
L\varphi &= \varphi z, \\
\partial_k(\varphi) &= \pi_{\geqslant 0}(L^k)\varphi \text{ for all } k \geqslant 1.
\end{aligned}
$$

- Linearization of the strict KP hierarchy:

$$
\begin{aligned}
M\psi &= \psi z \\
\partial_r(\psi) &= \pi_{>0}(M^r)\psi \text{ for all } r \geqslant 1.
\end{aligned}
$$

- φ resp. ψ **wave functions** of the KP resp. strict KP hierarchy

$$
\varphi = \{1 + \sum_{i<0} a_i z^i\} \exp(\sum_{k=1}^{\infty} t_k z^k) \text{ all } a_i \in R,
$$

$$
\psi = \{\sum_{i\leqslant 0} b_i z^i\} \exp(\sum_{k=1}^{\infty} t_k z^k), \text{ all } b_i \in R, b_0 \in R^*.
$$

Geometric construction of solutions 1

- To get $\mathbb{Z} \times \mathbb{Z}$-matrices: take a Hilbert space H with Hilbert basis $\{e_i \mid i \in \mathbb{Z}\}$. For each bounded operator $b : H \to H$, a $\mathbb{Z} \times \mathbb{Z}$-matrix $[b] = (b_{ij})$ by the formula

$$b(e_j) = \sum_{i \in \mathbb{Z}} b_{ij} e_i.$$

- Choice of $\mathcal{H}$ for 3 PsΔ-hierarchies:

$$\mathcal{H} = \{\vec{x} = \sum_{n \in \mathbb{Z}} x_n \vec{e}(n) \mid x_n \in \mathbb{R} \text{ or } \mathbb{C}, \sum_{n \in \mathbb{Z}} |x_n|^2 < \infty\}.$$

- We put the standard inner product on $\mathcal{H}$

$$< \vec{x} \mid \vec{y} >= \sum_{n \in \mathbb{Z}} x_n y_n \text{ or } < \vec{x} \mid \vec{y} >= \sum_{n \in \mathbb{Z}} x_n \overline{y}_n$$

- $\{\vec{e}(n) \mid n \in \mathbb{Z}\}$ an orthonormal basis of $\mathcal{H}$

Geometric construction of solutions 2

- For each $b \in B(\mathcal{H})$,

$$b(\vec{x}) = [b]\vec{x} = M_{[b]}\vec{x},$$

where [b] is the matrix of b w.r.t. this basis

- For $j \geqslant 1$, operator norms of M_{Λ^j} and $M_{\Lambda^j - \Lambda^{-j}}$ satisfy

$$||M_{\Lambda^j}|| = 1, ||M_{\Lambda^j - \Lambda^{-j}}|| \leqslant 2.$$

- Choose our parameters $t = (t_j)$ out of the space

$$\ell_1(\mathbb{N}) = \{t = (t_j) \mid \text{ all } t_j \in \mathbb{R} \text{ or } \mathbb{C} \text{ and } \sum_{j=1}^{\infty} |t_j| < \infty\},$$

equipped with the norm $||t||_1 = \sum_{j=1}^{\infty} |t_j|$.

- Define analytic maps $\gamma_{1,2}$ and γ_3 from $\ell_1(\mathbb{N})$ to $\mathrm{GL}(\mathcal{H})$ by

$$\gamma_{1,2}(t) = \exp(\sum_{j=1}^{\infty} t_j M_{\Lambda^j}) \text{ resp. } \gamma_3(t) = \exp(\sum_{j=1}^{\infty} -t_j M_{\Lambda^j - \Lambda^{-j}})$$

Geometric construction of solutions 3

- Matrices van $\gamma_{1,2}(t)$ and $\gamma_3(t)$:

$$[\gamma_{1,2}(t)] = \exp(\sum_{i=1}^{\infty} t_i \Lambda^i), [\gamma_3(t)] = \exp(\sum_{i=1}^{\infty} -t_i(\Lambda^i - \Lambda^{-i}))$$

- Relevant group in all cases

$$G(2) = \left\{ g = (g_{ij}) \in \mathrm{GL}(\mathcal{H}) \;\middle|\; g - \mathrm{Id} \in S_2(\mathcal{H}) \right\},$$

where the ideal $S_2(\mathcal{H})$ of Hilbert Schmidt operators, consists of all bounded operators $A : \mathcal{H} \mapsto \mathcal{H}$ such that

$$||A||_2^2 := \mathrm{trace}(A^* A) = \mathrm{trace}(|A|^2) < \infty.$$

Geometric construction of solutions 4

- Each $b \in B(H)$ decomposes as $b = u_-(b) + p_+(b)$, with

$$[u_-(b)] = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \mathbf{0} & 0 & 0 & \ddots \\ \ddots & b_{n\ n-1} & \mathbf{0} & 0 & \ddots \\ \ddots & b_{n+1\ n-1} & b_{n+1\ n} & \mathbf{0} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix},$$

$$[p_+(b)] = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \boldsymbol{b_{n-1\ n-1}} & b_{n-1\ n} & b_{n-1\ n+1} & \ddots \\ \ddots & 0 & \boldsymbol{b_{n\ n}} & b_{n\ n+1} & \ddots \\ \ddots & 0 & 0 & \boldsymbol{b_{n+1\ n+1}} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Geometric construction of solutions 5

- subgroups of $G(2)$:
  - $U_- := \{g = \mathrm{Id} + u_-(g) \mid g \in G(2)\}$,
  - $B_+ := \{g = p_+(g) \mid g \in G(2)\}$,
  - $U_+ := \{g \in B_+ \mid g_{ii} = 1, \text{ for all } i \in \mathbb{Z}\}$,
  - $B_- := \{g \in G(2) \mid g_{ij} = 0 \text{ for all } i < j\}$
  - $B_-^+ := \{g \in B_- \mid g_{ii} > 0 \text{ for all } i\}$
  - $O(G(2)) = \{g \in G(2) \mid gg^T = \mathrm{Id}\}$
- Big cell in $G(2)$: $\Omega = U_- B_+ = B_- U_+$,
- Relevant decomposition for LTT-hierarchy: for all $g \in \Omega$

$$g = u_-(g).b_+(g)$$

- Relevant decomposition for $SLTT$-hierarchy: for all $g \in \Omega$

$$g = b_-(g).u_+(g)$$

- Basic decomposition for ITC-hierarchy: $G(2) = B_-^+ O(G(2))$
- Each $g \in G(2), g = b_-^+(g).o(g)$.

Geometric construction of solutions 6

- For each $g \in G(2)$, the following set is non-empty, open and dense
$$\{t \in \ell_1(\mathbb{N}) \mid \gamma_{1,2}(t) g \gamma_{1,2}(t)^{-1} \in \Omega\}.$$
- In the *LTT*- and *SLTT*-case, choose the algebra of coefficients
$$R_g := C^\infty(\{t \in \ell_1(\mathbb{N}) \mid \gamma_{1,2}(t) g \gamma_{1,2}(t)^{-1} \in \Omega\}),$$
with the derivations $\partial_i = \frac{\partial}{\partial t_i}, i \geq 1$.
- In the *ITC*-case we take
$$R_g := C^\infty(\ell_1(\mathbb{N})),$$
with the derivations $\partial_i = \frac{\partial}{\partial t_i}, i \geq 1$.
- Define $\Phi_1(t) = u_-(\gamma_{1,2}(t) g \gamma_{1,2}(t)^{-1}).[\gamma_{1,2}(t)]$.
- Define $\Phi_2(t) = b_-(\gamma_{1,2}(t) g \gamma_{1,2}(t)^{-1}).[\gamma_{1,2}(t)]$.
- Define $\Phi_3(t) = b_-^+(\gamma_3(t) g \gamma_3(t)^{-1})[\gamma_3(t)]$.

Geometric construction of solutions 7

Theorem

There holds:

(a) *Let $g \in G(2)$. Then Φ_1 is a wave matrix for the LTT-hierarchy and for each coset $gB_+ \in G(2)/B_+$ there is a $\mathcal{L}_{gB_+}$ in $Ps\Delta$ that is a solution of the LTT-hierarchy.*

(b) *Let $g \in G(2)$. Then Φ_2 is a wave matrix for the SLTT-hierarchy and for each coset $gU_+ \in G(2)/U_+$ there is a $\mathcal{M}_{gU_+}$ in $Ps\Delta$ that is a solution of the SLTT-hierarchy.*

(c) *Let $g \in G(2)$. Then Φ_3 is a wave matrix for the ITC-hierarchy and for each coset $gO(G(2)) \in G(2)/O(G(2))$ there is a set $\{(\mathcal{F}_j)_{gU_+} \mid j \geqslant 1\}$ in $Ps\Delta$ that forms a solution of the ITC-hierarchy.*

Geometric construction of solutions 8

- For $i \in \mathbb{Z}$, define the subspace
$$\mathcal{H}_i := \{\sum_{n \leq i} a_n \vec{e}_n \in \mathcal{H}\}.$$
- The $\{\mathcal{H}_i\}$ form the basic flag
$$\cdots \mathcal{H}_{i-1} \subset \mathcal{H}_i \subset \mathcal{H}_{i+1} \cdots ,$$
corresponding to Id B_+.
- To gB_+ corresponds the flag $\mathcal{F}_{gB_+} = \{W_i = g\mathcal{H}_i\}$:
$$\cdots gH_{i-1} \subset gH_i \subset gH_{i+1} \cdots$$
- To gU_+ corresponds the flag $\mathcal{F}_{gB_+} = \{W_i = g\mathcal{H}_i\}$ and the basis $\{f_i\}$,
$$f_i \neq 0, f_i \in W_i / W_{i-1}.$$
- $O(G(2))$ is the fixed point set in $G(2)$ of the involution $\sigma(g) = (g^T)^{-1}$. Thus $G(2)/O(G(2))$ is a symmetric space.

Geometric construction of solutions 9

- Hilbert space for KP and strict KP:

$$H = \{\sum_{n\in\mathbb{Z}} a_n z^n \mid a_n \in \mathbb{C}, \sum_{n\in\mathbb{Z}} \mid a_n \mid^2 < \infty\},$$

- Decomposition $H = H_- \oplus H_+$, where

$$H_- = \{\sum_{n<0} a_n z^n \in H\} \quad \text{and} \quad H_+ = \{\sum_{n\geq 0} a_n z^n \in H\}$$

- The inner product $< \cdot \mid \cdot >$ is given by

$$< \sum_{n\in\mathbb{Z}} a_n z^n \mid \sum_{m\in\mathbb{Z}} b_m z^m >= \sum_{n\in\mathbb{Z}} a_n \overline{b_n}.$$

- Relevant Grassmanian : $\mathrm{Gr}^{(0)}(H) = \{gH_+ \mid g \in G(2)\}$

Geometric construction of solutions 10

- Thus $\mathrm{Gr}^{(0)}(H)$ equals the homogeneous space $G(2)/P_1$ with P_1 the stabilizer of H_+ in $G(2)$.
- P_1 is w.r.t. the decomposition $H = H_- \oplus H_+$ given by

$$P_1 = \{g = \begin{pmatrix} g_{--} & 0 \\ g_{+-} & g_{++} \end{pmatrix} \in G(2)\}.$$

- $G(2)$ acts on the pairs (W, ℓ), where ℓ is a line in W, by

$$(W, \ell) \mapsto (gW, g\ell).$$

- The stabilizer P_2 of the pair $(H_+, < z^0 >)$ is given by

$$\{g \in P_1 \mid g_{++} < z^0 >=< z^0 >\}$$

THANK YOU FOR YOUR ATTENTION