Volodya Roubtsov

"Real Monge-Ampère operators and (almost) complex, (almost) -product... and other geometric structures - part 2

Applications
(mainly in th. Meteorology)

talk at J. Krasilschik
Seminar

"Geometry of Differential Equations"

Moscow independent
University,
Control Problems Institute
Russian Academy of Sciences

March. 3, 2021, Moscow

Plan:

1. Short reminder and then
2. Shallow water equations and MAO
3. Elementary 2D hyperkahler triple & 2D S/geostr. models
4. S/Geostrophic eq's in 3D.
5. Diagnostic MAE in 2D and 3D (Dritschel-Viudez) and various geometries

Symplectic Equivalence-1

DEFINITION
Two SMAE $\Delta_{\omega_1} = 0$ and $\Delta_{\omega_2} = 0$ are locally equivalent iff there is exist a local symplectomorphism $F : (T^*M, \Omega) \to (T^*M, \Omega)$ such that

$$F^*\omega_1 = \omega_2.$$

REMARK: L is a generalized solution of $\Delta_{F^*\omega_1} = 0$ iff $F(L)$ is a generalized solution of $\Delta_{\omega} = 0$.

Legendre partial transformation

Figure: Legendre

$$
\begin{tikzcd}
\boxed{u_{q_1q_1} + u_{q_2q_2} = 0} \arrow[r, dashed, leftrightarrow] \arrow[d] & \boxed{v_{q_1q_1} v_{q_2q_2} - v_{q_1q_2}^2 = 1} \arrow[d] \\
\boxed{\omega = dq_1 \wedge dp_2 - dq_2 \wedge dp_1} & \boxed{\tilde{\omega} = dp_1 \wedge dp_2 - dq_1 \wedge dq_2} \arrow[l, "\Phi^*"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
\boxed{L_u = (q_1, q_2, u_{q_1}, u_{q_2})} \arrow[r, "\Phi"] \arrow[d] & \boxed{\begin{array}{l} L_v = (\tilde{q}_1, \tilde{q}_2, v_{\tilde{q}_1}, v_{\tilde{q}_2}) \\ = (q_1, -u_{q_2}, u_{q_1}, q_2) \end{array}} \arrow[d] \\
\varphi = \boxed{e^{q_1}\cos(q_2)} \arrow[r, dashed, leftrightarrow] & \psi = \boxed{\begin{array}{c} q_2 \arcsin(q_2 e^{-q_1}) \\ +\sqrt{e^{2q_1} - q_2^2} \end{array}}
\end{tikzcd}
$$

with $\Phi : T^*\mathbb{R}^2 \to T^*\mathbb{R}^2$, $(q_1, q_2, p_1, p_2) \mapsto (q_1, -p_2, p_1, q_2)$.

$\Gamma_\varphi = \Phi(L_{u=\varphi})$ - generalised sol

$\Gamma_\psi = L_{v=\psi}$ regular!

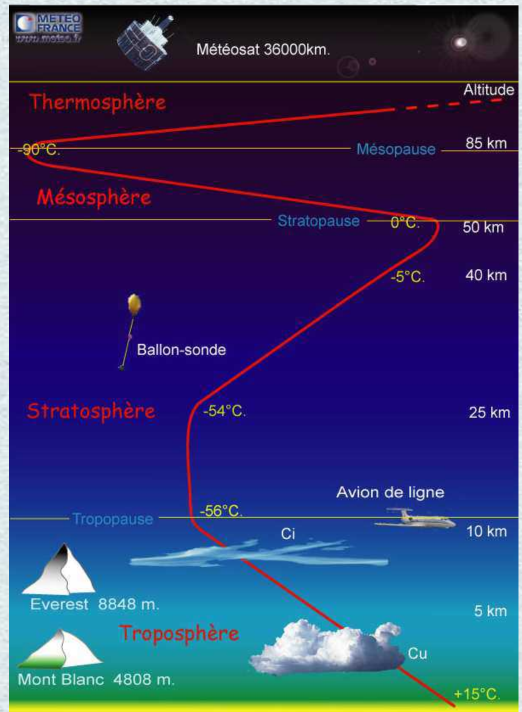


Figure: Atmosphere structure

Semi-Geostrophic model

Starting with Shallow water equations with f constant

$$\dot{\mathbf{u}} + f\mathbf{k} \times \dot{\mathbf{x}} + \nabla\phi = 0$$

and considering typical scales of order

$$U \approx 10ms^{-1}$$
$$L \approx 1000km$$
$$f \approx 10^{-4}s^{-1}$$
$$H \approx 10km$$

$$Ro = \frac{\left|\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}\right|}{|f\mathbf{u}|} \sim \frac{U}{fL} = 0.1$$

the geostrophic approximation $\dot{\mathbf{u}} \approx \dot{\mathbf{u}}_g$ is valid, where the geostrophic wind is defined by

$$\mathbf{u}_g = f^{-1}\mathbf{k} \times \nabla\phi$$

and Semi-Geostrophic equations as defined by B. Hoskins (75) are

$$\dot{\mathbf{u}}_g + f\mathbf{k} \times \dot{\mathbf{x}} + \nabla\phi = 0$$

Geostrophic coordinates

Introducing the so called geostrophic coordinates, and the Bernoulli potential

$$\mathbf{X} = \mathbf{x} - f^{-1}\mathbf{k} \times \mathbf{u}_g \quad , \quad \Phi(\mathbf{X}, t) = \phi(\mathbf{x}, t) + \frac{f^2}{2}|\mathbf{X} - \mathbf{x}|^2$$

Semi Geostrophic equations reduce to $\dot{\mathbf{X}} = f^{-1}\nabla^{\perp}_{\mathbf{X}}\Phi$

The potential vorticity, conserved quantity, reduces to

$$q_{SG} := \frac{1}{h}\left[f + \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} + f^{-1}\frac{\partial(u_g, v_g)}{\partial(x, y)}\right] = \frac{f}{h}\frac{\partial(X, Y)}{\partial(x, y)}$$

or to the equivalent Monge Ampère equations

$$q_{SG} = gf \,\mathrm{Hess}_{\mathbf{x}}\left(\phi + \frac{1}{2}|\mathbf{x}|^2\right)$$

$$\frac{1}{q_{SG}} = \frac{1}{gf}\left[\Phi - \frac{1}{2}\left|\frac{\partial\Phi}{\partial\mathbf{X}}\right|^2\right]\mathrm{Hess}_{\mathbf{X}}\left(\Phi - \frac{1}{2}|\mathbf{X}|^2\right)$$

Semi-geostrophic Model

In this model f is a constant and the acceleration vector $(\ddot{x}, \ddot{y})$ is replaced by the derivative of another vector (the geostrophic velocity) (u_g, v_g) with

$$u_g = -\frac{g}{f}\frac{\partial h}{\partial y}, \qquad v_g = \frac{g}{f}\frac{\partial h}{\partial x}, \tag{1}$$

The geostrophic approximation is given by

$$\begin{cases} \dot{u}_g + g\dfrac{\partial h}{\partial x} - \dot{y}f = 0, \\ \\ \dot{v}_g + g\dfrac{\partial h}{\partial y} + \dot{x}f = 0, \end{cases} \tag{2}$$

Potential vorticity

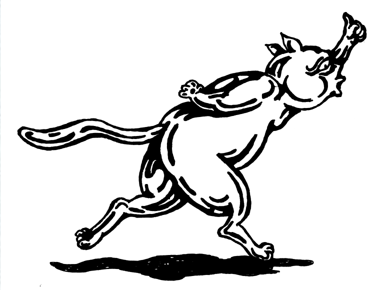
Another important cinematic notion is the potential vorticity:

$$\xi = \frac{1}{h}\left(\frac{\partial \dot{y}}{\partial x} - \frac{\partial \dot{x}}{\partial y} + f\right) \tag{3}$$

The potential vorticity associated with (1) is written as:

$$\Xi = \frac{1}{h}\left[f + \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} + \frac{1}{f}\frac{\partial(u_g, v_g)}{\partial(x, y)}\right] \tag{4}$$

Vorticity



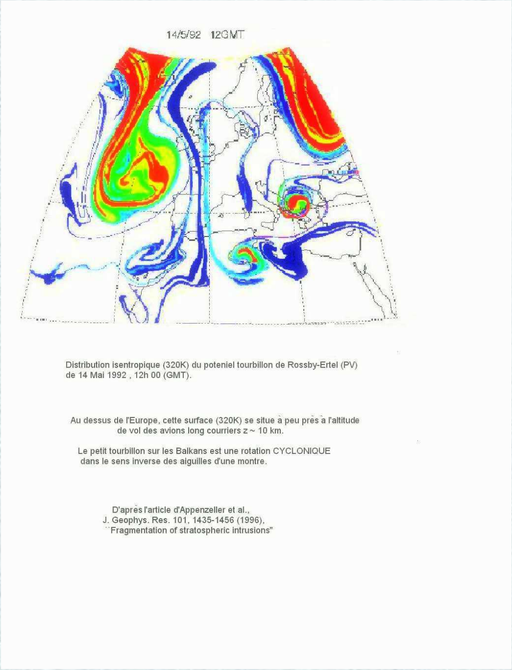


Figure: Real Potential Vorticity = Cyclons and anti-cyclons, Europe, 15 may 1992

Geostrophic Transformation

$$X = x + \frac{g}{f^2}\frac{\partial h}{\partial x}, \qquad Y = y + \frac{g}{f^2}\frac{\partial h}{\partial y}. \tag{5}$$

The coordinates (X, Y) are called *geostrophic coordinates*:
when f is a constant

$$\dot{X} = u_g, \dot{Y} = v_g.$$

Potential Vorticity and MAE

The potential vorticity in the coordinates (5) has a jacobian form (using (1)):

$$\Xi = \frac{f}{h}\frac{\partial(X,Y)}{\partial(x,y)} =$$

$$\frac{1}{h}\left[f + \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} + \frac{1}{f}\frac{\partial(u_g,v_g)}{\partial(x,y)}\right]. \tag{6}$$

This equation can be re-written as

$$\Xi = \frac{gf}{(H - \frac{1}{2}f(x^2+y^2))}\,\mathrm{det\,Hess}\,(H)\ , \tag{7}$$

où

$$H = \frac{1}{2}f(x^2+y^2) + gh$$

This a departure point of the MAE and the related geometry.

Geometric approach -2

The transformation

$$F : (q_1,\ q_2,\ p_1,\ p_2) \mapsto$$

$$(Q_1 = q_1 + p_1,\ Q_2 = q_2 + p_2)\,,$$

$$(P_1 = p_1,\ P_2 = p_2)$$

is a canonical.

The function

$$f \stackrel{\text{def}}{=} \frac{1}{2}(p_1^2 + p_2^2)$$

is a generating function to F.

Geometric approach - 3

The "contactization" of $(\mathcal{V}, \Omega)$:

$$(q_1,\ q_2,\ p_1,\ p_2) \hookrightarrow (q_1,\ q_2, u,\ p_1,\ p_2) \in \mathbb{R}^5$$
$$= J^1(\mathcal{D}).$$

A "lift" F_c of the transformation F to the contact space of 1-jets:

$$F_c : (q_1,\ q_2,\ p_1,\ p_2)$$
$$\mapsto (Q_1 = q_1 + p_1,\ Q_2 = q_2 + p_2)\,,$$
$$\left(V = u + \frac{1}{2}(p_1^2 + p_2^2),\ P_1 = p_1,\ P_2 = p_2\right)$$

The transformation F_c is a contact:
$F_c^*(U) = U$,
where

$$U = du - p_1 dq_1 - p_2 dq_2.$$

Contact Structure

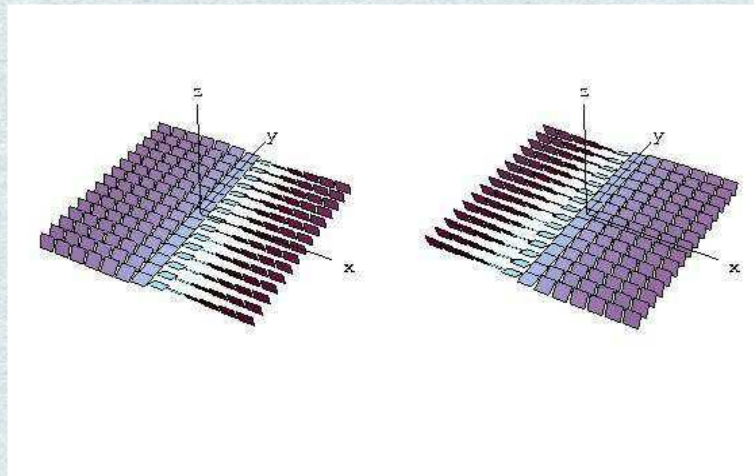


Figure: Contact structure in $\mathbb{R}^3$

The graph of the geopotential ϕ is a Legendre submanifold

$$L = \left\{ u = \phi,\ p_1 = \frac{\partial \phi}{\partial q_1}, p_2 = \frac{\partial \phi}{\partial q_2} \right\}$$

in $J^1(\mathcal{D})$.
Let ω be an effective 2-form on $J^1(\mathcal{D})$:

$$\omega = E dq_1 \wedge dq_2 + B\left(dq_1 \wedge dp_1 - dq_2 \wedge dp_2\right) + \tag{8}$$

$$+\ C dq_1 \wedge dp_2 - A dq_2 \wedge dp_1 + D dp_1 \wedge dp_2 .$$

The evaluation of the form ω on the graph of ϕ defines a MAE:

$$\Delta_\omega(\phi) = 0.$$

The Hamiltonian Form

$$\dot{Q}_1 = -\frac{\partial \Phi}{\partial Q_2}, \qquad \dot{Q}_2 = \frac{\partial \Phi}{\partial Q_1}, \tag{9}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \dot{q}_1 \frac{\partial}{\partial q_1} + \dot{q}_2 \frac{\partial}{\partial q_2} = \frac{\partial}{\partial t} + \dot{Q}_1 \frac{\partial}{\partial Q_1} + \dot{Q}_2 \frac{\partial}{\partial Q_2}. \tag{10}$$

The map $\{q_1, q_2\} \mapsto \{Q_1, Q_2\}$ is a Legendre transformation:

$$Q_1 = \psi_{q_1}, \;\; Q_2 = \psi_{q_2},$$

where

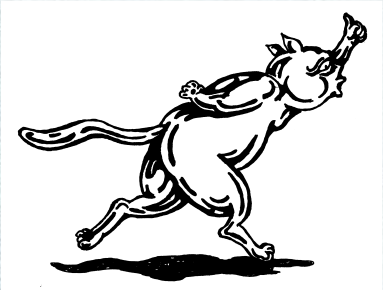
$$\psi = \phi + \frac{1}{2}(q_1^2 + q_2^2)$$

and

$$q_1 = \Psi_{Q_1}, \;\; q_2 = \Psi_{Q_2}$$

with $\Psi = 1/2(Q_1^2 + Q_2^2) - \Phi$.
Singularities of the application are interpreted as atmospheric wave fronts (Chynnoweth and Sewell 1989, 1991).

Wave fronts



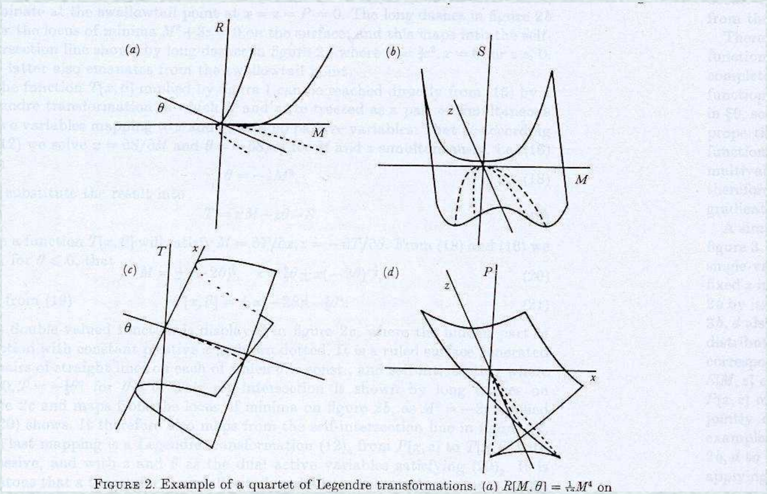


Figure: Wave fronts and legendrian singularities

The conservation law (the Ertel's theorem) of the potential vorticity obtains (using the Hamiltonian representation of the system):

$$\frac{d}{dt}\left(\frac{\partial(q_1, q_2)}{\partial(a, b)}\right) =$$

$$\frac{d}{dt}(1 + \phi_{q_1 q_1} + \phi_{q_2 q_2} + \det \mathrm{Hess}\, \phi\ ) = 0,$$

This equation is a part of the HyperKähler triple of MAEs (R. and Roulstone 1997, 2001):

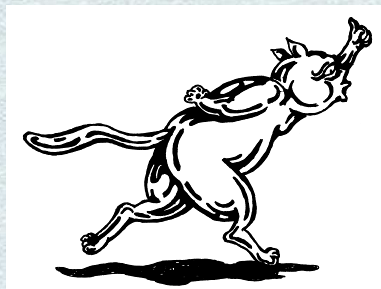
$$F^*(dI \wedge d\bar{I}) = . \tag{11}$$

$$\Omega - 2k[1 + a(p_{11} + p_{22}) + (a^2 - c^2)(p_{11}p_{22} - p_{12}^2)dq_1 \wedge dq_2]$$

The general family of (elliptic) MAE with constant coefficients carries all flat balanced models:

$$1 + \phi_{q_1 q_1} + a\phi_{q_2 q_2} + (a^2 - c^2)\,\mathrm{det\,Hess}\,\phi = \zeta^{\mathbf{C}}/f, \qquad (12)$$

Among them are:
The semi-geostrophic model($a = 1,\ c = 0$ with $\zeta^{\mathbf{C}}/f$ positive);
The L_1 Salmon dynamics with $a = c = 1$;
The $\sqrt{3}$ dynamics of McIntyre - Roulstonefor $a = 1, c = \sqrt{3}$ and $\zeta^{\mathbf{C}}/f < 3/2$;
Our classification theorem in $2d$ gives a classification of all "almost-balanced"($0 < c < \sqrt{3}$) models with a uniform potential vorticity.



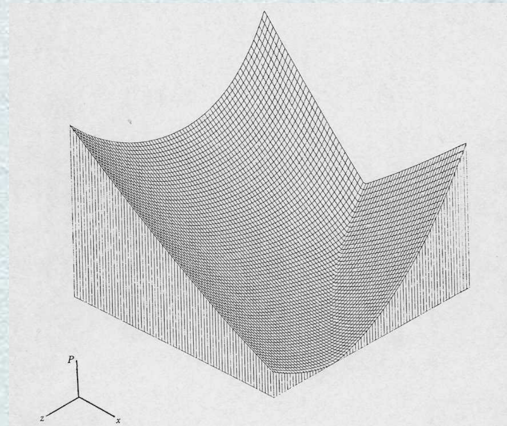


Figure: Numerical Solution of the semi-geostrophic $3d$ equation

$$hess_{x,y}u + \frac{\partial^2 u}{\partial z^2} = hessu \tag{13}$$

Sewell-Chynoweth MAO form and its equivalence

- The effective form of (1):

$$\omega = dp \wedge dq \wedge dz + dx \wedge dy \wedge dr - \gamma dx \wedge dy \wedge dz,$$

$(x, y, z, p, q, r)-$ canonical coordinates system of $T^*\mathbb{R}^3$.

- This form is a sum of two decomposable 3-forms:

$$\omega = dp \wedge dq \wedge dz + dx \wedge dy \wedge (dr - \gamma dz).$$

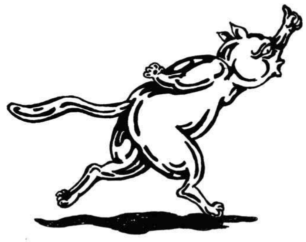
- $\phi^*(\omega) = dp \wedge dq \wedge dr - dx \wedge dy \wedge dz$ where ϕ is the symplectomorphism

$$\phi(x, y, z, p, q, r) = (x, y, r, p, q, \gamma r - z).$$

- The equation (1) is symplectically equivalent to the equation

$$hess(u) = 1. \qquad (2)$$

An exact solution of the SG 3D equation



$$f(x, y, z) = \int_a^{\sqrt{xy+yz+zx}} (b + 4\xi^3)^{1/3} d\xi$$

is a regular solution of (2). Therefore,

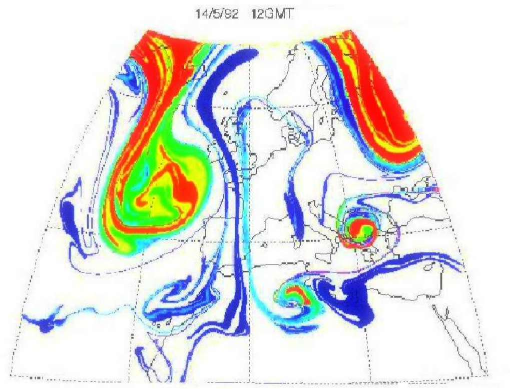
$$L = \Big\{(x, y, (x + y)\alpha, (y + z)\alpha, (z + x)\alpha, \gamma(x + y)\alpha - z)\Big\}$$

is a generalised solution of (1) with

$$\alpha = \frac{1}{2}\Big(\frac{b}{(xy + yz + zx)^{\frac{3}{2}}} + 4\Big)^{\frac{1}{3}}.$$

Hoskins geostrophic coordinate transformation

- The SG equations are used like a good approximation to the Boussinesq primitive equations when the rate of the flow momentum is smaller than the Coriolis force, or in other words, when the Rossby number $\mathrm{Ro} << 1$.
- Potential vorticity is a fundamental concept for understanding the generation of vorticity in cyclogenesis (the birth and development of a cyclone), especially along the polar front, and in analyzing flow in the ocean.
- B. Hoskins (1975) had proposed a remarkable coordinate transformation (a passage to geostrophic coordinates in $x - y$ directions) such that the geostrophic velocity and potential temperature may be represented in terms of one function both in the transformed coordinates as in physical ones

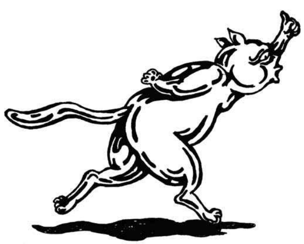


Distribution isentropique (320K) du poteniel tourbillon de Rossby-Ertel (PV)
de 14 Mai 1992 , 12h 00 (GMT).

Au dessus de l'Europe, cette surface (320K) se situe à peu près à l'altitude
de vol des avions long courriers z ~ 10 km.

Le petit tourbillon sur les Balkans est une rotation CYCLONIQUE
dans le sens inverse des aiguilles d'une montre.

D'après l'article d'Appenzeller et al.,
J. Geophys. Res. 101, 1435-1456 (1996),
"Fragmentation of stratospheric intrusions"



$$\begin{cases} X := x + \frac{v_g}{f} = x + \frac{1}{f^2}\frac{\partial \phi}{\partial x} \\ Y := y - \frac{u_g}{f} = y + \frac{1}{f^2}\frac{\partial \phi}{\partial y} \\ Z := z; \quad T := t. \end{cases}$$

Hoskins geostrophic 3D equation

- Let $\Phi := \phi + \frac{1}{2}(u_g^2 + v_g^2)$ then $\nabla\Phi = \nabla\phi$ and
- if the potential vorticity is uniform ($q_g = \frac{f\theta_0}{g}N^2$) then one have in the interior of the fluid for any time $T = t$

$$\frac{1}{f^2}(\Phi_{XX} + \Phi_{YY}) - \frac{1}{f^4}(\Phi_{XX}\Phi_{YY} - \Phi_{XY}^2) + \frac{1}{N^2}\Phi_{ZZ} = 1. \quad (3)$$

- Here (and in what follows) f is the *Coriolis parameter* taking as a constant and N is the *Brunt - Väisälä* frequency:

$$N = \sqrt{\frac{q_g g}{f\theta_0}},$$

for the uniform potential vorticity q_g and the constant potential temperature θ_0.

Hoskins geostrophic MA effective form : equivalence

- Consider the symplectomorphism

$$F(x,y,z,p,q,r) = (p,q,z,-x+f^2p,-y+f^2q,r). \tag{4}$$

- The new canonical coordinate system $(\tilde{x},\tilde{y},\tilde{z},\tilde{p},\tilde{q},\tilde{r})$

$$\begin{cases} \tilde{p} := -x+f^2p; \quad \tilde{x} := p; \\ \tilde{q} := -y+f^2q; \quad \tilde{y} := q; \\ \tilde{r} := r; \quad \tilde{z} := z \end{cases}$$

with $\tilde{\Omega} = \Omega$, provides the following effective form:

- $$\tilde{\omega} = \frac{1}{N^2} d\tilde{p} \wedge d\tilde{q} \wedge d\tilde{r} - \frac{1}{f^4} d\tilde{x} \wedge d\tilde{y} \wedge d\tilde{z}.$$

- The Hoskins SG (3) is equivalent to the (1):

$$\mathrm{hess}(u) = \frac{N^2}{f^4} = \frac{(q_g g)^2}{f^6(\theta_0)^2} \tag{5}$$

by the symplectomorphism (4).

HyperKäler triple of MAE

The conservation law of the potential vorticity (the Ertel's theorem) obtains (using the Hamiltonian representation of the system):

$$\frac{d}{dt}\left(\frac{\partial(q_1,q_2)}{\partial(a,b)}\right) =$$

$$\frac{d}{dt}(1+\phi_{q_1q_1}+\phi_{q_2q_2}+\det\mathrm{Hess}\,\phi\ )=0,$$

This equation is a part of the HyperKähler triple of MAEs (R. and Roulstone 1997, 2001):

$$\begin{cases}\omega_I=\left[1+a(p_{11}+p_{22})+(a^2-c^2)(p_{11}p_{22}-p_{12}^2)dq_1\right]\wedge dq_2 & ,\\ \omega_J=\left[2cp_{12}+ac(p_{11}p_{22}-p_{12}^2)\right]dq_1\wedge dq_2 & ,\\ \omega_K=-c\Omega\end{cases}$$

$2D$ balanced model MAE

- The general family of (elliptic) MAE with constant coefficients carries all flat balanced models:

$$1 + \phi_{q_1 q_1} + a\phi_{q_2 q_2} + (a^2 - c^2)\det \mathrm{Hess}\,\phi = \zeta^{\mathbf{C}}/f, \qquad (6)$$

  Among them are:
- *The semi-geostrophic model* ($a = 1,\ c = 0$ with $\zeta^{\mathbf{C}}/f$ positive);
- *The L_1 Salmon dynamics* with $a = c = 1$;
- *The $\sqrt{3}$ dynamics of McIntyre - Roulstone* for $a = 1, c = \sqrt{3}$ and $\zeta^{\mathbf{C}}/f < 3/2$;

  Our classification theorem in $2D$ gives a classification of all "almost-balanced"($0 < c < \sqrt{3}$) models with a uniform potential vorticity.

Dritschel-Viudez MAE

Recently a new approach to modelling stably-stratified geophysical flows was proposed by Dritschel and Viudez. This approach is based on the explicit conservation of potential vorticity and uses a change of variables from the usual primitive variables of velocity and density to the components of ageostrophic horizontal vorticity and a Monge-Ampère-like nonlinear equation with non-constant coefficients arises. The equation changes the type from elliptic to hyperbolic:

$$E\left(\Phi_{xx}\Phi_{zz}-\Phi_{xz}^2\right)+A\Phi_{xx}+2B\Phi_{xz}+C\Phi_{xz}+D=0 \qquad (8)$$

with

$$E=1\,,\quad A=1+\varphi_{xz}\,,\quad B=\frac{1}{2}\left(\varphi_{zz}-\varphi_{xx}\right)$$

$$C=1-\varphi_{xz}\,,\quad D=\varphi_{xx}\varphi_{zz}-\varphi_{xz}^2-\varpi$$

where φ is a given potential and the dimensionless PV anomaly ϖ may be also considered as a given quantity.

Integrability of the complex/product structure

- **THEOREM** (B.Banos, V.R.) $2D$ Dritschel-Viudez equation is locally equivalent to a Monge-Ampère equation with constant coefficients if and only if $\begin{cases} \Delta\varphi = 2c_1 \\ R = c_2 \end{cases}$
- for $R > 0$, we see that

$$\omega + i\sqrt{R}\,\Omega = du \wedge dv$$

with

$$\begin{cases} u = x - (c_1 - ic_2)z - \varphi_z + p \\ v = -(c_1 + ic_2)x + \varphi_x + r \end{cases}$$

- $\varphi = 2c_1$ and $R = c_2 > 0$ then $2D$ Dritschel-Viudez equation is equivalent to Laplace equation

$$\varphi_{xx} + \varphi_{zz} = 0$$

modulo the Legendre transform

$$F(x, z, p, r) = \frac{1}{\sqrt{R}}(x - c_1 z - \varphi_z, c_2 z, -c_2 x, -c_1 x + \varphi_x + r).$$

The corresponding Monge-Ampère structure

$$\begin{cases} \Omega = dx \wedge dp + dz \wedge dr \\ \omega = Edp \wedge dr + Adp \wedge dz + B(dx \wedge dp - dz \wedge dr) + \\ + Cdx \wedge dr + Ddx \wedge dz \end{cases}$$

The pfaffian is $\mathrm{pf}(\omega) = R$ with R the Rellich's parameter:

$$R = AC - ED - B^2 = 1 + \varpi - \left(\frac{\Delta\varphi}{2}\right)^2$$

A direct computation gives

$$d\omega = d\left(\frac{\Delta\varphi}{2}\right) \wedge \Omega \tag{9}$$

Hitchin hypersymplectic geometry-1

- if our 4-dimensional manifold M, endowed with the Monge-Ampère structure (Ω, ω) admits a lagrangian fibration (main example: M is the cotangent bundle of a smooth $2D-$manifold), then it exists a conformal split metric on M^4.
- When the corresponding Monge-Ampère equation is given by (8), this metric writes as

$$g = C(dx)^2 - 2Bdxdz + A(dz)^2 + E/2(dpdx + dqdz), \quad (10)$$

Using this metric, we get an additional 2-form $\hat{\omega}$ defined by

-
$$\hat{\omega}(\cdot\,,\,\cdot) = g(A_\omega\cdot\,,\,\cdot) \quad \text{with} \quad \omega(\cdot\,,\,\cdot) = \Omega(A_\omega\cdot\,,\,\cdot)$$

In coordinates,

$$\hat{\omega} = \left(-2AC + 2B^2 + D\right) dx \wedge dz - Bdx \wedge dp - Cdx \wedge dr$$
$$+ Adz \wedge dp + Bdz \wedge dr - dp \wedge dr$$

Underlying hypersymplectic geometry-2

In coordinates,

$$\hat{\omega} = \left(-2AC + 2B^2 + D\right) dx \wedge dz - Bdx \wedge dp - Cdx \wedge dr +$$

$$+ Adz \wedge dp + Bdz \wedge dr - dp \wedge dr$$

Introducing $\Theta = \dfrac{\Omega}{\sqrt{|R|}}$, we get an hypersymplectic triple $(\Theta, \omega, \hat{\omega})$ satisfying

$$\omega^2 = -\hat{\omega}^2 = \pm\Theta^2$$

$$\omega \wedge \hat{\omega} = \omega \wedge \Theta = \hat{\omega} \wedge \Theta = 0$$

Reminder: Recursion (Nijenhuis) operator

1° M; $\omega, \eta \in \Omega^2(M)$ - non-degenerate

$\exists!$ field $R \in End(TM)$ s.t.

$$\iota_X \omega = \iota_{RX} \eta \quad , \quad X \in \Gamma(TM)$$

For closed (symplectic) ω, η - important (bi-hamiltonian systems, etc.)

The field R - recursion or Nijenhuis operator

2° Symplectic pairs $R^2 = -Id$

Thm If the recursion operator R, s.t. $R^2 = -Id_{TM}$ then it defines integrable complex structure with holomorphic symplectic form Ω:

$$\Omega \in \Lambda^{2,0}(T_{\mathbb{C}} M) \ , \ \Omega = \omega + i\eta$$

closed non-degenerate.

Triples of symplectic forms:

M, $\omega_1, \omega_2, \omega_3 \in \Omega^2(M)$ symplectic

$$i_X \omega_i = i_{R_{i+2}X}\, \omega_{i+1} \quad \text{mod } 3$$

and all recursion operators satisfy

$R_i^2 = \pm Id$ $(R \neq \pm Id)$. From the definition any cyclic composition

$$R_{i+2} \circ R_{i+1} \circ R_i = Id$$

1° When all three $R_i^2 = -Id \Longrightarrow$

hypercomplex structure

2° If exactly only one $R_i^2 = -Id \Longrightarrow$

hyper-symplectic structure

Underlying hypersymplectic geometry-3

Equivalently, we obtain 3 tensors I, S and T satisfying

$$
\begin{aligned}
I^2 &= -1,\ S^2 = 1,\ T^2 = 1 \\
ST &= -TS = -I \\
TI &= -IT = S \\
IS &= -SI = T
\end{aligned}
$$

Moreover we have

$$d\hat{\omega} = -d\left(\frac{\Delta\varphi}{2}\right) \wedge \Omega$$

Underlying hypersymplectic geometry-4

Hence, when $\Delta\varphi = 0$, then ω and $\hat{\omega}$ are closed and satisfy $\omega^2 = -\hat{\omega}^2$: they define then an integrable product structure. Indeed, in the new coordinates:

$$X = \int R(x,z)\mathrm{d}x \qquad U = x - \varphi_z + p$$

$$Z = z \qquad V = z + \varphi_x + r$$

we see that

$$\begin{cases} \omega = dU \wedge dV - dX \wedge dZ \\ \\ \hat{\omega} = -(dU \wedge dV + dX \wedge dZ) \\ \\ \Omega = \dfrac{1}{R}(dX \wedge dU - SdZ \wedge dU + RdZ \wedge dV) \quad \text{with } S = \displaystyle\int R_z dx \end{cases}$$

Underlying hypersymplectic geometry-5

In other words, when φ is harmonic, a submanifold

$$L = \left\{ (\psi_Z, Z, U, \psi_U),\ (Z, U) \in \mathbb{R}^2 \right\}$$

is a generalized solution of $2D$ - Dritschel Viudez equation if and only if

$$\psi_{ZZ} + R\psi_{UU} = S$$

Special case of $2D-$diagnostic equation of Driitschel-Viduez

In the following partial case this diagnostic equation which corresponds to the choice

$$A = 1,\ B_1 = B_2 = 0,\ C = E = \epsilon^2,\ D = -\varpi, \tag{10}$$

becomes two-dimensional:

$$u_{xx} + u_{zz} + u_{xx}u_{zz} - u_{xz}^2 = \varpi. \tag{11}$$

It describes a geostrophically-balanced steady $2D$ flow which is closed to the QG models described in other lectures.
The corresponding effective form

$$\omega = dp \wedge dx + dr \wedge dz + dp \wedge dr - \varpi dx \wedge dz$$

has the Pfaffian $\operatorname{pf}\omega = 1 + \varpi$. In the classical notations the Pfaffian is nothing but the Rellich's parameter $R = AC - DE - B^2$.

We denote as usually by Π the dimensionless potential vorticity which relates to the PV anomaly ϖ as

$$\varpi \equiv \Pi - 1,$$

hence we had obtained the following meaning of the Pfaffian for $2D$ flow MAE

$$\mathrm{pf}\, \omega = \Pi.$$

or, in the case of the $2D$ diagnostic equation this metric depends on the potential θ and on the PV anomaly ϖ:

$$g = (1+\theta_{xz})(dx)^2 - (\theta_{zz} - \theta_{xx}) dxdz + (1-\theta_{xz})(dz)^2 + 1/2(dpdx + dqdz).$$

Constant coefficient case-1

Now we will discuss a reducibility of this equation to a normal form with constant coefficients.
Let us calculate the symplectic invariant for given θ and ϖ and check the criteria. The direct computation gives that

$$dw = 1/2d(\Delta\theta) \wedge \Omega.$$

Proposition

If the given potential function θ is a harmonic ($\Delta\theta = 0$) then the 2D diagnostic equation of Dritschel and Viduez is reducible by a local symplectomorphism to a MA equation with constant coefficients.

Constant coefficient case-2

We will check it by a direct computation: the Pfaffian $Pf(\omega)$ of the corresponded effective 2-form ω can be expressed as

$$\mathrm{pf}(\omega) = (1 - \theta_{xz}^2) - 1/4(\theta_{zz} - \theta_{xx})^2 - (\mathrm{hess}\,\theta - \varphi).$$

Then we can easily verify that this Pfaffian (which is also is the Rellich invariant of the diagnostic equation with a harmonic potential θ) is equal to $1 + \varpi = \Pi$. This is exactly the same value as it was in the above-mentioned $2D$ model with constant coefficients.

Constant coefficient case-3

A "Legendre-dual" potential satisfying to hess $\theta = 1$ provides us also with an example of reducible to constant coefficients MAE diagnostic equation.
The Pfaffian in this case is equal to $1 - 1/4(\Delta\theta)^2 + \cancel{\varphi}\,\varpi$. The Jorgens theorem implies that $\theta(x, z) = \alpha x^2 + 2\beta xz + \gamma z^2$ modulo linear terms with constant $\alpha, \beta, \gamma, \alpha\beta - \gamma^2 = 1/4$. The Laplacian value in this case is $2(\alpha + \gamma)$ and the P~~a~~faffian is equal to

$$\Pi - (\alpha + \gamma)^2 = \Pi - (\alpha^2 + 2\beta^2 + \gamma^2 + 1/2).$$

$D = 1 - \varpi,\ E = 1,\ A = 1 + \theta_{xz}\quad B = \frac{1}{2}(\theta_{zz} - \theta_{xx})$

$C = 1 - \theta_{xz}$

$Pf = 1 + \varpi - \frac{1}{4}(\Delta\theta)^2\ ;\quad d\omega = \frac{1}{2}\, d \log|Pf(\omega)| \wedge \Omega = 0;$

3D diagnostic Dritschel-Viudez

MAE: rotating stratified flows

Let us consider the 3d-Dritschel-Viudez equation:

$$E\left(\Phi_{zz}(\Phi_{xx}+\Phi_{yy})-\Phi_{xz}^2-\Phi_{yz}^2\right)+$$
$$A(\Phi_{xx}+\Phi_{yy})+2B_1\Phi_{xz}+2B_2\Phi_{yz}+$$
$$C\,\Phi_{zz}+D=0$$

With

$$A=1+\Theta_z\,,\quad B_1=\frac{1}{2}\Delta\varphi-\Theta_x\,,$$
$$B_2=\frac{1}{2}\Delta\psi-\Theta_y\,,\quad C=1-\Theta_z\,,\quad E=1$$
$$D=\Theta_x\Delta\varphi+\Theta_y\Delta\psi-|\nabla\Theta|^2-\overline{\omega}$$

φ, ψ - potentials, $\Theta=\varphi_x+\psi_y$

$\overline{\omega}$ dimensionless PV-anomaly

$$\omega = E(dx\wedge dq\wedge dr - dy\wedge dp\wedge dr) +$$
$$+ A(dy\wedge dz\wedge dp - dx\wedge dz\wedge dq) +$$
$$+ B_1(dx\wedge dy\wedge dp + dy\wedge dz\wedge dr) +$$
$$+ B_2(dx\wedge dy\wedge dq - dx\wedge dz\wedge dp)$$
$$+ C\, dx\wedge dy\wedge dr + D\, dx\wedge dy\wedge dz$$

Independently of coefficients:

the Hitchin pfaffian $h_\omega \equiv 0$

The Dritschel-Viudez 3D diagnostic equation belongs to the class of degenerated MA geometries (potentially very interesting for Doubrov-Ferapontov-Novikov-Kruglikov "hydrodynamic integrability)

Generalized Rellich's invariant

$$R := AC - ED - B_1^2 - B_2^2$$

This parameter (analogously 2D) separates <u>elliptic</u> and <u>hyperbolic</u> cases. Remind that

$R = AC - ED - B^2 = pf(\omega)$ in 2D case.

g_ω has non-zero eigenvalues

$$\lambda_1 = 2R, \quad \lambda_{2,3} = S \pm \sqrt{S^2 - 4(1+A^2)R}$$

where $S := A^2 + B_1^2 + B_2^2 + R + 1$

<u>Proposition</u> 1) If $R > 0$ $\operatorname{sign} g_\omega = (3,0)$

elliptic 3D-Dritschel-Vinder MAE is equivalent to $\Delta u = 0$

2) If $R < 0$ $\operatorname{sign} g = (1,2)$

hyperbolic 3D MAE is equivalent to $\Box u = 0$.

<u>Open question</u>: Invariant meaning of R

$2D$- Dritschel-Viudez equation and underlying generalized complex geometry

Relation (8) implies that $2D$- Dritschel Viudez equation is of divergent type since

$$d(\omega + \lambda\Omega) = 0 \quad \text{with} \quad \lambda = -\frac{\Delta\varphi}{2}$$

and to any 2D Monge - AmpΓïre equation $\Delta_\omega = 0$ of divergent type corresponds an integrable generalized complex structure $\mathbb{J}_\omega : M^4 \to \mathrm{End}(TM \oplus T^*M)$ (B.Banos).
It is defined by:

$$\mathbb{J}_\omega = \left(\begin{array}{c|c} A_\omega - \lambda & \Omega^{-1} \\ \hline 2\lambda\omega - (1 - R + \lambda^2)\Omega & \lambda - A^*_\omega \end{array} \right)$$

A conservation law of $\Delta_\omega = 0$ is a 1-form $\alpha \in \Omega^1(M^4)$ such that $d\alpha|_L = 0$ on any generalized solution L.
The Hodge-Lepage-Lychagin theorem implies that α is a conservation law if and only if $d\alpha = f\omega + g\Omega$. The function f is called a generating function with conjugate g

It is proved by B/Banos that a function f is a generating function if and only it is pluriharmonic on $(M, \mathbb{J}_\omega)$, that is

$$\partial_\omega \overline{\partial}_\omega f = 0$$

here $\overline{\partial}_\omega = d + i\mathbb{J}_\omega \circ d\,\mathbb{J}_\omega$ (Gualtieri)

Example

$f(x, z, p, r) = x$ is a generating function for 2D - Dritschel Viudez equation with conjugate function

$$g(x, z, p, r) = \varphi_x + z + r$$

Hypercomplex geometry and Von Karman equation

Consider an elliptic Monge-Ampère equation $\Delta_\omega = 0$ with $d\omega = 0$ and $\Omega \wedge \omega = 0$. Assume moreover it exists a closed 2-form Θ such that

$$\Omega \wedge \Theta = \omega \wedge \Theta = 0$$

and

$$4\omega = \Omega^2 + \Theta^2.$$

Note that $\exp(\omega - i\Omega)$ and $\exp(-\omega - i\Theta)$ satisfy the conditions of the above lemma. We suppose also that $\Theta^2 = \lambda^2\Omega$ with λ a non vanishing function. This implies that $\omega^2 = \mu^2\Omega^2$ with

$$\mu = \frac{\sqrt{1+\lambda^2}}{2}.$$

The triple (ω, Ω, Θ) defines a metric G and an almost hypercomplex structure (I, J, K) such that

$$\omega = \mu G(I\cdot, \cdot), \quad \Omega = G(J\cdot, \cdot), \quad \Theta = \lambda G(K\cdot, \cdot).$$

Define now the two almost complex structures

$$I_+ = \frac{K + \lambda J}{\mu}, \quad I_- = \frac{K - \lambda J}{\mu}.$$

From

$$\omega = \frac{\Omega + \Theta}{2}(I_- \cdot, \cdot)$$

and

$$\omega = \frac{\Omega - \Theta}{2}(I_+ \cdot, \cdot)$$

we deduce that I_+ and I_- are integrable.

Consider again the Von Karman equation

$$v_x v_{xx} - v_{yy} = 0.$$

with corresponding primitive and closed form

$$\omega = p_1 dq_2 \wedge dp_1 + dq_1 \wedge dp_2.$$

Define then Θ by

$$\Theta = dp_1 \wedge dp_2 + (1 + 4p_1) dq_1 \wedge dq_2.$$

With the triple (ω, Ω, Θ) we construct I_+ and I_- defined by

$$I_+ = \frac{1}{2}\begin{pmatrix} 0 & -1 & 1 & 0 \\ -1/p_1 & 0 & 0 & -1/p_1 \\ -(1+4p_1)/p_1 & 0 & 0 & -1/p_1 \\ 0 & 1+4p_1 & -1 & 0 \end{pmatrix}$$

$$I_- = \frac{1}{2}\begin{pmatrix} 0 & -1 & -1 & 0 \\ -1/p_1 & 0 & 0 & 1/p_1 \\ (1+4p_1)/p_1 & 0 & 0 & -1/p_1 \\ 0 & -(1+4p_1) & -1 & 0 \end{pmatrix}$$

It is worth mentioning that I_+ and I_- are well defined for all $p_1 \neq 0$. But the metric G is definite positive only for $p_1 < -\frac{1}{}$

$p_1 \neq 0$ But the metric G is definite posi-tive only for $p_1 < -1/4$.

Thank you for your attention!