Gauge PDEs on manifolds with boundaries and asymptotic symmetries

Maxim Grigoriev

UMONS
&
Lebedev Inst., Lomonosov MSU

Based on: MG, Mikhail Markov 2023; MG 2022

Earlier relevant works: Bekaert MG 2012,2013, MG Kotov 2019, Barnich MG 2010

Background

- Very often we deal with gauge field theories on manifolds with boundaries: AdS/CFT, flat space holography, . . .
- The first-principle and probably most fundamental approach to gauge theories is provided by Batalin-Vilkovisky (BV) formalism. Locality is taken care of by merging BV with the jet-bundle langauge *Henneaux, Barnich, Brandt, . . .*
- BV on manifolds with boundaries *Cattaneo et. all 2012 . . .* .
- An alternative based on AKSZ-like representation *Barnich MG, 2003; Bekaert MG 2012*.
- More general object: gauge PDE *Barnich MG 2010, MG Kotov 2019*. Behave well with respect to restriction to submanifolds/boundaries.

- Aim of this talk: full-scale graded geometry approach to gauge theories with boundaries and their symmetries.

Other sources of ideas and methods:
- Geometric of PDEs *Vinogradov, Tulczyjew, . . .*
- Covariant phase-space *Kijowski, Tulczyjew; Lee, Wald, . . .*
- Unfolded approach in higher spin gauge theory *M.Vasiliev*
- FDA approach to SUGRA *d'Auria, Fre, Castellani, Grassi . . .*
- Conformal geometry, tractor calculus *Eastwood, Gover, Cap, . . .* . *Fefferman-Graham* construction.
- BRST first-quantized (L_∞) approach to SFT and HS gauge fields *Zwiebach; Thorn, Bochicchio, Bengtsson, Stern, Ouvry, Henneaux, Teitelboim, ...*
- *Fedosov* quantization and its variations

Boundary data

Suppose we are given with a local gauge theory on spacetime $X = \Sigma \times \mathbb{R}_{\geqslant 0}$, $\Sigma = \partial X$.

- What geometrical/physical data is induced on Σ?. What is the first principle and invariant approach to extract it?

More generally: low dimensional strata (corners, submanifolds, defects, . . .)

Digression: 1d, $\Sigma = pt$. No gauge symmetry specified

Equations of motion:

$$\dot{\psi}^A = V^A(\psi(t), t)$$

The structure induced on Σ is just the phase-space: manifold M with coordinates ψ^A which parameterize the initial data for the dynamics. In this case M is the phase space. In the variational case M comes equipped with a symplectic structure, giving a Hamiltonian formulation.

Message: the boundary data is described by a pullback of the ODE (understood in inrinsic terms) to the boundary.

PDE on $\Sigma \times \mathbb{R}_{\geqslant 0}$. No gauge symmetry specified

A given PDE can be equivalently represented as a fiber-bundle $E \to X$, base coordinates x^μ, fiber coordinates ψ^A. Solutions are sections $\psi^A = \sigma^A(x)$ satisfying:

$$\frac{\partial}{\partial x^\mu}\sigma^A(x) = \Gamma^A_\mu(\sigma(x), x)\,,$$

where $\Gamma^A_\mu(\psi, x)$ are coefficients of the flat Ehresmann connection $\Gamma = dx^\mu \Gamma^B_\mu \frac{\partial}{\partial \psi^B}$ on E describing the Cartan distribution. Invariant definition of PDE due to *Vinogradov*; cf. unfolded formulation by *Vasiliev*.

The boundary data is obtained by pulling back (E, Γ) to Σ. This is a new PDE on Σ whose connection is given by a pull-back of Γ. In coordinates: $i^*(\Gamma) = dx^i \Gamma^A_i \frac{\partial}{\partial \psi^A}$, where $x^\mu = \{x^0, x^i\}$ and Σ is singled out by $x^0 = 0$.

Gauge PDEs. Toy model $X = pt$ (0-dimensional).

Def. Q-manifold is a $\mathbb{Z}$-graded manifold M equipped with a degree 1 nilpotent vector field Q, $\mathrm{gh}(Q) = 1$.

Standard example: $M = T[1]X$, $Q = \theta^a \frac{\partial}{\partial x^a}$ – de Rham differential.

Gauge PDE in 0-dim is a Q-manifold equivalent to a nonegatively graded one.

Solutions: zero locus, i.e. $p \in M$ such that $Q(p) = 0$.
Gauge parameters: vector fields Y, $\mathrm{gh}(Y) = -1$.
Gauge transformations generated by $[Q, Y]$

Equivalence of Q-manifolds

Idea: restrict to local analysis. Let

$$M = N \times T[1]V\,, \qquad Q = Q_N + d_{T[1]V}$$

with V a graded space. Then (M, Q) and (N, Q_N) are equivalent. Q-manifold $(T[1]V, d_{T[1]V})$ is called contractible. In coordinates:

$$Q = Q_N + v^\alpha \frac{\partial}{\partial w^\alpha}\,, \qquad Q_N = q^i(\phi)\frac{\partial}{\partial \phi^i}\,.$$

Often one finds a "minimal" equivalent Q-man.

Geometric characterization: let w^a be independent functions such that w^a, Qw^a are independent, the surface $w^a = 0 = Qw^a$ is a Q-submanifold isomorphic to (N, Q_N). Simple geometric picture of the homotopy transfer. From gauge theory perspective w^α, v^α – are known as "generalized auxiliary fields" *Henneaux, 1990; Barnich, M.G. 2004*.

Def. *[Kotov, Strobl]* Locally trivial bundle $\pi : E \to M$ of Q-manifolds is called Q-bundle if π is a Q-map. Section $\sigma : M \to E$ is called Q-section if it's a Q-map.
In general, $\pi : E \to M$ is not a locally trivial Q-bundle.
Indeed, although locally $E \cong M \times F$ (product of manifolds) in general Q is not a product Q-structure of some Q_F and Q_M.

Example: let $\pi_X \colon E \to X$ be a fiber bundle then
$\pi = d\pi_X \colon (T[1]E, d_E) \to (T[1]X, d_X)$ is a Q-bundle.

Def. *[MG, Kotov]* (N, Q_N) is called an equivalent reduction of (M, Q) if (M, Q) is a locally trivial Q-bundle over (N, Q_N) with a contractible fiber and $(M, Q) \to (N, Q_N)$ admits a global Q-section.

This generates an equivalence relation for Q-manifolds.

Gauge PDEs (gPDE)

Def. Gauge PDE $(E, T[1]X, Q)$ is a Q-bundle over $T[1]X$. In addition: equivalent to nonnegatively graded.

In local coordinates $x^\mu, \theta^\mu, \psi^A$:

$$Q = \theta^\mu \frac{\partial}{\partial x^\mu} + Q^A(\psi, x, \theta) \frac{\partial}{\partial \psi^A}$$

Solutions: $\sigma : T[1]X \to E$ is a solution if

$$d_X \circ \sigma^* = \sigma^* \circ Q \,, \qquad d_X \psi^A(x,\theta) = Q^A(\psi^A(x,\theta), x, \theta)\,.$$

Gauge parameter: $Y = Y^A(\psi, x, \theta) \frac{\partial}{\partial \psi^A}$, $\mathrm{gh}(Y) = -1$.

Infinitesimal gauge transformations:

$$\delta_Y \sigma^* = \sigma^* \circ [Q, Y]$$

In a similar way one defines gauge (for gauge)^N symmetries.

Equivalence of gauge PDEs

Def. A sub-gPDE $(\tilde{E}, \tilde{Q}, T[1]X) \subset (E, Q, T[1]X)$ (i.e. $\tilde{E} \subset E$ is a subbundle, Q restricts to $\tilde{Q}$) is called an equivalent reduction if E is a locally trivial Q-bundle over $\tilde{E}$ (as bundles over $T[1]X$) with a contractible fiber.

In local coordinates: if in adapted coordinates $x^\mu, \theta^\mu, \phi^i, w^a, v^a$ one has $Qw^a = v^a$ and $\tilde{E}$ is singled out by $w^a = 0 = v^a$ then $\tilde{E}$ is an equivalent reduction.

A version of elimination of "generalized auxiliary fields" *Henneaux, 1990; Barnich, M.G. 2004; M.G. Kotov 2019*.

Example: PDE

Let $E_0 \to X$, Γ be a PDE defined in a geometrical way. Extend it to a bundle $E \to T[1]X$ and define Q as a covariant differential:

$$Q = \theta^\mu \left(\frac{\partial}{\partial x^\mu} + \Gamma_\mu^B \frac{\partial}{\partial \psi^B}\right), \qquad (\theta^a \equiv dx^a)$$

$Q^2 = 0$ thanks to the flatness of Γ. We arrive at Q-bundle $(E, T[1]X, Q)$.

Solutions of this gPDE are solutions of the underlying PDE (covariantly constant sections).

There are no gauge transformations encoded in the gPDE becuase $\mathrm{gh}(\psi^A) = 0$

Usual PDEs are gauge PDEs whose grading is horizontal

Example: BV formulation (EOM level)

Let $\mathcal{E} \to X$ be a fiber-bundle underlying BV (fiber coordinates are: fileds , ghosts, antifields, ..).

Take as E jets $J^\infty(\mathcal{E})$ pulled back to $T[1]X$ (horizontal forms on $J^\infty(\mathcal{E})$) and $Q = d_\mathsf{h} + s$ and the total degree ($\mathrm{gh}(\theta^\mu) = 1$). At least locally, the gauge system determined by $(E, T[1]X, Q)$ is equivalent to the one encoded in the BV formulation $(J^\infty(\mathcal{E}), s)$.
Barnich, MG 2010

The notion of gauge PDE includes BV at EOM level as a particular case and hence all reasonable gauge theories. Justifies definition.

Riemannian geometry as a gauge PDE. Off-shell GR

Take $G = S^2(T^*X) \oplus T[1]X$. Take E to be $J^\infty(G)$ pulled back to $T[1]X$. Local trivialization:

$$x^\mu, \theta^\mu, \qquad g_{ab}, g_{ab|c}, \dots, \qquad \xi^a, \xi^a{}_{|c} \cdots$$

In a suitable trivialization:

$$Q = \mathsf{d}_X + \gamma\,, \quad \gamma g_{ab} = \xi^c g_{ab|c} + \xi^c{}_{|a} g_{cb} + \xi^c{}_{|b} g_{ac}\,, \quad \gamma \xi^a = \xi^c \xi^a{}_{|c}\,, \dots$$

E.g. Lagrangians: $H^n(Q, local functions)$, $n = \dim X$.

Locally, $E = (T[1]X, d_X) \times (\mathcal{F}, q)$, i.e. Locally-trivial Q-bundle.

If one takes the subbundle of prolonged Einstein equations:

$$D_{(a)}(R_{ab} - \frac{1}{n} g_{ab} R) = 0$$

one gets gPDE which is not a jet-bundle.

Conformal-like off-shell GR

As an example of equivalent gPDEs, extend G by extra coordinate $\Omega > 0$ of degree 0 and extra coordinate λ of degree 1. Take $J^\infty(G)$ extended to $T[1]X$ and Q:

$$Qg_{bc} = \xi^a D_a g_{bc} + g_{ac} D_b \xi^a + g_{ba} D_c \xi^a + 2\lambda g_{bc}, \quad Q\Omega = \xi^a D_a \Omega + \lambda\Omega \,,$$
$$Q\xi^b = \xi^a D_a \xi^b, \qquad Q\lambda = \xi^a D_a \lambda \,, \qquad [Q, D_a] = 0$$

Equivalent to off-shell GR. Contractible pairs are $D_{(a)}(\Omega - 1)$ and $Q(D_{(a)}(\Omega - 1)) = \Omega D_{(a)}\lambda + \ldots$. Naive equivalent reduction: jets $\rightarrow$ jets

Minimal model for off-shell GR

$\Gamma^a_{(bc|d\dots)}$ form contractible pairs with $\xi^a_{bcd\dots}$ and g_{ab} with symmetric part of ξ^a_b. Resulting minimal model *Stora; Barnich, Brandt, Henneaux; Vasiliev . . .* :

Coordinates: x^μ, θ^μ, $\quad \xi^a, \rho^a{}_b$, $\quad R_{ab}{}^c{}_d, R_{a(b}{}^c{}_{de)}, \dots, R_{a(b}{}^c{}_{de\dots)}, \dots$

$$Qx^\mu = \theta^\mu\,, \quad Q\xi^a = \rho^a{}_c\,\xi^c\,, \qquad q\rho^{ab} = \rho^a{}_c\,\rho^{cb} + \lambda\xi^a\xi^b + \xi^c\xi^d R^{ab}_{cd}\,,$$
$$QR_{ab}{}^c{}_d = \xi^e R_{a(b}{}^c{}_{de)} + \rho_a{}^f R_{fb}{}^c{}_d + \dots\,, \qquad \dots$$

For instance $H^0(Q)$ immediately gives Riemannian invariants. On-shell version: R are totally traceless (only Weyl tensors).

Minimal model for off-shell GR

Sections are parameterized by:

$$\sigma^*(\xi^a) = e^a_\mu(x)\theta^\mu, \quad \sigma^*(\rho^{ab}) = \omega^{ab}_\mu(x)\theta^\mu, \quad \sigma^*(R_{ab}{}^c{}_d) = \mathsf{R}_{ab}{}^c{}_d(x), \ldots$$

Equations of motion $\mathsf{d}_X \circ \sigma^* = \sigma^* \circ Q$:

$$\mathsf{d}_X e^a + \omega^a{}_b e^b = 0\,, \quad \mathsf{d}_X \omega^{ab} + \omega^a{}_c \omega^{cb} = e^c e^d R_{cd}{}^{ab}\,, \quad \ldots$$

Cartan structure equations. Taking a total degree "gh+form degree" is crucial. Frame-like formulations.

On shell version – equivalent form of Einstein equations.

Equivalent reduction to minimal model: jets $\to$ non-jets.

Induced gPDE on submanifolds

Let $B \subset X$ be a submanifold. $T[1]B$ is naturally a submanifold in $T[1]X$. For instance, if x^0, x^i are coordinates on X, x^0, x^i and θ^0, θ^i are adapted coordinates on $T[1]X$, and $B \subset X$ is singled out by $x^0 = 0$, then $T[1]B$ is singled out by

$$x^0 = 0\,, \qquad \theta^0 = 0$$

Note that d_X on $T[1]X$ is tangent to $T[1]B \subset T[1]X$. Denote $i : T[1]B \to T[1]X$.

Let $(E, Q, T[1]X)$ be a gPDE. Then its pullback i^*E (as a bundle) to $T[1]B$ is again a gPDE $(i^*E, Q, T[1]B)$. Note that i^*E is naturally a submanifold in E ($x^0 = 0 = \theta^0$) and Q is tangent to $i^*E \subset E$. The same applies to the case where $B = \partial X$. $(i^*E, Q, T[1]B)$ describes a theory of "boundary values" of the fields described by $(E, Q, T[1]X)$.

Example: 1d AKSZ sigma model

Let (M, q) be a BFV extended phase space of a first order constrained system with constraints $T_\alpha = 0$. The BRST differential:

$$q = c^\alpha \{T_\alpha, \cdot\} + T_\alpha \frac{\partial}{\partial \mathcal{P}_\alpha} + \ldots = \{\Omega_{BFV}, \cdot\}$$

Consider gPDE $E = T[1]\mathbb{R}_{\geqslant 0} \times (M, q)$, $Q = \theta \frac{\partial}{\partial t} + q$.

Fields are: $\lambda^\alpha = \sigma^*(c^\alpha)$, $z^a(t) = \sigma^*(z^a)$. Note that $\sigma^*(\mathcal{P}_\alpha) = 0$ because $\mathrm{gh}(\mathcal{P}_\alpha) = -1$. Equations of motion and gauge transformations

$$\dot{z}^a = \{z^a, T_\alpha\}\,\lambda^\alpha\,, \qquad T_\alpha = 0\,,$$

$$\delta_Y \lambda^\alpha = \dot{\epsilon} + \ldots, \qquad \delta_Y z^a = \{z^a, T_\alpha\}\,\epsilon^\alpha, \qquad Y = \epsilon^\alpha(t) \frac{\partial}{\partial c^\alpha}$$

Boundary data: a gauge PDE in 0-dim. In this case: the BFV phase space of the system.

gPDE of bondary conditions

Let X be a manifold with boundary, $B = \partial X$. gPDE of boundary conditions $(E_B, Q, T[1]B)$ is a sub-gPDE of $(i^*E, Q, T[1]B)$. The extra constraints are boundary conditions for fields, ghosts (gauge-parameters), etc.

Remark. Even if $E_B \subset i^*E$ is a nontrivial subbundle it doesn't not mean that we have nontrivial boundary conditions. For instance, take E_B to be an equivalent reduction of i^*E.

gPDEs with boundaries

Def. gPDE with boundary is $(E, Q, T[1]X, E_B, T[1]B)$, where $E_B \subset i^*E$ is a sub gPDE of i^*E. E_B can be regarded as "gPDE of boundary conditions".

$\sigma : T[1]X \to E$ is a solution if σ is a solution to E and moreover $\sigma|_{T[1]B}$ is a solution to E_B.

A gauge parameter is a vertical vector field Y on E such that $\mathrm{gh}(Y) = -1$ and Y is tangent to E_B.

An infinitesimal gauge transformation of a solution $\sigma : X \to E$ is

$$\delta_Y \sigma^* = \sigma^* \circ [Q, Y]$$

It takes solutions to solutions. In particular, restrictions of Y to E_B define gauge symmetries of $(E_B, Q, T[1]B)$.

Symmetries

Let $(E, Q, T[1]X)$ be a gPDE. Its symmetry is a vertical vector field W satisfying $[Q, W] = 0$. Usually, one also asks gh$(W) = 0$. Symmetries of the form $W = [Q, Y]$ are considered trivial

A symmetry transformation of a section σ:

$\delta_W \sigma^* = \sigma^* \circ W$ in components: $\delta\psi^A(x, \theta) = W^A(\psi(x, \theta), x, \theta)$

For $W = [Q, Y]$ this indeed gives gauge transformations.

At least locally, the above def. is equivalent to the standard def. of symmetries in local BV

In the case of gPDE with boundary one in addition requires W to be tangent to E_B, i.e. to preserve boundary conditions.

Asymptotic symmetries

There are various approaches. The common lore is that asymptotic symmetries are those gauge-like symmetries that are not genuine gauge symmetries because their parameters do not satisfy boundary conditions.

More formally, $W = [Q, Y]$ is an asymptotic symmetry if W is tangent to E_B (preserves boundary conditions). If Y is itself tangent to E_B than W is a trivial asymptotic symmetry.

Note that asymptotic symmetries form a subalgebra of all symmetries of $(E_B, Q, T[1]B)$ and are a property of i^*E and its sub-gPDE $(E_B, Q, T[1]B)$.

Note also that if $E_B = i^*E$ there are no nontrivial asymptotic symmetries, i.e. they indeed arise from nontrivial boundary conditions.

Asymptotically simple GR as gPDE with boundaries

Penrose's approach:
$(\widetilde{X}, \tilde{g})$ is an asymptotically simple spacetime if there exists spacetime (X, g) with boundary $B = \partial X$ and a function $\Omega > 0$ such that:
- $\widetilde{X}$ is the interior of X
- $g = \Omega^2 \tilde{g}$ on $\widetilde{X}$
- $\Omega = 0$ on B and $\Omega > 0$ in the interior
- completeness condition

If in addition $\tilde{g}$ is Einstein near B then $(\widetilde{X}, \tilde{g})$ is said asymptotically flat if $\Lambda = 0$ (- AdS if $\Lambda < 0$ or - dS if $\Lambda > 0$).

It is clear that $i^* g$ is defined up to a conformal factor. If $i^* g$ is nondegenerate B gets a conformal structure.

Asymptotically simple GR leads to a manifold with boundary. From the field-theory perspective (X, g) is a particular solution of a gauge theory on a manifold with boundary.

Conformal-like on-shell GR on X

Fields g_{ab}, Ω. On the boundary

$$\Omega = 0, \quad Q\Omega = 0, \qquad D_a\Omega \neq 0$$

Field-theoretical realization of the asymptotically flat spacetime. In particular, the physical metric (which diverges on the boundary) is $\tilde{g}_{ab} = \Omega^{-2} g_{ab}$.

The respective gPDE formulation:

$$Qg_{bc} = \xi^a D_a g_{bc} + g_{ac} D_b \xi^a + g_{ba} D_c \xi^a + 2\lambda g_{bc}, \quad Q\Omega = \xi^a D_a \Omega + \lambda\Omega\,.$$
$$Q\xi^b = \xi^a D_a \xi^b, \qquad Q\lambda = \xi^a D_a \lambda$$

along with $[D_a, Q] = 0$

Einstein equations on $\tilde{g}$ can be rewritten in terms of g, Ω as:

$$D_{(a)}F_{bc} = 0, \qquad \Omega\rho + \frac{g^{ab}}{2}D_a\Omega D_b\Omega = -\frac{\Lambda}{(n-1)(n-2)}\,,$$

where

$$F_{bc} \equiv D_b D_c\Omega - \Gamma^d_{bc}D_d\Omega + \Omega P_{bc} + \rho g_{bc},$$
$$\rho \equiv -\frac{1}{n}g^{bc}(D_b D_c\Omega - \Gamma^d_{bc}D_d\Omega + P_{bc}\Omega)$$

$F_{bc} = 0$ is the celebrated almost Einstein equations *Eastwood, Gover, . . .* . For a fixed g_{ab} it admits nonvanishing solution only if g_{ab} is conformally Einstein. Emergence of tractor bundles.

Minimal model for the induced boundary gPDE

Recall that asymptotic symmetries are determined by boundary gPDE so let us concentrate on it. The strategy is then:

1. obtain a minimal model (maximal equivalent reduction) of the boundary gPDE
2. In addition, impose a suitable version of the BMS boundary conditions
3. Find asymptotic symmetries

The equivalent reduction in the sector of g_{ab}, ξ^a, λ is similar to the one for conformal geometry and is known in the literature *Boulanger*. Explains the emergence of Cartan formulation of conformal geometry and tractor calculus in the approach developed by *Herfray*.

Minimal model for boundary data (the case of $\Lambda = 0$)

$$Q\xi^A = \xi^B \rho_B{}^A - \xi^A \lambda, \qquad Q\lambda = \xi^A \lambda_A,$$

$$Q\rho_A{}^B = \rho_A{}^C \rho_C{}^B + \lambda_A \xi^B - \lambda^B \xi_A + \frac{1}{2}\xi^C \xi^D \mathsf{W}^B{}_{ACD}$$

$$Q\lambda^A = \rho^A{}_B \lambda^B - \lambda\lambda^A + \frac{1}{2}\xi^C \xi^D \mathsf{C}^A{}_{CD} + \xi^u \xi^D \mathsf{C}^A{}_{uD},$$

$$QC^A = C^B \rho_B{}^A + \lambda^u \xi^A - \lambda^A \xi^u + \frac{1}{2}\xi^C \xi^D \mathsf{W}^A{}_{\Omega CD},$$

$$Q\lambda^u = C^A \lambda_A - \lambda\lambda^u + \frac{1}{2}\xi^C \xi^D \mathsf{C}_{\Omega CD} + \xi^u \xi^D \mathsf{C}_{\Omega uD},$$

$$Q\xi^u = -\xi^u \lambda - \xi^A C_A, \qquad Q(\text{curvatures}) = \ldots .$$

The index split is $a = (u, \Omega, A)$, $A = 1, \ldots, d-2$. With curvatures set to zero this defines CE for $iso(n-1,1)$. First relations define $so(n-1,1) \subset iso(n-1,1)$.

Somewhat similar systems: *Nguyen, Salzer; Herfray*

BMS symmetries

Simplest vacuum solution: all vanish except for $\sigma^*(\xi^A) = e^A_\alpha dy^\alpha$ (define a degenerate metric) and $\sigma^*(\xi^u) = e^u_\alpha dy^\alpha$ (define a vector field in the kernel of the metric). Conformal Carrollian geometry.

BMS boundary conditions (define sub-gPDE of the minimal model):

$$\xi^A - e^A = 0, \qquad \xi^u - \theta^u = 0\,, \qquad \lambda = 0$$

along with Q(above conditions) $= 0$. The solution for the non-trivial "gauge parameter" Y reads:

$$Y = \epsilon^u \frac{\partial}{\partial \xi^u} + \epsilon^A \frac{\partial}{\partial \xi^A} + \dots$$

where $\epsilon^u = (u\bar{\lambda} + T(y))$ and $\epsilon^A = \epsilon^A(y)$ are components of BMS vector field $\epsilon^{BMS} = (u\bar{\lambda} + T(y))\frac{\partial}{\partial u} + \epsilon^A(y)\frac{\partial}{\partial y^A}$ in the adapted coordinate system y^A, u on B.

Boundary data for matter fields

PDE for matter fields coupled to gravity:

$$\widetilde{P}(\widetilde{\phi}_{(a)}, \widetilde{g}_{ab(c)}) = 0\,, \quad Q_{GR} \text{ is tangent to } D_{(a)}\widetilde{P} = 0\,,$$

$$Q_{GR}\widetilde{\phi} = \xi^a D_a \widetilde{\phi} + \ldots$$

Uplifting to conformal-like GR. In terms of $\phi = \Omega^{-w}\widetilde{\phi}$:

$$Q\phi = -w\lambda\phi + \xi^a D_a \phi + \ldots$$

Now $D_{(a)}\widetilde{P} = 0$ is Q-invariant. Taking minimal k such that $P = \Omega^k \widetilde{P}$ does not contain negative powers of Ω, gives a boundary data for $\widetilde{P}$.

Example:

$$\Lambda < 0 \qquad w = \frac{\dim B}{2} - 1$$

and scalar ϕ. Result in conformally-coupled scalar on the boundary$+$ subleading.

More generally: gauge theoretical realisation of *Fefferman-Graham* construction. GJMS conformal-invariant operators. For more general linear ϕ and flat backgrond the gPDE approach is known *Bekaert, MG, 2012*. Relation to Fefferman-Graham construction *Bekaert, MG, Skvortsov 2017*.

For $\Lambda = 0$, ϕ scalar, and flat background reproduces *Nguyen; Bekaert Oblak; Herfray; Donnay et all, . . .* .

More generally: any matter (possibly gauge) coupled to gravity gives a conformal (conformal Carrollian) boundary gPDE. It is usually interesting only if w is special (critical).

Lagrangian theories

If $(E, Q, T[1]X)$ is a gPDE, a possible Lagrangians is encoded in a compatible presymplectic structures:

$$d\omega = 0\,, \quad L_Q\omega \sim 0\,, \quad \mathrm{gh}(\omega) = n-1$$

It follows there exists $\mathcal{L}$, $\mathrm{gh}(\mathcal{L}) = n$:

$$i_Q\omega + d\mathcal{L} \sim 0$$

This defines an AKSZ-like action *Alkalaev MG 2013, MG 2016, 2022,...*

$$S[\sigma] = \int_{T[1]X} (\sigma^*(\chi)(\mathrm{d}_X) + \sigma^*(\mathcal{L}))\,, \qquad \omega = d\chi$$

Local BV formalism beyond jet bundles

Variational symmetries

A symmetry W, $[Q,W]=0$ is called variational if $L_W\omega + dL_Q\alpha \sim 0$ for some α. The associated Hamiltonian is defined as

$$i_W\omega - L_Q\alpha - dH_W \in \mathcal{I} \sim 0\,, \qquad QH_W = 0\,,$$

This gives a (generalised) Noether theorem. If $\mathrm{gh}(W) = -k$ then $\mathrm{gh}(H_W) = n-1-k$ and the conserved charge can be defined as

$$\mathbf{H_W}[\sigma] = \int_{T[1]C} \sigma^*(H_W)$$

where C is a submanifold of codimension $k+1$. $\mathbf{H_W}$ is invariant under deformations of C provided ∂C is intact and σ is a solution. $k=0$ conserved charges, $k>0$ lower degree conservation laws (e.g. surface charges). *Abbot, Deser,..* BRST cohomology approach *Barnich, Brandt*.

Conservation laws associated to asymptotic symmetries

Boundary terms in the action are important! Presymplectic potential χ, $\omega = d\chi$ is compatible with the boundary conditions if $\chi|_{E_B} = 0$. In this case the action

$$S[\sigma] = \int_{T[1]X} (\sigma^*(\chi)(\mathrm{d}_X) + \sigma^*(H))$$

is differentiable.

The conserved current associated to $W = [Q, Y]$ can be taken to be $H_W = i_Y\chi$ and hence vanishes for genuine gauge symmetries (for which Y is tangent to E_B).

Compatible with the alternative definition of asympototic symmetries as those gauge-like symmetries whose associated charges vanish *Brown, Henneaux; Banados; . . .*

Conclusions

- Geometrical (in the sense of geometry of PDEs) and BV-BRST extended version of the covariant phase space formalism

- Shown to be instrumental in the study of asymptotic symmetries of gravity

- Known to unify Lagrangian BV and Hamiltonian BFV into a unified AKSZ-like formalism (presymplectic gauge PDEs). Reduces to *De Donder-Weyl* covariant Hamiltonian formalism in the simplest cases.

- Gives a first principle derivation of the conformal geometry/tractor calculus approach to BMS by *Herfray*. Earlier: conformal boundary calculus *Gover, Waldron*

- Gives a field-theoretical realization of the *Fefferman-Graham* construction ($\Lambda < 0$) as well as its conformal Carollian analog ($\Lambda = 0$).

- Earlier simplified version of the approach *Bekaert, MG 2012, 2013* is very instrumental in studying holography for higher-spin fields. Mixed-symmetry generalization *Chekmenev, MG*

- Possible applications for flat-space HS holography. One of the main motivation.

Open:

constructing full-scale graded-geometrical theory of (generalized) symmetries and conservation laws in gauge theories with lower-dimensional strata.

Possible applications to flat space HS holography.