Extrinsic geometry and linear differential equations of $\mathfrak{sl}_3$-type

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Motivation

- Study projective embeddings of filtered manifolds that can be "approximated" by rational homogeneous varieties $G/P \to P^n$.
- Examples of such rational homogeneous varieties are rational normal curves $P^1 \to P^n$, conics $P^1 \times P^1 \to P^3$, classical Veronese, Segre, Plücker embeddings and many others.
- What if we consider embeddings of a smallest parabolic homogeneous space of depth ≥ 2, namely $\mathrm{Flag}_{1,2}(\mathbb{R}^3)$?
- The canonical moving frame for embeddings of such type was constructed in our earlier work (D.-Machida-Morimoto, 2021).
- A bit unexpectedly, we encounter many similarities with the projective geometry of surfaces.

Analogy with projective geometry of surfaces

- Consider a system of 2nd order linear PDEs:

$$\begin{aligned} u_{xx} &= A_1 u_x + B_1 u_y + C_1 u \\ u_{yy} &= A_2 u_x + B_2 u_y + C_2 u, \end{aligned}$$

  where A_i, B_i, C_i are functions of x and y.
- Assume that the compatibility conditions are satisfied. Then this system has 4-dimensional solution space. Each solution is uniquely determined by u, u_x, u_y, u_{xy} at a point.
- If $\{u_0, u_1, u_2, u_3\}$ is a basis in the solution space, then the surface $[u_0 : u_1 : u_2 : u_3]$ is a hyperbolic surface in P^3, whose asymptotic curves are given by $x = \text{const}$ and $y = \text{const}$.
- Conversely, any hyperbolic surface in P^3 can be represented this way. In particular, the trivial system $u_{xx} = u_{yy} = 0$ corresponds to the Segre embedding

$$P^1 \times P^1 \to P^3, \quad ([1 : x], [1 : y]) \mapsto [1 : x : y : xy].$$

Non-holonomic version of above PDEs

1. Replace ∂_x and ∂_y in the above equations by Lie derivatives along vector fields X and Y on a 3-dim manifold M that span a contact distribution. Let $Z = [X, Y]$.
2. Consider now linear systems of PDEs of the form:

$$X^2 u = A_1 Xu + B_1 Yu + C_1 u$$
$$Y^2 u = A_2 Xu + B_2 Yu + C_2 u,$$

   where u is an unknown function on M and A_i, B_i, C_i are arbitrary functional coefficients.
3. Assume that the compatibility conditions are satisfied. Then this system has **8-dimensional** solution space. Each solution is uniquely determined by u, Xu, Yu, XYu, Zu, XZu, YZu, Z^2u at a point.
4. If $\{u_0, u_1, \dots, u_7\}$ is a basis in the solution space, then we get an embedding of M to P^7 given by $[u_0 : u_1 : \dots : u_7]$.
5. What kind of embeddings to we get that way? What embedding corresponds to the "trivial" case, when $X = \partial_x + y\partial_z$, $Y = \partial_y$, $X^2u = Y^2u = 0$?

Submanifolds in parabolic homogeneous spaces

- Let G/P be an arbitrary parabolic homogeneous space: $\mathfrak{g} = \sum_{i\in\mathbb{Z}} \mathfrak{g}_i$ is a graded semisimple Lie algebra of the Lie group G and $\mathfrak{p} = \sum_{i\geq 0} \mathfrak{g}$ is a parabolic subalgebra of $\mathfrak{g}$.
- G/P is naturally equipped with a structure of a filtered manifold

$$0 \subset T^{-1} \subset \cdots \subset T^{-\nu} = T(G/P)$$

  defined as a flag of G-invariant vector distributions equal to $\oplus_{i\leq k}\mathfrak{g}_{-i}$ mod $\mathfrak{p}$ at $o = eP$.
- Given a submanifold $M \subset G/P$ we define its symbol at $x \in M$ as $\operatorname{gr} T_x M$ viewed as a graded subspace in $\mathfrak{g}_-$.
- The symbol is a graded subalgebra in $\mathfrak{g}_-$, viewed up to the action of G_0. In general, it depends on a point $x \in M$.

Embeddings of $\mathfrak{sl}_3$ type

- Consider $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{R})$ with the full grading $\mathfrak{g} = \sum_{i=-2}^{2} \mathfrak{g}_i$. Then $\mathfrak{g}_-$ is just a 3-dim Heisenberg Lie algebra, and the dimensions of $\mathfrak{g}_i$ are

$$(1, 2, 2, 2, 1).$$

- Take V be the adjoint representation of $\mathfrak{g}$. So, $\dim V = 8$ and $\mathfrak{sl}(V)$ is naturally equipped with the grading with degrees from -4 to 4.
- Consider the parabolic homogeneous space $\mathrm{Flag}_{1,3,5,7}(V) = PSL(V)/P$, where P the stabilizer of a fixed flag:

$$0 \subset \mathfrak{g}_2 \subset \mathfrak{g}_1 \oplus \mathfrak{g}_2 \subset \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \subset \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \subset \mathfrak{g} = V.$$

- Note that $\mathfrak{g}$ is naturally embedded into $\mathfrak{sl}(V)$ as a graded subalgebra. In particular, $\mathfrak{g}_-$ is a graded subalgebra in $\mathfrak{sl}(V)_-$.
- We consider 3-dim submanifolds in $\mathrm{Flag}_{1,3,5,7}(V)$ with symbol $\mathfrak{g}_-$. We call them embeddings of $\mathfrak{sl}_3$ type.

Projective embeddings of filtered manifolds

- Any submanifold M of type $\mathfrak{sl}_3$ is a 3-dimensional contact manifold. Denote by $T^{-1}M$ the contact distribution on M.
- There is a natural projection:

$$M \hookrightarrow \mathrm{Flag}_{1,3,5,7}(V) \rightarrowtail P(V) = P^7.$$

- Due to the relation

$$\mathfrak{sl}(V)_{k-1} = [\mathfrak{g}_{-1}, \mathfrak{sl}(V)_k]$$

the embedding of M into $\mathrm{Flag}_{1,3,5,7}(V)$ can be restored from the projective embedding $M \to P(V)$ via the (weak) osculating flag:

$$\mathcal{O}_x^{-1} = \hat{x}, \quad x \in M \subset P^7,$$
$$\mathcal{O}_x^{k-1} = \underline{T_x^{-1}M}(\mathcal{O}^k) + \mathcal{O}_x^k, \quad k \leq -2.$$

- We say that an embedding of a 3-dim contact manifold $M \to P^7$ is of type $\mathfrak{sl}_3$, if it lifts to a submanifold $M \subset \mathrm{Flag}_{1,3,5,7}$ with the prescribed symbol $\mathfrak{g}_- \subset \mathfrak{sl}(8,\mathbb{R})_-$.

Flat model: adjoint variety of $SL(3)$

- The flat (or most symmetric) example of an embedding of $\mathfrak{sl}_3$ type is the highest root orbit in $P(V) = P(\mathfrak{sl}_3)$, also called *the adjoint variety*. It consists of all 3×3 matrices conjugate to the highest root space:
$$\begin{pmatrix} 0 & 0 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$
- Projectivized set of all trace-free rank one 3×3 matrices. It is a 3-dim manifold that can be identified with $\mathrm{Flag}_{1,2}(\mathbb{R}^3)$.
- Each such matrix is of the form $\alpha \otimes v$, $\alpha \in \mathbb{R}^{3,*}$, $v \in \mathbb{R}^3$ and $\langle \alpha, v \rangle = 0$. Up to a constant such matrices are in 1-1 correspondence with flags
$$0 \subset \langle v \rangle \subset \alpha^{\perp} \subset \mathbb{R}^3.$$
- Note that $\mathrm{Flag}_{1,2}(\mathbb{R}^3) = PSL(3)/B$ carries a natural contact structure and its embedding into P^7 naturally possesses $PSL(3)$ as a symmetry group.

Theorem

To each embedding of $\mathfrak{sl}_3$ type $M \to P^7$ there canonically corresponds the pair (P, ω), where

1. *P is a principal frame bundle over M with the structure group $G^0 = B = ST(3, R)$;*
2. *ω is an $\mathfrak{sl}(V)$-valued 1-form satisfying*
   - **(i)** $\langle \tilde{A}, \omega \rangle = A$, $A \in \mathfrak{g}^0$;
   - **(ii)** $R_a^* \omega = \mathrm{Ad}(a^{-1})\omega$, $a \in G^0$;
   - **(iii)** $L_{\tilde{A}} \omega = -\mathrm{ad}(A)\omega$, $A \in \mathfrak{g}^0$;
   - **(iv)** $d\omega + \frac{1}{2}[\omega, \omega] = 0$;
3. *if we decompose ω as $\omega = \omega_I + \omega_{II}$ according to the direct sum decomposition $\mathfrak{sl}(V) = \mathfrak{g} \oplus \mathfrak{g}^\perp$, then $\omega_I \colon T_z P \to \mathfrak{g}$ is a linear isomorphism for any $z \in P$;*
4. *if we write $\omega_{II} = \chi \omega_I$, then χ is a $\mathrm{Hom}(\mathfrak{g}_-, \mathfrak{g}^\perp)$-valued function on P and $\partial^* \chi = 0$.*

Fundamental invariants and cohomology

Theorem

The set of fundamental invariants of embeddings $M \to P^7$ of $\mathfrak{sl}_3$ type is described by the Lie algebra cohomology $H^1_+(\mathfrak{g}_-, \mathfrak{sl}(V)/\mathfrak{g})$.

- Algebraically, we have:

$$\mathfrak{sl}(V) = 2\Gamma_{1,1} + \Gamma_{3,0} + \Gamma_{0,3} + \Gamma_{2,2},$$

where $\Gamma_{1,1} = \mathfrak{sl}(3)$.

- Using Kostant theorem we get:

$$\begin{aligned} H^1_+(\mathfrak{g}_-, \Gamma_{1,1}) &= H^1_+(\mathfrak{g}_-, \Gamma_{2,2}) = 0, \\ H^1_+(\mathfrak{g}_-, \Gamma_{3,0}) &= H^1_1(\mathfrak{g}_-, \Gamma_{3,0}) = \langle \xi_1^R \rangle, \\ H^1_+(\mathfrak{g}_-, \Gamma_{0,3}) &= H^1_1(\mathfrak{g}_-, \Gamma_{0,3}) = \langle \xi_1^S \rangle. \end{aligned}$$

- ξ_1^R, ξ_1^S define *two fundamental invariants* of embeddings of $\mathfrak{sl}_3$ type. The embedding is locally flat if and only if they both vanish.

Intermediate parabolic space

- The adjoint action of $SL(3)$ preserves the Killing form K on $\mathfrak{sl}(3,\mathbb{R})$ of signature (5,3). The adjoint variety, lifted to $\mathrm{Flag}_{1,3,5,7}(V)$, $V = \mathfrak{sl}(3)$, consists of isotropic and co-isotropic flags in V:

$$W_1 \subset W_3 \subset W_3^{\perp} \subset W_1^{\perp}.$$

The set of all such flags forms parabolic homogeneous space $\mathrm{IFlag}_{1,3}(V,K)$ of the group $L = SO(5,3)$.

Theorem

For any embeddings $M \to \mathrm{Flag}_{1,3,5,7}(V)$ of $\mathfrak{sl}_3$ type there exists a symmetric form K on V such that $M \subset \mathrm{IFlag}_{1,3}(V,K)$.

- This is the direct consequence of $H^1_+(\mathfrak{g}_-,\Gamma) = 0$ for $\Gamma = \Gamma_{1,1}$, $\Gamma_{2,2}$.
- And the following decompositions of $\mathfrak{sl}(3)$ modules:

$$\mathfrak{sl}(3) \subset \mathfrak{so}(5,3) \subset \mathfrak{sl}(V) = \mathfrak{sl}(8)$$

$$\mathfrak{so}(5,3) = \mathfrak{sl}(3) + \Gamma_{3,0} + \Gamma_{0,3},$$

$$\mathfrak{sl}(8) = \mathfrak{so}(5,3) + \Gamma_{2,2} + \Gamma_{1,1}.$$

What we classify

1. Osculating embeddings $\varphi\colon M \hookrightarrow \mathrm{Flag}_{1,3,5,7}(\mathbb{R}^8)$ or the corresponding embedding $M \hookrightarrow P^7$.
2. Symmetry algebra $\mathrm{sym}(M)$ is defined as a set of all vector fields from $\mathfrak{sl}(8,\mathbb{R})$ tangent to M. We say that M has a locally transitive symmetry algebra if $\mathrm{sym}(M)$ is transitive on M, i.e. spans TM at all points.
3. We would like to describe (up to the action of $PSL(8,\mathbb{R})$) all embeddings $M \to P^7$ of $\mathfrak{sl}_3$ type with transitive symmetry algebra.
4. The corresponding systems of PDEs can be written as:
$$Z_1^2 u = a_1 Z_1 u + b_1 Z_2 u + c_1 u,$$
$$Z_2^2 u = a_2 Z_1 u + b_2 Z_2 u + c_2 u,$$
in terms of left-invariant vector fields Z_1, Z_2 on a 3-dimensional Lie group H, such that $\langle Z_1, Z_2\rangle$ is a (left-invariant) contact distribution on H. Here a_i, b_i and c_i are constants.
5. The main question is when such systems are compatible, that is have exactly 8-dim solution space.

Main classification

Theorem

Let $M^3 \hookrightarrow P^7$ be an osculating embedding of $\mathfrak{sl}_3$ type with a locally transitive symmetry algebra. Then, up to equivalence, it corresponds to one of the following systems of PDEs:

	Equation	Symmetry algebra
(O)	$Z_1^2 u = Z_2^2 u = 0$	$\mathfrak{sl}(3, \mathbb{R})$
(I_0)	$Z_1^2 u = 0, Z_2^2 u = 6Z_1 u$	4-dim solvable
(I_1)	$Z_1^2 u = 0,$ $Z_2^2 u = 6Z_1 u + 2P_2 Z_2 u - \left(\frac{24P_2^2}{25} \pm 1\right) u$	3-dim solvable
(I_2)	$Z_1(Z_1 \pm 2)u = 0, Z_2^2 u = 6Z_1 u \pm 9u$	3-dim solvable
(II_0)	$Z_1^2 u = -6Z_2 u, Z_2^2 u = 6Z_1 u$	$\mathfrak{sl}(2, \mathbb{R})$
(II_1)	$(Z_1 - P_1)^2 u = -6(Z_2 - P_2)u + (P_1^2 + 3P_2)u,$ $(Z_2 - P_2)^2 u = 6(Z_1 - P_1)u + (P_2^2 - 3P_1)u,$ $P_1 P_2 = -9$	3-dim solvable
(II_2)	$(Z_1 - P_1)^2 u = -6(Z_2 - P_2)u + (\frac{1}{4}P_1^2 + 3P_2)u,$ $(Z_2 - P_2)^2 u = 6(Z_1 - P_1)u + (\frac{1}{4}P_2^2 - 3P_1)u,$ $P_1 P_2 = -144$	3-dim solvable

Contact Cayley surface

- **Cayley's ruled cubic** is a surface in P^3 given by the following equation (in affine coordinates):

$$z = xy - x^3/3.$$

- It corresponds to the following system of 2nd order PDEs:

$$u_{xx} = u_y, \quad u_{yy} = 0.$$

- Take $X = \partial_x$ and $Y = \partial_y + x\partial_z$ and consider the system

$$X^2 u = Yu, \quad Y^2 u = 0.$$

- It has 8-dim solution space, which defines an embedding $M \to P^7$ of $\mathfrak{sl}_3$ type. It has a transitive 4-dim symmetry. We call it *contact Cayley surface*.
- Classical Cayley surface is a unique submaximal model for non-degenerate surfaces in P^3. Similarly, contact Cayley surface is a unique submaximal model for embeddings of $\mathfrak{sl}_3$ type. Both have transitive symmetry algebras with 1-dim stabilizer.

- Another remarkable example of transitive $\mathfrak{sl}_3$ type embeddings is a unique non-flat case with simple symmetry algebra $SL(2)$.
- Take M to be the Lie group $SL(2)$ and define X and Y to be left-invariant vector fields corresponding to $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. It is clear that $\langle X, Y \rangle$ is a contact structure on $SL(2)$.
- Consider the following system of PDEs:

$$X^2 u = -\alpha Y u, \quad Y^2 u = \beta X u, \quad \alpha\beta \neq 0.$$

  It turns out that this system is compatible if and only if $\alpha\beta = 10$. Up to the equivalence $(\alpha, \beta) \mapsto (t\alpha, t^{-1}\beta)$ one can assume that $\alpha = \beta = \sqrt{10}$.
- The 8-dim solution space defines an embedding $SL(2) \to P^7$ of $\mathfrak{sl}_3$ type. The symmetry algebra is $\mathfrak{sl}(2, \mathbb{R})$ (simply transitive). It acts on $\mathbb{R}^8$ as a sum of two irreducible representations of dimensions 7 and 1.

Embedding with $SL(2)$ symmetry in coordinates

- Choose a local coordinate system (x, y, z) on $SL(2)$ such that the left-invariant vector fields are:

$$Z_1 = \partial_x + y^2\partial_y + y\partial_z,$$
$$Z_2 = x^2\partial_x + \partial_y - x\partial_z,$$
$$Z_0 = -[Z_1, Z_2] = -2(x\partial_x - y\partial_y + \partial_z).$$

- The corresponding right-invariant vector fields have the form:

$$Z_1' = e^z\Big((xy+1)\partial_y + x\partial_z\Big),$$
$$Z_2' = e^{-z}\Big(-(xy+1)\partial_x + y\partial_z\Big),$$
$$Z_0' = \partial_z.$$

- The solution space is spanned by the constants and

$$(Z_2')^k\left[\frac{x^6+\sqrt{10}x^3+1}{(xy+1)^3}e^{3z}\right], \quad k = 0, \dots, 6.$$

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