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Inverse Variational Problems: Examples

Plan
- Introduction
- Examples of 2nd order ODE (systems)
- Conformal geodesics equ
- Discussion

$$\text{Lagrangian} \xrightarrow{\text{Euler}} \text{PDE}$$

Examples — ODEs

Helmholtz conditions $\ell_E^* = \ell_E$ — Differential Equation $F=0$

Issues:
- Multipliers
- Parametrized vs unparametrized sols (curves)
- Not one eqn but a family depending a functional parameters
- Symmetries (invariant variational bicomplex)

$\tilde{F}_j = \sum \Pi^i F_i$

C. Spencer sequence — A.M. Vinogradov

I. Anderson; T. Kogan; P. Olver

1 Second order ODEs

$$x^i_{tt} + \ell^i(x, x_t) = \nu x^i_t \qquad 0 \le i \le n$$

ν - parameter to be eliminated

$(x^0, x^1, \dots, x^n) = (x, y^1, \dots, y^n)$ $\quad y^i = \frac{dy^i}{dx}$

Eq: $\ell^i(x, x_t) = \sum_{0}^{n} \Gamma^i_{jk}(x)\, x^j_t x^k_t$

Un-parametrized sols $\equiv$ pathes: Path geom.

Que: when is path geom variational?

For $n+1=2$: always variational 1885 N. Sonin

For $n+1=3$: Davis, 1929; Douglas, 1941

$\ddot{y}^i = f^i(x, y^j, \dot{y}^j_k)$

$1 \le i, j \le n$

Idea of sol.

- Vainberg, Tonti trick: ODE has ord ≥ 2 $\Rightarrow$ lagrangian can be chosen of ord < 2: $\int$ by parts $\equiv$ total diverg. $\frac{d}{dx}\psi$

$\int L(x, y, \dot{y})\,dx \longrightarrow$ extremum

EL equs: $\frac{\partial L}{\partial y^j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}^j} = 0 \qquad 1 \le j \le n$

Non-deg cond: $\det\Big[\underbrace{\frac{\partial^2 L}{\partial \dot{y}^a \partial \dot{y}^b}}_{\Phi_{ab}}\Big]_{n\times n} \neq 0$

EL $\Big|_{\text{solved on ODE}} \equiv 0$

Douglas cond:

$\Leftrightarrow \frac{\partial \Phi_{ik}}{\partial \dot{y}^j} = \frac{\partial \Phi_{jk}}{\partial \dot{y}^i}$

$\Leftrightarrow \frac{d}{dx}\Phi_{ij} + \frac{1}{2}\frac{\partial \Phi^k}{\partial \dot{y}^i}\Phi_{kj} + \frac{1}{2}\frac{\partial \Phi^k}{\partial \dot{y}^j}\Phi_{ki} = 0$

$\Leftrightarrow A^k_i \Phi_{kj} = A^k_j \Phi_{ki}$

where $A^k_i = \frac{d}{dx}\frac{\partial \Phi^k}{\partial \dot{y}^i} - 2\frac{\partial \Phi^k}{\partial y^i} + \frac{1}{2}\frac{\partial \Phi^k}{\partial \dot{y}^l}\frac{\partial \Phi^l}{\partial \dot{y}^i}$

$\Leftrightarrow \Phi(A\xi, \eta) = \Phi(\xi, A\eta)$

$\Leftrightarrow \Phi(A'\xi, \eta) = \Phi(\xi, A'\eta)$

where $A' = \frac{d}{dx}A - \frac{1}{2}AJ - \frac{1}{2}J^*A$, $\quad J = \frac{\partial \Phi^k}{\partial x^k}$

$\Rightarrow \Phi(A''\xi, \eta) = \Phi(\xi, A''\eta)$

for Matrix $[A \mid A' \mid \dots]$ linear system Φ generic data rank $= n\lambda \Rightarrow \Phi = 0$

Hence a generic path structure in C^1 topology for $n \ge 3$ and C^3 topology for $n=2$ is not variational $\Rightarrow$ not Finsler metrizable $\Rightarrow$ not metrizable

Example of variational path geom

projective connection

christoffel $\Gamma^k_{ij} \rightsquigarrow$ Thomas $\Pi^k_{ij} = \Gamma^k_{ij} - \frac{1}{n+1}\left(\Gamma^p_{ip}\delta^k_j + \Gamma^p_{jp}\delta^k_i\right)$

$\Gamma^0_{21} = x^1$ all others Γ's vanish

ODE: $\ddot{y}^i = 2\dot{y}^1\dot{y}^i y^2 \dot{y}^0$, $1 \le j \le n$

linearizable

$\ddot{y}^1 = \dot{y}^2, \quad \dddot{y}^2 = 0, \dots, \ddot{y}^n = 0$

$L = (\dot{y}^1 - xy^2)\dot{y}^2 + \sum_3^n (\dot{y}^i)^2$

$\hat{L} = \left(\frac{x_1^1}{x_1^0} - x^0 x^2\right)x_1^2 + \frac{(x_1^3)^2 + \dots + (x_1^n)^2}{x_1^0}$

1-homog in velocities $\quad \int \hat{L}\,dt$ reparametrization invariant

Sophus proj structure is variational but is not Finsler metrizable

Lagrangian is of Kropina type

2 Conformal geodesic equ $(M^n, [g])$

$\ddot{b} = \frac{3g(u,a)}{|u|^2}a - \frac{3|a|^2}{2|u|^2}u + |u|^2 P^\sharp u - 2P(u,u)u$ $\quad$ $x = x(t)$ curve

Hitchin, Ivanova

$P = \frac{1}{n-2}\left(Ric_g - \frac{1}{2(n-1)}R_g g\right)$ H. Friedrich (elliptic) G. Schmidt $\quad$ $u = \dot{x}$ velocity

Schouten 1923, 2005

We assume $|u|^2 \ge 0$ $\quad$ $a: \nabla_u u$ acceleration

T. Bailey, M. Eastwood (1990) $\quad$ $b = \nabla_u a$ jerk

(Tam) $u \cdot b = 3\frac{(u \cdot a)^2}{|u|^2} - \frac{3}{2}|a|^2 - |u|^2 P(u,u)$

(non-zero) $u \cdot v = 3\frac{u \cdot a}{|u|^2}\, u \cdot a - |u|^2 P^\sharp u \cdot u$

Rks (1) A curve is a conf geodesic iff $\exists\, g \in [g]$ s.t. the curve is an affinely parametrized geodesic of g and $P^\sharp u = 0$.

$g \to \tilde{g} = e^{2\varphi} g \qquad P \to \tilde{P} = P - Hess(\varphi) + d\varphi \otimes d\varphi - \frac{1}{2}|d\varphi|^2 g$

$P^\sharp u = 0 \Rightarrow \det(P^\sharp) = 0$ Monge-Amp equ

(2) Eqn $\ddot{b}$ is covariant and preserved under conformal rescalings $g \to \tilde{g} = e^{2\varphi} g$

(3) Each b enters with projective parameter

Theorem (1) The eqn for unparametrized conf geodesics is variational

$t \to \tau(t)$ Schwartzian

(joint w. Vladimir Matveev, Wilhelm Stannat)

(2) The eqn for parametrized conf geod. is not variational

Rks 1: Thm (1) was posted to arXiv one week prior to DVCS by T. Marugame (derive flow claims for CR structure)

2: Phenomenon unpara/para: var / non-variat.

has to be compared w/ Riemannian situation. There geodesics are variational both as parametrized $L = g(u,u)$ action $= |u|^2$

and un-parametrized $L = \sqrt{g(u,u)}$ length

1-homogen in velocities

$\int L\,dt$ reparametrizable

Lagrangian for parametrized case

Vainberg-Tonti: L of ord < 3

(flat ex) $L = L(x, \dot{x}, \ddot{x})$

if $\frac{\partial^2 L}{\partial \ddot{x}^a \partial \ddot{x}^b} \neq 0$ for some a, b

then the corresponding Euler-Lagrange equ is of ord $= 4$

$\Rightarrow L$ has to be affine in $\ddot{x}$

$L = L_0(x, \dot{x}) + \sum L_i(x, \dot{x})\ddot{x}^i$

$\left[\frac{\partial EL(L)}{\partial \ddot{x}^a \partial \ddot{x}^b}\right]_{n\times n}$ $\leftarrow$ skew-symm. $\quad \det(\cdot) = (Pf(\cdot))^2$

non-deg $Pf \neq 0$

uparametrized 3D story: in flat case this was known J. Math Biology 2014

3rd ord 1st order

$L = \frac{\ell}{\sqrt{A^2}}$

$\ell = |u| = \sqrt{g(u,u)}$

$A = |u \times a| = \begin{vmatrix} g(u,u) & g(u,a) \\ g(u,a) & g(a,a) \end{vmatrix}^{1/2}$

$V = (u \times a) \cdot b = \begin{vmatrix} u^1 & u^2 & u^3 \\ a^1 & a^2 & a^3 \\ b^1 & b^2 & b^3 \end{vmatrix}$

extremals = circles in $\mathbb{R}^3$

$$\int L\,dt = \int \tau\, ds$$

torsion of curve in $\mathbb{R}^3$ $\quad$ invariant differential form

$\frac{V}{A^2}$

Rk: Action is conformally invariant

Conf. change $\tau \to \bar{\tau} = \tau + \partial\psi$