Singularity Categories of Differential Equations

Jacob Kryczka

School of Mathematics, Harbin Institute of Technology



Geometry of Differential Equations Seminar, IUM

November 04, 2026

Outline

1. Introduction
2. Reminders: homological algebra
3. Quasi-smooth PDEs
4. Prolongation and formal integrability
5. Singularity categories
6. Summary

Outline

1. Introduction
2. Reminders: homological algebra
3. Quasi-smooth PDEs
4. Prolongation and formal integrability
5. Singularity categories
6. Summary

Singularities of PDEs: Two reconstruction problems

Let $\mathcal{E}$ be a system of PDEs of order $\leq k$, let Σ be a *type* of singularity, and let $\mathcal{E}_\Sigma$ be the PDE whose solutions are the solutions of $\mathcal{E}$ with Σ-type singularities.

There are two classes of singularities:

(i) Keep (R-)manifolds smooth. Consider a fiber bundle $\pi : E \to M$ and W an n-dimensional (maximal) integral submanifold of Cartan distribution on $J^k_M E$. One is studying singularities of $\pi^k_{k-1}|_W : W \to J^{k-1}_M(E)$.

(ii) Allow R-manifolds / equation manifolds *themselves* to be singular.

Class (i) leads to the first type of reconstruction problem (due to A.M Vinogradov, 1986):

$$\text{Given } (\Sigma, \mathcal{E}_\Sigma), \quad \text{find } \mathcal{E}.$$

For which Σ and $\mathcal{E}$ is reconstruction unique?

Today: The questions we have in mind relate to class (ii):

(a) What information about $\mathcal{E}$ is encoded by its singularities?

(b) Can we attach invariants to these singularities that distinguish equations?

Set-up

Fix $n, m > 0$ and $k \geq 0$.

Analytic PDE of order $\leq k$ in n variables and m functions: an open subset

$$U_k \hookrightarrow \mathbb{C}^n \times \prod_{|\sigma| \leq k} \mathbb{C}^m,$$

together with a coherent ideal $I_k \subset \mathcal{O}_{U_k}$. Equivalently, a closed analytic subscheme

$$Z_0 = \left(|Z_0|, \mathcal{O}_{Z_0}\right) \hookrightarrow J^k_X(E),$$

for X a complex n-manifold and $E \to X$ a holomorphic vector bundle of rank m.

If Z_0 is cut out by N equations, present it as the zero locus of a bundle map $F_P : J^k_X(E) \to V$ ($\operatorname{rank} V = N$):

$$\begin{array}{ccc} \mathcal{E}_{F_P} & \xrightarrow{\iota} & J^k_X(E) \\ \downarrow & & \downarrow{\scriptstyle F_P} \\ X & \xrightarrow{0_V} & V. \end{array}$$

Impose regularity conditions so that equation manifolds are *smooth*.

Set-up

Fix $n, m > 0$ and $k \geq 0$.

Analytic PDE of order $\leq k$ in n variables and m functions: an open subset

$$U_k \hookrightarrow \mathbb{C}^n \times \prod_{|\sigma| \leq k} \mathbb{C}^m,$$

together with a coherent ideal $I_k \subset \mathcal{O}_{U_k}$. Equivalently, a closed analytic subscheme

$$Z_0 = \left(|Z_0|, \mathcal{O}_{Z_0}\right) \hookrightarrow J^k_X(E),$$

for X a complex n-manifold and $E \to X$ a holomorphic vector bundle of rank m.

If Z_0 is cut out by N equations, present it as the zero locus of a bundle map $F_P : J^k_X(E) \to V$ ($\operatorname{rank} V = N$):

$$\begin{array}{ccc} \mathcal{E}_{F_P} & \xrightarrow{\iota} & J^k_X(E) \\ \downarrow & & \downarrow{\scriptstyle F_P} \\ X & \xrightarrow{0_V} & V. \end{array}$$

Impose regularity conditions so that equation manifolds are *smooth*.

Question

What happens if we relax this and allow the equation manifold itself to be singular?

Regularity conditions

Let $\xi : E \to M$ be a vector bundle and $F : \Gamma(\pi) \to \Gamma(\xi)$ be a nonlinear differential operator. View it also as a section $F \in \Gamma(\pi^*_\infty(\xi))$, of the induced vector bundle $\pi_\infty(\xi)$ over $J^\infty(\pi)$:

$$J(F) : J^\infty(\pi) \to J^\infty(\xi), \quad [s]^\infty_x \mapsto \big[F(s)\big]^\infty_x.$$

Let $\mathcal{E} = \mathcal{E}(F)$ be the differential equation defined by F,

$$\mathcal{E}(F) = \ker(J(F)) := \{\theta \in J^\infty(\pi) | J(F)(\theta) = 0\}. \tag{1}$$

Locally, $F = (F_1, \ldots, F_N)$ with $F_\ell = F_\ell(x_i, p^j_\sigma)$, then (1) is the infinite prolongation of the system

$$\begin{cases} F_1 = 0 \\ \quad \vdots \\ F_N = 0. \end{cases}$$

Let $\mathcal{I}_\mathcal{E} \subset \mathcal{F}(\pi)$ be the ideal defined by (1).

Definition

The PDE $\mathcal{E}_F$ is *regular with respect to* $F = (F_1, \ldots, F_N)$ if:

- The ideal $\mathcal{I}_\mathcal{E}$ is generated by $D_I(F_\ell), 1 \leq \ell \leq N, 0 \leq |I| < \infty$ (differentially finitely-generated).
- Λ^1 is projective. This implies $\overline{\Lambda}^1, \mathcal{C}\Lambda^1$ are both projective.

A mild generalization: quasi-smooth equations

Suppose the equation is locally described by a section

$$s : J^k(E) \longrightarrow V$$

of a vector bundle. Its classical solution locus is

$$\mathcal{E}_0 = \{s = 0\} \subset J^k(E).$$

In the transverse case, $\mathcal{E}_0$ is smooth, but we drop this condition.

Guiding principle

Replace the smooth equation manifold by a quasi-smooth equation manifold:

$$\mathcal{E}_0 = F^{-1}(0) \subset J^k_M E, \rightsquigarrow \quad \mathbb{R}Z(F) =: \boldsymbol{\mathcal{E}} \quad + \mathbf{Sing}(\boldsymbol{\mathcal{E}}).$$

Quasi–smooth: A space $\mathcal{E}$ is quasi–smooth if it has a *cotangent complex* $\mathbb{L}_{\mathcal{E}}$ in degrees $[-1, 0]$:

- (Zariski-locally) it is $\mathcal{O}^{\oplus m} \to \mathcal{O}^{\oplus n}$ in degrees $-1, 0$;
- It is expressible as

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & \mathbb{A}^n \\ \downarrow & & \downarrow f \\ \{0\} & \longrightarrow & \mathbb{A}^m. \end{array} \quad \Rightarrow \mathbb{L}_{\mathcal{E}} \simeq [k^m \xrightarrow{df^\vee} k^n].$$

Categorical reconstruction

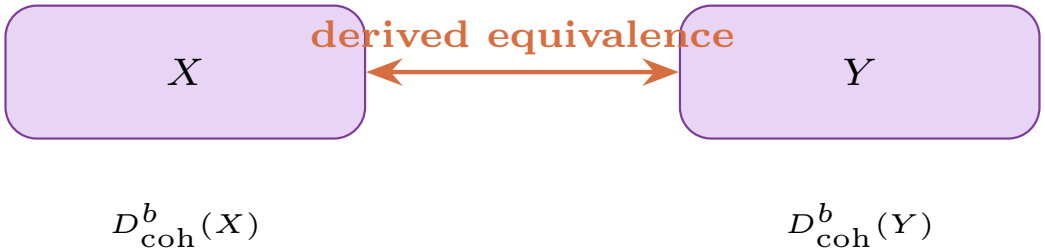
Let X be a smooth projective variety. Consider its bounded derived category of coherent sheaves:

$$D^b_{\mathrm{coh}}(X) := D^b\big(\mathrm{Coh}(X)\big).$$

Bondal–Orlov. If K_X is ample or anti-ample, then X is determined up to isomorphism by $D^b_{\mathrm{coh}}(X)$.

Q. Can a category attached to a PDE $\mathcal{E} \hookrightarrow J^k(E^{n+m}, n)$ determine the equation's geometry, and vice versa?

Different spaces may have equivalent categories: a geometric transformation can become a categorical equivalence.



Q. Could two PDEs have equivalent categorical geometry even when no point transformation or classical Bäcklund transformation connects them? (S. Igonin)

Further evidence: categories encode geometry

Let k be algebraically closed. Non-commutative algebraic geometry studies k-schemes and varieties via

$$D^b(X) := D^b(\mathrm{Coh}(X)),$$

as a replacement for the underlying space.

It can reflect structure (e.g orientations) and properties (e.g. smooth, properness) of a variety.

1. X is smooth and proper $\iff$ $\mathrm{Perf}(X)$ is smooth and proper.
2. If X is a smooth projective $K3$ surface ($\mathcal{O}_X \simeq K_X$), then $D^b_{\mathrm{coh}}(X)$ is 2-Calabi–Yau.
3. If X, Y are smooth projective and $D^b_{\mathrm{coh}}(X) \simeq D^b_{\mathrm{coh}}(Y)$, then X is hyperkähler $\iff$ Y is hyperkähler.
4. If X is a complex analytic space, then $\mathrm{Perf}(X)$ smooth and proper $\iff$ X is algebraizable.

Q. Can something interesting be said for differential equations?

How should we study the singularities of an equation?

Let Z be a possibly singular space.

Categorical approach

Study the singularities of Z through the behaviour of coherent sheaves on Z.

Serre's criterion. A noetherian scheme X is regular if and only if every coherent sheaf E admits a finite locally free resolution:

$$0 \to V^n \to V^{n-1} \to \cdots \to V^1 \to V^0 \to E.$$

Equivalently, $\mathrm{Perf}(X) = D^b_{\mathrm{coh}}(X)$.

For a singular space X this equality fails; $\mathrm{Perf}(X) \subseteq D^b_{\mathrm{coh}}(X)$.

The failure of coherent sheaves to be perfect detects the singularities of the underlying space.

The category of singularities

Category of singularities is designed to measure the 'excess':

$$D_{\mathrm{sg}}(X) := \frac{D^b_{\mathrm{coh}}(X)}{\mathrm{Perf}(X)}.$$

Here:

- $D^b_{\mathrm{coh}}(X)$ consists of bounded coherent complexes;
- $\mathrm{Perf}(X)$ consists of complexes locally represented by bounded complexes of vector bundles.

Thus $D_{\mathrm{sg}}(X)$ discards objects admitting finite locally free resolutions, retaining only those which give obstructions to smoothness.

Fundamental property

For a noetherian scheme of finite type over a field,

$$D_{\mathrm{sg}}(X) = 0 \quad \Longleftrightarrow \quad X \text{ is regular.}$$

In other words, excess vanishes if and only if X is regular.

Classical invariants of hypersurface singularities

Let U be a smooth variety over $\mathbb{C}$ and $f : U \to \mathbb{A}^1_{\mathbb{C}}$.

If f has isolated singularities, the **Milnor number**

$$\mu(f) := \dim_{\mathbb{C}} \mathcal{O}_U/(\partial_1 f, \ldots, \partial_n f)$$

is an invariant of the singularity.

We compare two refined invariants:

1. a **cohomological invariant**: cohomology of vanishing cycles;
2. a **categorical invariant**: the Eisenbud–Orlov category of singularities, equivalently the category of matrix factorizations $\mathrm{MF}(U, f)$.

Serre and Eisenbud: finite resolutions and 2-periodicity

Eisenbud. Let A be a regular local ring, $f \in A$, and $R = A/(f)$. Then every minimal R-projective resolution of a finitely generated R-module M becomes **2-periodic** after $\dim A + 1$ steps:

$$\cdots \to F \to Q \to F \to Q \to V_n \to \cdots \to V_0 \to M.$$

Tool of matrix factorizations are to give a presentation of this 2-periodic tail.
R regular local ring, $f \in R$ nonzero divisor consider
$Z(f) := \operatorname{Spec}(R/f) \hookrightarrow \operatorname{Spec}(R)$.

$$\mathrm{MF}(U, f) := \left\{ Q \leftrightarrow P : \delta_0\delta_1 = \delta_1\delta_0 = f \right\}.$$

Category of singularities and Orlov's theorem

Let U be smooth affine, $f : U \to \mathbb{A}^1$ regular, and $U_0 = f^{-1}(0)$.

The category of singularities is the Verdier quotient

$$D_{\mathrm{sg}}(U_0) := \frac{D^b_{\mathrm{coh}}(U_0)}{\mathrm{Perf}(U_0)}.$$

Matrix factorizations $\mathrm{MF}(U, f)$ capture the 2-periodic part of the resolution after the finite vector-bundle resolution.

Orlov

Let U be a scheme over $\mathbb{C}$, $f : U \to \mathbb{A}^1$ flat, and assume the only critical value of f is 0. Then $\mathrm{Perf}(X) \hookrightarrow D^b_{coh}(X) \to D_{sg}(U_0) \simeq \mathrm{MF}(U, f)$:

$$\mathrm{MF}(U, f) \longrightarrow D_{\mathrm{sg}}(U_0), \qquad (E_1, E_0, \delta_0, \delta_1) \longmapsto \operatorname{coker} \delta_0$$

is an equivalence of triangulated categories.

Idea: $\mathrm{MF}(U, f)$ controls all singularities of hypersurface $U_0 \subseteq U$. The statement adapts when f has more critical values, and to $f = (f_1, \ldots, f_r)$.

Singular support: location and direction

Many objects in mathematics have a notion of smoothness and of **singularity**. A *singular support* records the location of singularities and *also* remembers in which direction they occur.

Ex.0 Wavefronts of generalized functions.

Ex.1 Sheaves on manifolds.

Ex.2 Linear PDEs.

Ex.3 Nonlinear PDEs: characteristic varieties.

Ex.4 Derived characteristic variety: cohomological support of the tangent *complex* of a PDE.

Our aim: pass from Ex.3 to Ex.4 in the mildest possible way.

Ex.5 Arinkin–Gaitsgory support of coherent sheaves: measures how far a coherent complex is from being a perfect complex.

Rough idea: if a space Z is singular, we can detect/observe its singularities by studying sheaves on it, together with their singular supports.

Singular support: where and in which directions?

The category of singularities detects failure of regularity. Can we refine this information?

Singular support theory. For a quasi-smooth derived space Z, one associates a classical space of singular directions

$$\mathrm{Sing}(Z).$$

For a coherent object $\mathcal{F}$, its singular support

$$\mathrm{SS}(\mathcal{F}) \subseteq \mathrm{Sing}(Z)$$

records the directions in which $\mathcal{F}$ fails to be perfect.

Analogy

- Wavefront sets record where a distribution is singular and in which cotangent directions.
- Singular support of coherent complexes records where and in which singular directions their homological behaviour fails to be regular.

Categories of singularities with support conditions

Let Z be quasi-smooth and let $\Lambda \subseteq \mathrm{Sing}(Z)$ be a conical closed subset.

Consider the category

$$\boxed{\mathrm{IndCoh}_\Lambda(Z),}$$

the category of ind-coherent sheaves on Z whose singular support is contained in Λ:

$$\mathrm{SS}(\mathcal{F}) \subseteq \Lambda.$$

Why IndCoh? For singular spaces, Coh and QCoh have different homological behaviour.

$$\mathrm{IndCoh}(Z) := \mathrm{Ind}\big(\mathrm{Coh}(Z)\big).$$

Krause: If Z is smooth, $\mathrm{IndCoh}(Z)$ is the category of injective complexes.
Z smooth: $\mathrm{Perf}(Z) = \mathrm{Coh}(Z)$, hence $\mathrm{IndCoh}(Z) = \mathrm{QCoh}(Z)$.

Central object

$$(Z, \Lambda) \longmapsto \mathrm{IndCoh}_\Lambda(Z).$$

We want to understand the geometry of Z through this category.

Slogan: the more singular Z, the farther $\mathrm{IndCoh}(Z)$ is from $\mathrm{QCoh}(Z)$.

Towards singularity categories of differential equations

Can we study a differential equation through the category of its singularities?

Plan of the talk.

1. Homological algebra: cdgas, Koszul complexes, cotangent complexes, and homotopical jet functors.
2. Quasi-smooth PDEs: singular schemes and homotopy-prolongation.
3. Singularity categories of PDEs: basic properties, examples, and open questions.

Based on:

- *Derived Moduli Spaces of Non-linear PDEs: Singular Propagations*, Kryczka–Sheshmani–Yau, Adv. Math. 2026.
- *Derived Moduli Spaces of Non-linear PDEs: Variational Bicomplex and the BV Formalism*, Kryczka–Sheshmani–Yau, Moduli (Compos. Math.) 2026.

Outline

1 Introduction

2 Reminders: homological algebra

3 Quasi-smooth PDEs

4 Prolongation and formal integrability

5 Singularity categories

6 Summary

Commutative dg-algebras

A **commutative differential-graded $\mathbb{C}$-algebra** is a triple

$$A^\bullet := \big(A^\sharp, \mu_A, d_A\big),$$

where $A^\sharp = \bigoplus_i A^i$ with $A^i \in \mathrm{Mod}_k$, with multiplications

$$\mu_A : A^i \otimes_k A^j \to A^{i+j}, \forall i, j,$$

and unit element $1_A \in A^0$ (degree zero) with k-linear differential:

$$d_A : A^i \to A^{i+1}, d_A \circ d_A = 0.$$

The differential d_A is a graded derivation with respect to multiplication:

$$d_A(a \cdot_A b) = d_A(a) \cdot_A b + (-1)^{\deg(a)} a \cdot d_A(b).$$

A dg-algebra $A^\bullet$ is *discrete* if $A^i \simeq 0$, for all $i \neq 0$.

Cdga's

To any commutative dg-algebra $A^\bullet$ we have its cohomology algebra:

$$H^*(A^\bullet) := \bigoplus_i H^i(A^\bullet), \qquad H^i(A^\bullet) := \ker(d_A^i)/\mathrm{im}(d_A^{i-1}).$$

Cdga's

To any commutative dg-algebra $A^\bullet$ we have its cohomology algebra:

$$H^*(A^\bullet) := \bigoplus_i H^i(A^\bullet), \qquad H^i(A^\bullet) := \ker(d_A^i)/\mathrm{im}(d_A^{i-1}).$$

The cohomology ring $H^*(A^\bullet)$ is a graded commutative k-algebra.

Cdga's

To any commutative dg-algebra $A^\bullet$ we have its cohomology algebra:

$$H^*(A^\bullet) := \bigoplus_i H^i(A^\bullet), \qquad H^i(A^\bullet) := \ker(d_A^i)/\mathrm{im}(d_A^{i-1}).$$

The cohomology ring $H^*(A^\bullet)$ is a graded commutative k-algebra.
Morphisms: $f : A^\bullet = (A^\sharp, \cdot_A, d_A) \to B^\bullet = (B^\sharp, \cdot_B, d_B)$ preserve differentials and multiplication:

$$f \circ d_A = d_B \circ f : A^i \to B^{i+1}, \quad f(a \cdot_A b) = f(a) \cdot_B f(b) \in B^{i+j}.$$

Cdga's

To any commutative dg-algebra $A^\bullet$ we have its cohomology algebra:

$$H^*(A^\bullet) := \bigoplus_i H^i(A^\bullet), \qquad H^i(A^\bullet) := \ker(d_A^i)/\mathrm{im}(d_A^{i-1}).$$

The cohomology ring $H^*(A^\bullet)$ is a graded commutative k-algebra.
Morphisms: $f : A^\bullet = (A^\sharp, \cdot_A, d_A) \to B^\bullet = (B^\sharp, \cdot_B, d_B)$ preserve differentials and multiplication:

$$f \circ d_A = d_B \circ f : A^i \to B^{i+1}, \quad f(a \cdot_A b) = f(a) \cdot_B f(b) \in B^{i+j}.$$

Quasi-isomorphisms: A morphism $f : A^\bullet \to B^\bullet$ is a quasi-isomorphism (weak-equivalence) if it induces an isomorphism $H^*(A^\bullet) \cong H^*(B^\bullet)$.

Cdga's

To any commutative dg-algebra $A^\bullet$ we have its cohomology algebra:

$$H^*(A^\bullet) := \bigoplus_i H^i(A^\bullet), \qquad H^i(A^\bullet) := \ker(d_A^i)/\mathrm{im}(d_A^{i-1}).$$

The cohomology ring $H^*(A^\bullet)$ is a graded commutative k-algebra.
Morphisms: $f : A^\bullet = (A^\sharp, \cdot_A, d_A) \to B^\bullet = (B^\sharp, \cdot_B, d_B)$ preserve differentials and multiplication:

$$f \circ d_A = d_B \circ f : A^i \to B^{i+1}, \quad f(a \cdot_A b) = f(a) \cdot_B f(b) \in B^{i+j}.$$

Quasi-isomorphisms: A morphism $f : A^\bullet \to B^\bullet$ is a quasi-isomorphism (weak-equivalence) if it induces an isomorphism $H^*(A^\bullet) \cong H^*(B^\bullet)$.

Definition

A derived $\mathbb{C}$-scheme is a pair $\mathbf{X} = (X, \mathcal{O}_\mathbf{X}^\bullet)$ where X is a topological space and $\mathcal{O}_\mathbf{X}^\bullet$ is a sheaf of cdga's such that:

$$t_0(\mathbf{X}) = (X, H^0(\mathcal{O}_\mathbf{X}^\bullet))$$

is a usual scheme/variety (classical truncation) and $H^i(\mathcal{O}_\mathbf{X}^\bullet)$ are quasi-coherent $H^0(\mathcal{O}_\mathbf{X}^\bullet)$-modules.

A **affine** is $\mathbf{Spec}(A^\bullet)$, so $t_0(\mathbf{Spec}(A^\bullet) \simeq \mathrm{Spec}\big(H^0(A^\bullet)\big)$.

Differential–graded modules

Let $A^\bullet$ be a commutative dg-algebra over k.

Differential–graded modules

Let $A^\bullet$ be a commutative dg-algebra over k.

A **differential–graded $A^\bullet$-module** $M^\bullet$ is a complex of k-modules $M^\bullet = (M^\sharp, d_M)$ with a multiplcation:

$$(A \otimes_k M)^\bullet \to M^\bullet,$$

compatible with differentials.

Differential–graded modules

Let $A^\bullet$ be a commutative dg-algebra over k.

A **differential–graded $A^\bullet$-module** $M^\bullet$ is a complex of k-modules $M^\bullet = (M^\sharp, d_M)$ with a multiplcation:

$$(A \otimes_k M)^\bullet \to M^\bullet,$$

compatible with differentials.
Explicitly, for all i, j there are k-bilinear maps

$$\nu : A^i \times M^j \to M^{i+j},$$

such that $1_A m = m$ and

$$(a \cdot_A b)m = a(bm), \quad d_M(am) = d_A(a)m + (-1)^{\deg(a)} a d_M(m).$$

Differential–graded modules

Let $A^\bullet$ be a commutative dg-algebra over k.

A **differential–graded $A^\bullet$-module** $M^\bullet$ is a complex of k-modules $M^\bullet = (M^\sharp, d_M)$ with a multiplcation:

$$(A \otimes_k M)^\bullet \to M^\bullet,$$

compatible with differentials.
Explicitly, for all i, j there are k-bilinear maps

$$\nu : A^i \times M^j \to M^{i+j},$$

such that $1_A m = m$ and

$$(a \cdot_A b)m = a(bm), \quad d_M(am) = d_A(a)m + (-1)^{\deg(a)} a d_M(m).$$

A morphism $f : M^\bullet \to N^\bullet$ is $A^\bullet$-linear and compatible with differentials.

Differential–graded modules

Let $A^\bullet$ be a commutative dg-algebra over k.

A **differential–graded $A^\bullet$-module** $M^\bullet$ is a complex of k-modules $M^\bullet = (M^\sharp, d_M)$ with a multiplcation:

$$(A \otimes_k M)^\bullet \to M^\bullet,$$

compatible with differentials.
Explicitly, for all i, j there are k-bilinear maps

$$\nu : A^i \times M^j \to M^{i+j},$$

such that $1_A m = m$ and

$$(a \cdot_A b)m = a(bm), \quad d_M(am) = d_A(a)m + (-1)^{\deg(a)} a d_M(m).$$

A morphism $f : M^\bullet \to N^\bullet$ is $A^\bullet$-linear and compatible with differentials.

A morphism of dg-$A^\bullet$-modules is a *quasi-isomorphism* if it inducess an isomorphism in cohomologies: $H^*(M^\bullet) \cong H^*(N^\bullet)$.

Cotangent complexes

Let $R^\bullet \to A^\bullet$ be a morphism in $\mathbf{cdga}_k$.

Cotangent complexes

Let $R^\bullet \to A^\bullet$ be a morphism in $\mathbf{cdga}_k$.

The *complex of Kahler differentials*: $\Omega^1_{A^\bullet/R^\bullet} \in \mathbf{DGMod}(A^\bullet)$, defined by setting:

$$\Omega^1_{A^\bullet/R^\bullet} := I/I^2, \quad I := \ker\big((A \otimes_R A)^\bullet \to A^\bullet)\big).$$

Cotangent complexes

Let $R^\bullet \to A^\bullet$ be a morphism in $\mathbf{cdga}_k$.

The *complex of Kahler differentials*: $\Omega^1_{A^\bullet/R^\bullet} \in \mathbf{DGMod}(A^\bullet)$, defined by setting:

$$\Omega^1_{A^\bullet/R^\bullet} := I/I^2, \quad I := \ker\big((A \otimes_R A)^\bullet \to A^\bullet)\big).$$

The *cotangent complex* of $A^\bullet$ is the object:

$$\mathbb{L}_{A/R} := \big(\Omega^1_{A/R} \otimes_{QA} A\big)^\bullet \in \mathbf{DGMod}(A^\bullet).$$

Cotangent complexes

Let $R^\bullet \to A^\bullet$ be a morphism in $\mathbf{cdga}_k$.

The *complex of Kahler differentials*: $\Omega^1_{A^\bullet/R^\bullet} \in \mathbf{DGMod}(A^\bullet)$, defined by setting:

$$\Omega^1_{A^\bullet/R^\bullet} := I/I^2, \quad I := \ker\big((A \otimes_R A)^\bullet \to A^\bullet)\big).$$

The *cotangent complex* of $A^\bullet$ is the object:

$$\mathbb{L}_{A/R} := \big(\Omega^1_{A/R} \otimes_{QA} A\big)^\bullet \in \mathbf{DGMod}(A^\bullet).$$

By construction: $H^0\big(\mathbb{L}_{A^\cdot/R^\cdot}\big) \simeq \Omega^1_{H^0(A^\cdot)/H^0(R^\cdot)}$.

Cotangent complexes

Let $R^\bullet \to A^\bullet$ be a morphism in $\mathbf{cdga}_k$.

The *complex of Kahler differentials*: $\Omega^1_{A^\bullet/R^\bullet} \in \mathbf{DGMod}(A^\bullet)$, defined by setting:

$$\Omega^1_{A^\bullet/R^\bullet} := I/I^2, \quad I := \ker\big((A \otimes_R A)^\bullet \to A^\bullet\big).$$

The *cotangent complex* of $A^\bullet$ is the object:

$$\mathbb{L}_{A/R} := \big(\Omega^1_{A/R} \otimes_{QA} A\big)^\bullet \in \mathbf{DGMod}(A^\bullet).$$

By construction: $H^0\big(\mathbb{L}_{A^\cdot/R^\cdot}\big) \simeq \Omega^1_{H^0(A^\cdot)/H^0(R^\cdot)}$.

Any $f : A^\bullet \to B^\bullet$ induces: $\mathcal{A} \otimes^L_{\mathcal{B}} \mathbb{L}_{\mathcal{B}} \to \mathbb{L}_{\mathcal{A}} \to \mathbb{L}_f$,
Globalize: to any morphism $f : \mathcal{X} \to \mathcal{Y}$ of derived schemes, the relative cotangent complex is given by the distinguished triangle

$$\mathbb{L}_{\mathcal{Y}/\mathbb{C}} \otimes^L_{\mathcal{O}_\mathcal{Y}} \mathcal{O}_\mathcal{X} \to \mathbb{L}_{\mathcal{X}/\mathbb{C}} \to \mathbb{L}_{\mathcal{X}/\mathcal{Y}} \to \big(\mathbb{L}_{\mathcal{Y}/\mathbb{C}} \otimes^L_{\mathcal{O}_\mathcal{Y}} \mathcal{O}_\mathcal{X}\big)[1].$$

Cotangent complexes

Let $R^\bullet \to A^\bullet$ be a morphism in $\mathbf{cdga}_k$.

The *complex of Kahler differentials*: $\Omega^1_{A^\bullet/R^\bullet} \in \mathbf{DGMod}(A^\bullet)$, defined by setting:

$$\Omega^1_{A^\bullet/R^\bullet} := I/I^2, \quad I := \ker\left((A \otimes_R A)^\bullet \to A^\bullet\right).$$

The *cotangent complex* of $A^\bullet$ is the object:

$$\mathbb{L}_{A/R} := \left(\Omega^1_{A/R} \otimes_{QA} A\right)^\bullet \in \mathbf{DGMod}(A^\bullet).$$

By construction: $H^0\left(\mathbb{L}_{A^\cdot/R^\cdot}\right) \simeq \Omega^1_{H^0(A^\cdot)/H^0(R^\cdot)}$.

Any $f : A^\bullet \to B^\bullet$ induces: $\mathcal{A} \otimes^L_{\mathcal{B}} \mathbb{L}_{\mathcal{B}} \to \mathbb{L}_{\mathcal{A}} \to \mathbb{L}_f$,
Globalize: to any morphism $f : \mathcal{X} \to \mathcal{Y}$ of derived schemes, the relative cotangent complex is given by the distinguished triangle

$$\mathbb{L}_{\mathcal{Y}/\mathbb{C}} \otimes^L_{\mathcal{O}_\mathcal{Y}} \mathcal{O}_\mathcal{X} \to \mathbb{L}_{\mathcal{X}/\mathbb{C}} \to \mathbb{L}_{\mathcal{X}/\mathcal{Y}} \to \left(\mathbb{L}_{\mathcal{Y}/\mathbb{C}} \otimes^L_{\mathcal{O}_\mathcal{Y}} \mathcal{O}_\mathcal{X}\right)[1].$$

Upshot: In the world of derived algebras (i.e. cdga's) we can study arbitrary intersections (higher Tor's) which have well-behaved tangent complexes.

A simple example: a derived thickening of the origin

Let $A = \mathbb{C}[x]$ and $B = A/(x) \cong \mathbb{C}$. Consider the derived self-intersection

$$B \otimes^{\mathbb{L}}_{A} B.$$

Koszul resolution
Resolve B as an A-module:

$$K_A(x) = \left[A \xrightarrow{x} A \right].$$

Tensoring with B gives

$$B \otimes_A K_A(x) = \left[B \xrightarrow{0} B \right].$$

Hence

$$\mathrm{Tor}_i^A(B, B) = \begin{cases} \mathbb{C} & i = 0, 1, \\ 0 & \text{otherwise.} \end{cases}$$

Ordinary versus derived
The ordinary tensor product sees only

$$B \otimes_A B \cong \mathbb{C}.$$

The derived tensor product remembers

$$B \otimes^{\mathbb{L}}_{A} B \simeq \mathbb{C} \oplus \mathbb{C}[1]$$

(as a graded vector space).

Moral: The derived intersection retains the hidden Tor-information.

Examples

Let $L, L' \hookrightarrow M$ be smooth subvarieties in a smooth variety M.

$$L \times^h_M L' := \left(L \cap L', \mathcal{O}_L \otimes^L_{\mathcal{O}_M} \mathcal{O}_{L'}\right).$$

Its tangent complex:

$$\mathbb{T}_{L \times^h_M L', s} = \left[T_{L,s} \oplus T_{L',s} \to T_{M,s}\right]$$

in degrees $0, 1$.

$H^0(\mathbb{T}_{L \times^h_M L', s} = T_{L \cap L', s}$ while H^1 encodes the failure of transversality.

Examples

$f : X \to \mathbb{A}^1$ a function on a smooth scheme.

Examples

$f : X \to \mathbb{A}^1$ a function on a smooth scheme.

Its derived critical locus:

$$\begin{array}{ccc} \mathbb{R}\mathrm{Crit}(f) & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle df} \\ X & \xrightarrow{0_X} & T^*X \end{array}$$

Examples

$f : X \to \mathbb{A}^1$ a function on a smooth scheme.

Its derived critical locus:

$$\begin{array}{ccc} \mathbb{R}\mathrm{Crit}(f) & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle df} \\ X & \xrightarrow{0_X} & T^*X \end{array}$$

Derived ring of functions modelled by

$$\left[\cdots \wedge^2 T_X \to T_X \to \mathcal{O}_X\right],$$

with differential $d := i_{df}$ given by contraction with df.

Examples

$f : X \to \mathbb{A}^1$ a function on a smooth scheme.

Its derived critical locus:

$$\begin{array}{ccc} \mathbb{R}\mathrm{Crit}(f) & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle df} \\ X & \xrightarrow{0_X} & T^*X \end{array}$$

Derived ring of functions modelled by

$$\left[\cdots \wedge^2 T_X \to T_X \to \mathcal{O}_X\right],$$

with differential $d := i_{df}$ given by contraction with df.

$H^0(\mathcal{O}_{\mathbb{R}\mathrm{Crit}(f)}) \simeq \mathcal{O}_X/(\partial_1 f \cdots \partial_n f)$.

Examples

$f : X \to \mathbb{A}^1$ a function on a smooth scheme.

Its derived critical locus:

$$\begin{array}{ccc} \mathbb{R}\mathrm{Crit}(f) & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle df} \\ X & \xrightarrow{0_X} & T^*X \end{array}$$

Derived ring of functions modelled by

$$\left[\cdots \wedge^2 T_X \to T_X \to \mathcal{O}_X\right],$$

with differential $d := i_{df}$ given by contraction with df.

$H^0(\mathcal{O}_{\mathbb{R}\mathrm{Crit}(f)}) \simeq \mathcal{O}_X/(\partial_1 f \cdots \partial_n f)$.

Note. In general $\mathbb{R}\mathrm{Crit}(f)$ and $Z(f)$ do not coincide.

Examples

$f : X \to \mathbb{A}^1$ a function on a smooth scheme.

Its derived critical locus:

$$\begin{array}{ccc} \mathbb{R}\mathrm{Crit}(f) & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle df} \\ X & \xrightarrow{0_X} & T^*X \end{array}$$

Derived ring of functions modelled by

$$\left[\cdots \wedge^2 T_X \to T_X \to \mathcal{O}_X\right],$$

with differential $d := i_{df}$ given by contraction with df.

$H^0(\mathcal{O}_{\mathbb{R}\mathrm{Crit}(f)}) \simeq \mathcal{O}_X/(\partial_1 f \cdots \partial_n f)$.

Note. In general $\mathbb{R}\mathrm{Crit}(f)$ and $Z(f)$ do not coincide.
Even in these cases, tangent complexes can have non-trivial structure (e.g. $Z(f : \mathbb{A}^2_{[x,y]} \to \mathbb{A}^1_{[t]}, f(x,y) = xy$ (Compute $\mathbb{C} \otimes^1_{\mathbb{C}[t]} \mathbb{C}[x,y]$, via a Koszul resolution of $\mathbb{C}$ over $\mathbb{C}[t]$).

Examples

Let $E \to M$ be a vector bundle. Fix $s : M \to E$ a global section ($s \in H^0(M, E)$).

Consider:

$$\begin{array}{ccc} \mathbb{R}Z(s) & \longrightarrow & M \\ \downarrow & & \downarrow{\scriptstyle s} \\ M & \xrightarrow{0_M} & E \end{array}$$

Derived zero locus: take usual scheme-theoretic zero locus $i_Z : Z(s) \hookrightarrow M$ and the pull-back of dg-algebra of functions on total space of E:

$$M \times^h_E M = \left(Z(s), i_Z^{-1}\mathrm{Sym}^\bullet(E^\vee[1]), s^\flat\right).$$

$s^\flat$: the operator of contraction with s.

Koszul cdga's

R : a commutative algebra over k, with $\mathsf{f} = (f_1, \ldots, f_r) \in R$.

Koszul cdga's

R : a commutative algebra over k, with $\mathsf{f} = (f_1, \dots, f_r) \in R$.
$K_\bullet(R, \mathsf{f}) : \left(\bigwedge^\bullet (R^{\oplus r}), d \right)$:

$$0 \to \wedge^r R^{\oplus r} \to \wedge^{r-1} R^{\oplus r} \to \cdots \to \wedge^2 R^{\oplus r} \to R^{\oplus r} \to 0.$$

$$d(e_1 \wedge \cdots \wedge e_j) = \sum_{1 \leq i \leq j} (-1)^{i+1} (f_i) e_i \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e_j.$$

Koszul cdga's

R : a commutative algebra over k, with $\mathsf{f} = (f_1, \ldots, f_r) \in R$.
$K_\bullet(R, \mathsf{f}) : \big(\bigwedge^\bullet(R^{\oplus r}), d\big)$:

$$0 \to \wedge^r R^{\oplus r} \to \wedge^{r-1} R^{\oplus r} \to \cdots \to \wedge^2 R^{\oplus r} \to R^{\oplus r} \to 0.$$

$$d(e_1 \wedge \cdots \wedge e_j) = \sum_{1 \leq i \leq j} (-1)^{i+1} (f_i) e_i \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e_j.$$

Properties:

1. $K_\bullet(R, \mathsf{f})$ is a commutative dg-R-algebra.
2. $K_\bullet(R, f) \simeq [R \xrightarrow{f} R]$ in $R - Mod$ i.e $K_\bullet(R, f) \xrightarrow{\simeq} R/(f)$ if and only if f is a non-zero divisor.
   - $\bigotimes_{i=1}^r K_\bullet(R, f_i) \xrightarrow{\simeq} K_\bullet(R; \mathsf{f})$.
3. If $(g_1, \ldots, g_m)$ generates the same ideal as f, then

   $$K_\bullet(R, \mathsf{f}) \xrightarrow{\simeq} K_\bullet(R; \mathsf{g})$$

   i.e. $H_*(K_\bullet(R; \mathsf{f})) \cong H_*\big(K_\bullet(R, \mathsf{g})\big)$.
4. Augmented: there is a map $K_\bullet(R; \mathsf{f}) \to R/(f_1, \ldots, f_r)$ such that

   $$H_0\big(K_\bullet(R; \mathsf{f})\big) \simeq R/(\mathsf{f}).$$

Koszul cdga's

R : a commutative algebra over k, with $\mathsf{f} = (f_1, \dots, f_r) \in R$.
$K_\bullet(R, \mathsf{f}) : \big(\bigwedge^\bullet(R^{\oplus r}), d\big)$:

$$0 \to \wedge^r R^{\oplus r} \to \wedge^{r-1} R^{\oplus r} \to \cdots \to \wedge^2 R^{\oplus r} \to R^{\oplus r} \to 0.$$

$$d(e_1 \wedge \cdots \wedge e_j) = \sum_{1 \leq i \leq j} (-1)^{i+1} (f_i) e_i \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e_j.$$

Properties:

1. $K_\bullet(R, \mathsf{f})$ is a commutative dg-R-algebra.
2. $K_\bullet(R, f) \simeq [R \xrightarrow{f} R]$ in $R - Mod$ i.e $K_\bullet(R, f) \xrightarrow{\simeq} R/(f)$ if and only if f is a non-zero divisor.
   - $\bigotimes_{i=1}^r K_\bullet(R, f_i) \xrightarrow{\simeq} K_\bullet(R; \mathsf{f})$.
3. If $(g_1, \dots, g_m)$ generates the same ideal as f, then

   $$K_\bullet(R, \mathsf{f}) \xrightarrow{\simeq} K_\bullet(R; \mathsf{g})$$

   i.e. $H_*(K_\bullet(R; \mathsf{f})) \cong H_*\big(K_\bullet(R, \mathsf{g})\big)$.
4. Augmented: there is a map $K_\bullet(R; \mathsf{f}) \to R/(f_1, \dots, f_r)$ such that

   $$H_0\big(K_\bullet(R; \mathsf{f})\big) \simeq R/(\mathsf{f}).$$

Suggests: $K_\bullet$ are functions on derived enhancement of the classical space $Spec(R/(f))$.

Quasi–smoothness

A sequence $(f_1, \ldots, f_N)$ of elements of A determines a homomorphism $\mathbb{C}[T_1, \ldots, T_N] \to A$, given by $T_i \mapsto f_i, i = 1, \ldots, N$.

Quasi–smoothness

A sequence $(f_1, \ldots, f_N)$ of elements of A determines a homomorphism $\mathbb{C}[T_1, \ldots, T_N] \to A$, given by $T_i \mapsto f_i, i = 1, \ldots, N$.

Koszul complex: the derived tensor product,

$$K_A(\langle f \rangle) \xrightarrow{\simeq} A \otimes^{\mathbb{L}}_{\mathbb{C}[T_1,\ldots,T_n]} \mathbb{C}[T_1, \ldots, T_n]/\langle T_1, \ldots, T_n \rangle.$$

Quasi–smoothness

A sequence $(f_1, \ldots, f_N)$ of elements of A determines a homomorphism $\mathbb{C}[T_1, \ldots, T_N] \to A$, given by $T_i \mapsto f_i, i = 1, \ldots, N$.

Koszul complex: the derived tensor product,

$$K_A(\langle f \rangle) \xrightarrow{\simeq} A \otimes^{\mathbb{L}}_{\mathbb{C}[T_1,\ldots,T_n]} \mathbb{C}[T_1, \ldots, T_n]/\langle T_1, \ldots, T_n \rangle.$$

Remark

Derived geometrically, the Koszul complex can be seen as the dg-ring of functions on a certain derived sub-scheme of $\mathrm{Spec}(A)$.

Let $A_\bullet$ be a simplicial ring or cdga.

Quasi–smoothness

A sequence $(f_1, \ldots, f_N)$ of elements of A determines a homomorphism $\mathbb{C}[T_1, \ldots, T_N] \to A$, given by $T_i \mapsto f_i, i = 1, \ldots, N$.

Koszul complex: the derived tensor product,

$$K_A(\langle f \rangle) \xrightarrow{\simeq} A \otimes^{\mathbb{L}}_{\mathbb{C}[T_1,\ldots,T_n]} \mathbb{C}[T_1, \ldots, T_n]/\langle T_1, \ldots, T_n \rangle.$$

Remark

Derived geometrically, the Koszul complex can be seen as the dg-ring of functions on a certain derived sub-scheme of $\mathrm{Spec}(A)$.

Let $A_\bullet$ be a simplicial ring or cdga.
Fix $\langle f \rangle := (f_1, \ldots, f_n) \in A_\bullet$, i.e. a collection of points of the underlying space of $A_\bullet$, and consider the homotopy-pushout:

$$\begin{array}{ccc} \mathbb{C}[T_1, \ldots, T_n] & \xrightarrow{\mathsf{f}} & A_\bullet \\ \downarrow & & \downarrow \\ \mathbb{C}[T_1, \ldots, T_n]/\langle T \rangle & \longrightarrow & A_\bullet /\!/ \langle f \rangle. \end{array} \tag{2}$$

Fact. $\pi_0\big(A_\bullet /\!/ \langle f \rangle\big) \simeq \pi_0(A_\bullet)/\langle f \rangle$.

Quasi–smoothness II

A *closed immersion of derived schemes*: $i : \mathbf{X} \to \mathbf{Z}$ means the underlying classical map of schemes $X = t_0(\mathbf{X}) \to t_0(\mathbf{Z})$ is a closed immersion.

Quasi–smoothness II

A *closed immersion of derived schemes*: $i : \mathbf{X} \to \mathbf{Z}$ means the underlying classical map of schemes $X = t_0(\mathbf{X}) \to t_0(\mathbf{Z})$ is a closed immersion.

Definition

A quasi-smooth closed immersion is a morphism of the form $\mathbf{Spec}(A^\bullet//(f_1, \ldots, f_n)) \hookrightarrow \mathbf{Spec}(A^\bullet)$.

Quasi–smoothness II

A *closed immersion of derived schemes*: $i : \mathbf{X} \to \mathbf{Z}$ means the underlying classical map of schemes $X = t_0(\mathbf{X}) \to t_0(\mathbf{Z})$ is a closed immersion.

Definition

A quasi-smooth closed immersion is a morphism of the form $\mathbf{Spec}(A^\bullet//(f_1, \ldots, f_n)) \hookrightarrow \mathbf{Spec}(A^\bullet)$.

All quasi-smooth closed immersions are affine-locally of this form.

Quasi–smoothness II

A *closed immersion of derived schemes*: $i : \mathbf{X} \to \mathbf{Z}$ means the underlying classical map of schemes $X = t_0(\mathbf{X}) \to t_0(\mathbf{Z})$ is a closed immersion.

Definition

A quasi-smooth closed immersion is a morphism of the form $\mathbf{Spec}(A^\bullet//(f_1, \ldots, f_n)) \hookrightarrow \mathbf{Spec}(A^\bullet)$.

All quasi-smooth closed immersions are affine-locally of this form.

- A quasi-smooth map between classical schemes is the same as a local complete intersection map of schemes.
- A quasi-smooth closed immersion between classical schemes is the same as a (Koszul) regular immersion: $\mathrm{Spec}(A/I) \hookrightarrow \mathrm{Spec}(A)$ with A smooth and I an ideal in A generated by Koszul regular sequence $(a_1, \ldots, a_r) \in A^r$.

Quasi–smoothness II

A *closed immersion of derived schemes*: $i : \mathbf{X} \to \mathbf{Z}$ means the underlying classical map of schemes $X = t_0(\mathbf{X}) \to t_0(\mathbf{Z})$ is a closed immersion.

Definition

A quasi-smooth closed immersion is a morphism of the form $\mathbf{Spec}(A^\bullet//(f_1, \ldots, f_n)) \hookrightarrow \mathbf{Spec}(A^\bullet)$.

All quasi-smooth closed immersions are affine-locally of this form.

- A quasi-smooth map between classical schemes is the same as a local complete intersection map of schemes.
- A quasi-smooth closed immersion between classical schemes is the same as a (Koszul) regular immersion: $\mathrm{Spec}(A/I) \hookrightarrow \mathrm{Spec}(A)$ with A smooth and I an ideal in A generated by Koszul regular sequence $(a_1, \ldots, a_r) \in A^r$.

These notions allow us to study differential equations which are zero-loci of not-necessarily regular sequences of functions on (dg)-jet schemes.

Homotopical jet functors I

Fix $k \in \mathbb{N}$.
The k-jet functor of $A^\bullet \in \mathbf{cdga}_{\mathbb{C}}$:

Homotopical jet functors I

Fix $k \in \mathbb{N}$.
The k-jet functor of $A^\bullet \in \mathbf{cdga}_{\mathbb{C}}$:

$$LJ^k(A^\bullet) : \mathbf{cdga}_k \to \mathbf{Spc}$$

sending $R^\bullet$ to an ∞-groupoid $\mathrm{Maps}\big(A^\bullet, R^\bullet \otimes^L_{\mathbb{C}} \mathbb{C}[x]/(x^{n+1})\big)$.

Homotopical jet functors I

Fix $k \in \mathbb{N}$.
The k-jet functor of $A^\bullet \in \mathbf{cdga}_{\mathbb{C}}$:

$$LJ^k(A^\bullet) : \mathbf{cdga}_k \to \mathbf{Spc}$$

sending $R^\bullet$ to an ∞-groupoid $\mathrm{Maps}\big(A^\bullet, R^\bullet \otimes^L_{\mathbb{C}} \mathbb{C}[x]/(x^{n+1})\big)$.

Usual intuition: This gives $R^\bullet$-valued families of truncated Taylor expansions of elements of $A^\bullet$.
It is corepresented by an object $L\mathcal{J}^k(A^\bullet)$ in **cdga**.

Homotopical jet functors I

Fix $k \in \mathbb{N}$.
The k-jet functor of $A^\bullet \in \mathbf{cdga}_{\mathbb{C}}$:

$$LJ^k(A^\bullet) : \mathbf{cdga}_k \to \mathbf{Spc}$$

sending $R^\bullet$ to an ∞-groupoid $\mathrm{Maps}\big(A^\bullet, R^\bullet \otimes^L_{\mathbb{C}} \mathbb{C}[x]/(x^{n+1})\big)$.

Usual intuition: This gives $R^\bullet$-valued families of truncated Taylor expansions of elements of $A^\bullet$.
It is corepresented by an object $L\mathcal{J}^k(A^\bullet)$ in **cdga**.

Lemma (K-SY)

- LJ^k is a left-adjoint,
- If X is a smooth classical scheme of finite type so is $L\mathcal{J}^k(X)$.
- In particular, $\pi_0\big(L\mathcal{J}^k(A)\big) \simeq \mathcal{J}^k\big(\pi_0(A)\big)$.

$$L\mathcal{J}^k\big(\mathbb{C}[x_1, \ldots, x_d]\big) \simeq J^k\big(\mathbb{C}[x_1, \ldots, x_d]\big) = \mathbb{C}\big[x_i^{(\sigma)} || \sigma| \leq k\big]. \tag{3}$$

This is the *standard k-jet $\mathbb{C}$-scheme*, whose elements are differential polynomials.

Homotopical jet functors II

Lemma (K-SY)

For every $k \leq \ell \leq \infty$, there are canonical jet-projection maps $L\mathcal{J}^{\ell}(\mathsf{X}) \to L\mathcal{J}^{k}(\mathsf{X})$. Dually, we obtain $\pi^{*}_{\ell,k} : L\mathcal{J}^{k}(A_{\bullet}) \to L\mathcal{J}^{\ell}(A_{\bullet})$.

By general properties of cotangent sequences: we have an induced map

$$\mathbb{L}_{L\mathcal{J}^{k}(A_{\bullet})/\mathbb{C}} \otimes^{L}_{L\mathcal{J}^{k}(A_{\bullet})} \mathbb{L}_{L\mathcal{J}^{\ell}(A_{\bullet})/\mathbb{C}} \to \mathbb{L}_{L\mathcal{J}^{\ell}(A_{\bullet})/\mathbb{C}}.$$

Lemma (K-SY)

If $i : \mathbf{X} \to \mathbf{Z}$ is a quasi–smooth closed immersion, then for any $k \in \mathbb{N}$,

$$L\mathcal{J}^{k}(i) : L\mathcal{J}^{k}(\mathbf{X}) \to L\mathcal{J}^{k}(\mathbf{Z}),$$

is quasi–smooth closed immersion.

Useful result

Theorem (K-SY)

Let $A^\bullet//(f_1,\dots,f_r)$ be a Koszul cdga of functions. Then,

$$LJ^k\big(A^\bullet//(f_1,\dots,f_r)\big) \xrightarrow{\simeq} LJ^k(A^\bullet)//(f_1^{(\sigma)},\dots,f_r^{(\sigma)}),$$

is an equivalence.

Idea of proof

Since $L\mathcal{J}^k(X) \simeq J^k(X)$ for a classical $\mathbb{C}$-scheme X we have that

$$L\mathcal{J}^k\big(\mathbb{C}[T_1,\dots,T_n]\big) \simeq J^k\big(\mathbb{C}[T_1,\dots,T_n]\big) \simeq \mathbb{C}[T,T^{(1)},\dots,T^{(k)}],$$

and thus

$$L\mathcal{J}^k\big((2)\big) \xrightarrow{\simeq} \begin{array}{ccc} \mathbb{C}[T,T^{(1)},\dots,T^{(k)}] & \xrightarrow{L\mathcal{J}^k(\mathsf{f})} & L\mathcal{J}^k(A_\bullet) \\ \downarrow & & \downarrow \\ J^k(\mathbb{C}) & \longrightarrow & L\mathcal{J}^k(A_\bullet /\!/ \langle f \rangle). \end{array}$$

Via abstract nonsense: this diagram is also a homotopy push-out!

$$L\mathcal{J}^k(\mathsf{f}) : \mathbb{C}[T,T_j^{(i)}] \to L\mathcal{J}^k(A_\bullet),$$

is given by $T_i \mapsto f_i$ and $T_i^{(j)} \mapsto f_i^{(j)}$, for $, i = 1,\dots,n, j = 1,\dots,k$.

Example

Let $A = \mathbb{C}[x]/(x^2)$ be the classical non-reduced point. Its first jet algebra is

$$J^1(A) = \mathbb{C}[x, x^{(1)}]/(x^2,\, 2xx^{(1)}).$$

The derived jet is the Koszul cdga

$$LJ^1(A) = \mathbb{C}[x, x^{(1)}]//(x^2,\, 2xx^{(1)}) = K_\bullet\big(x^2,\, 2xx^{(1)}\big).$$

Example

Let $A = \mathbb{C}[x]/(x^2)$ be the classical non-reduced point. Its first jet algebra is

$$J^1(A) = \mathbb{C}[x, x^{(1)}]/(x^2,\, 2xx^{(1)}).$$

The derived jet is the Koszul cdga

$$LJ^1(A) = \mathbb{C}[x, x^{(1)}]//(x^2,\, 2xx^{(1)}) = K_\bullet\big(x^2,\, 2xx^{(1)}\big).$$

The sequence $(x^2, 2xx^{(1)})$ is not regular: in $R/(x^2)$ one has $x \cdot (2xx^{(1)}) = 0$.

Example

Let $A = \mathbb{C}[x]/(x^2)$ be the classical non-reduced point. Its first jet algebra is

$$J^1(A) = \mathbb{C}[x, x^{(1)}]/(x^2,\, 2xx^{(1)}).$$

The derived jet is the Koszul cdga

$$LJ^1(A) = \mathbb{C}[x, x^{(1)}]//(x^2,\, 2xx^{(1)}) = K_\bullet\big(x^2,\, 2xx^{(1)}\big).$$

The sequence $(x^2, 2xx^{(1)})$ is not regular: in $R/(x^2)$ one has $x \cdot (2xx^{(1)}) = 0$.
Koszul homology:

$$\pi_0(LJ^1(A)) = J^1(A), \qquad \pi_1(LJ^1(A)) = H_1(K_\bullet) \simeq \mathbb{C}[x^{(1)}].$$

Example

Let $A = \mathbb{C}[x]/(x^2)$ be the classical non-reduced point. Its first jet algebra is

$$J^1(A) = \mathbb{C}[x, x^{(1)}]/(x^2,\, 2xx^{(1)}).$$

The derived jet is the Koszul cdga

$$LJ^1(A) = \mathbb{C}[x, x^{(1)}]//(x^2,\, 2xx^{(1)}) = K_\bullet\big(x^2,\, 2xx^{(1)}\big).$$

The sequence $(x^2, 2xx^{(1)})$ is not regular: in $R/(x^2)$ one has $x \cdot (2xx^{(1)}) = 0$.
Koszul homology:

$$\pi_0(LJ^1(A)) = J^1(A), \qquad \pi_1(LJ^1(A)) = H_1(K_\bullet) \simeq \mathbb{C}[x^{(1)}].$$

Hence $LJ^1(A)$ is not classical.

Outline

1 Introduction

2 Reminders: homological algebra

3 Quasi-smooth PDEs

4 Prolongation and formal integrability

5 Singularity categories

6 Summary

Systems of NLPDEs

A system of N differential equations for a vector-valued function $u = (u^1, \dots, u^m)$ of variables $x = (x_1, \dots, x_n)$ is:

$$Z_k := \begin{cases} F_1(x, u, \frac{\partial u}{\partial x}, \dots, \frac{\partial^k u}{\partial x^k}, \dots, \frac{\partial^{|\sigma|} u}{\partial x_\sigma}) = 0, \\ \vdots \\ F_N(x, u, \frac{\partial u}{\partial x}, \dots, \frac{\partial^k u}{\partial x^k}, \dots, \frac{\partial^{|\sigma|} u}{\partial x_\sigma}) = 0. \end{cases} \tag{4}$$

Derivatives $u^\alpha_\sigma := \frac{\partial^{|\sigma|} u^\alpha}{\partial x_\sigma}$, for $\alpha = 1, \dots, m$: multi-index notation
$\sigma = (\sigma_1, \dots \sigma_n) \in \mathbb{N}^n$ with *length* $|\sigma| := \sigma_1 + \cdots + \sigma_n \leq k$ (*order*).
Simplifying assumption: The orders of each $F_A, A = 1, \dots, N$ are the same.
• Z_k is the zero locus of (4) in k-jet space $J^k_X E \ni (x^i, u^\alpha, u^\alpha_\sigma)$ i.e. $dim = n + m\binom{n+k}{k}$

$$\mathcal{O}_{Z_k} = \mathcal{O}_{J^k_X E}/I_k, \text{with } I_k = span\{(4)\}.$$

Remark 1. The structure sheaf $\mathcal{O}_{J^k_X(E)}$ is the k'th-jet sheaf $J^k(E^\vee)$.

In local charts

Consider $\iota_k : Z_k = \{F(x, u, u_{[\sigma]}) = 0\} \hookrightarrow J^k_X(E)$:

$$T_\theta J^k_X E = \text{span}\{v := \dot{x}^i \frac{\partial}{\partial x_i} + \dot{u}^\alpha_\sigma \frac{\partial}{\partial u^\alpha_\sigma}\}, \quad \text{and } T_\theta Z_k = \{v \in T_\theta J^k_X E | TF(v) = 0\}.$$

A linear system for the coefficients of v:

$$\sum_{i=1}^{n} \frac{\partial F}{\partial x^i}(\theta) \cdot x^i + \sum_{1 \leq \alpha \leq m, 1 \leq \sigma \leq \text{ord}(F)} \frac{\partial F}{\partial u^\alpha_\sigma}(\theta) \dot{u}^\alpha_\sigma = 0.$$

The geometric symbol $\mathcal{N}_k$, of Z_k is the family of vector spaces:

$$(\mathcal{N}_k)_z := T_z Z_k \cap T_{J^k_X E / J^{k-1}_X E, z} \simeq T_{\pi^k_{k-1}|_{Z_k}}, \text{ (relative tangents)}.$$

Coordinates

For $Z_k = \{F = 0\}$, then $\mathcal{N}_k = \sum_{1 \leq \alpha \leq m, |\sigma| = \text{ord}(F)} \frac{\partial F}{\partial u^\alpha_\sigma}(\theta) \dot{u}^\alpha_\sigma = 0$.

Symbols and Characteristic Modules, Algebro-geometrically

Consider relative differentials of dg of holomorphic functions $g \in \mathcal{O}_{J^\ell_X E}$ i.e.
$d(g) \in \Omega^1_{J^\ell_X E/J^{\ell-1}_X E} \simeq \mathcal{O}_{J^\ell_X E}[\xi_1, \ldots, \xi_n]_k$, with $(x;\xi) \in T^*X$.
Put $\mathcal{O}_{Z_r} = \mathcal{O}_{J^r_X E}/I_r$.
Order r symbol: $\mathcal{N}_r \subset \oplus_\alpha \mathcal{O}_{Z_r}[\xi_1, \ldots, \xi_n]_r d(u^\alpha)$, generated by classes dF mod I_r of $F \in I_r$.
Symbol: the associated graded module generated by $\mathcal{N}_r$.
Characteristic module: graded quotient $\mathcal{M} := \bigoplus_r \mathcal{O}_{Z_r}[\xi_1, \ldots, \xi_n]/\mathcal{N}_r$.
$*$ Alternatively, look at conormal I_r/I_r^2 of Z_r, as sub-sheaf of $\Omega^1_{J^r_X E} \otimes \mathcal{O}_{Z_r}$. Then,

$$\mathcal{M}_r \simeq \Omega^1_{Z_r/Z_{r-1}}.$$

From $Z^\infty \Rightarrow F^r Z := Z_r \subset J^r_X(E)$,

$$\mathcal{O}_{F^r Z} \otimes_{\mathcal{O}_{F^{r-1} Z}} \Omega^1_{F^{r-1} Z/X} \to \Omega^1_{F^r Z/X} \to \Omega^1_{F^r Z/F^{r-1} Z} \to 0. \tag{5}$$

Applying $\mathcal{O}_{Z^\infty} \otimes_{\mathcal{O}(F^r Z)} (-)$:

$$\mathcal{O}_{Z^\infty} \otimes_{\mathcal{O}_{F^{r-1} Z}} \Omega^1_{F^{r-1} Z/X} \to \mathcal{O}_{Z^\infty} \otimes_{\mathcal{O}_{F^r Z}} \Omega^1_{F^r Z/X} \to \mathcal{O}_{Z^\infty} \otimes_{\mathcal{O}_{F^r Z}} \Omega^1_{F^r Z/F^{r-1} Z} \to 0.$$

Quasi-smooth PDEs

Let $E \to X$ be a vector bundle and $J^k_X(E)$ the k-jet bundle.

Definition

A quasi-smooth algebraic PDE of order $\leq k$ is a quasi-smooth closed immersion

$$\iota_{\mathcal{E}} : \mathcal{E} \hookrightarrow LJ^k_X(E).$$

In particular, $t_0(\mathcal{E}) \hookrightarrow J^k_X(E)$ is PDE in the usual sense, with a closed embedding $t_0(\mathcal{E}) \hookrightarrow \mathcal{E}$. The same definition works for analytic PDEs.

- If $\mathcal{E}$ is quasi-smooth algebraic with analytification $\mathcal{E}^{\mathrm{an}}$, then $\mathcal{E}^{\mathrm{an}}$ is quasi-smooth analytic.

(Zariski-)Local presentations

Proposition (K-SY)

Let $\iota : \mathcal{E} \hookrightarrow L\mathcal{J}^k(\mathbf{X})$ be a quasi–smooth closed-immersion where $\mathbf{X} = \mathsf{Spec}(A_\bullet)$ for a simplicial $\mathbb{C}$-algebra. Then, we can always write

$$\mathcal{E} \simeq \mathbf{Spec}_{\mathbb{C}}\big(L\mathcal{J}^k(A_\bullet)//\langle G\rangle\big), \qquad G := \big(g_1, \ldots, g_c\big),$$

for a finite sequence of elements in $L\mathcal{J}^k(A_\bullet)$, such that

$$\mathcal{O}^\bullet_{\mathcal{E}} \xrightarrow{\sim} L\mathcal{J}^k(A_\bullet) \otimes^{\mathbb{L}}_{\mathbb{C}[T_1,\ldots,T_c]} \mathbb{C},$$

via the map $T_A \mapsto g_A, A = 1, \ldots, c$.

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\ \iota\ } & L\mathcal{J}^k(\mathbf{X}) \\ \downarrow & & \downarrow{\scriptstyle g} \\ \{0\} & \longrightarrow & \mathbb{A}^c_{\mathbb{C}} \end{array}$$

is a homotopy Cartesian diagram.

'Higher' information

$$\pi_0(\mathcal{O}^\bullet_{\mathcal{E}}) \simeq \pi_0\big(\mathcal{O}_{L\mathcal{J}^k(\mathbf{X})}\big)/\big(g_1,\dots,g_c\big).$$

In general $\pi_i(\mathcal{O}^\bullet_{\mathcal{E}})$ are higher cohomologies of the Koszul sequence for the functions $(g_1,\dots,g_c)$.
Namely,

$$\pi_i(\mathcal{O}^\bullet_{\mathcal{E}}) = H^{-i}(K^\bullet_{L\mathcal{J}^k(A_\bullet)}(G)\big).$$

Thus, derived PDEs $\mathcal{E}$ carry higher cohomological information in the $\pi_0(\mathcal{O}^\bullet_{\mathcal{E}})$-modules $\pi_i(\mathcal{O}^\bullet_{\mathcal{E}})$.

Lemma

If $A = \mathbb{C}[x_1,\dots,x_d]/\langle f\rangle$, is a complete intersection algebra, so that $f_1,\dots,f_c$ is a regular sequence in $\mathbb{C}[x_1,\dots,x_d]$, then

$$\pi_*\big(L\mathcal{J}^k(A)\big) \simeq H^*\big(K^\bullet(f_1,\dots,f_c,f_A^{(1)},\dots,f_A^{(k)})\big),$$

where $f_A^{(j)} := (f_1^{(j)}, f_2^{(j)},\dots,f_c^{(j)})$, where $j = 1,\dots,k$.

Singular schemes of quasi-smooth PDEs

Definition

The singularity scheme of $\mathcal{E}$ is

$$\mathbf{Sing}_X(\mathcal{E}) := \mathrm{Spec}_{\mathcal{O}_{t_0(\mathcal{E})}}\left(\mathrm{Sym}^*\left(H^1(\mathbb{T}_{\mathcal{E}})\right)\right).$$

Via $\pi : \mathsf{Sing}(\mathcal{E}) \to \mathcal{E}$ the fibers $\pi^{-1}(x)$ over x in $\mathcal{E}$ are the vector spaces $H^{-1}(\mathbb{L}_{\mathcal{E},x})$ specifying how singular $\mathcal{E}$ is at x.

Fact

If $\mathcal{E} \simeq \mathcal{E}$ is classical, then $\mathbf{Sing}_X(\mathcal{E}) = \varnothing$.

Support condition: a conical subset $\Lambda \subseteq \mathbf{Sing}(\mathcal{E})$.
Complexes of coherent sheaves F on $\mathcal{E}$ with support in Λ:

$$\mathrm{Coh}_\Lambda(\mathcal{E}).$$

Singularity scheme of a quasi-smooth PDE

Let M be a smooth manifold, $E, V \to M$ vector bundles, and

$$F : J^k_M(E) \longrightarrow V$$

a system of $r = \mathrm{rk}(V)$ PDEs. Write

$$Z_0 := F^{-1}(0), \qquad \mathcal{E} := \mathbb{R}Z(F).$$

The derived zero locus is quasi-smooth, with cotangent complex

$$\mathbb{L}_{\mathcal{E}} \simeq \left[\pi^* V^\vee \xrightarrow{dF^\vee} \pi^* \Omega_{J^k_M(E)}\right], \qquad \text{degrees } [-1, 0].$$

Since

$$H^1(\mathbb{T}_{\mathcal{E}}) = \mathrm{coker}\left(T_{J^k_M(E)}|_{Z_0} \xrightarrow{dF} \pi^* V|_{Z_0}\right),$$

we obtain the fibrewise description

$$\mathbf{Sing}(\mathcal{E})_\theta = \ker\left(dF^\vee_\theta : V^\vee_{\pi_k(\theta)} \longrightarrow T^*_\theta J^k_M(E)\right).$$

Interpretation

The fibre consists of linear relations among the differentials of the defining PDEs at a solution θ. It is measuring the failure of $dF_\theta : T_\theta J^k_M(E)|_{Z_0} \to V_{\pi_k(\theta)}$ to be surjective.

Algebraic versus analytic singularities

Analytification preserves tor-amplitude, hence flatness.

Proposition

Let $\mathcal{E}$ be an algebraic quasi-smooth PDE of order $\leq k$ with quasi-smooth analytification $\mathcal{E}^{\mathrm{an}}$. Then

$$\mathbf{Sing}_X(\mathcal{E})^{\mathrm{an}} \simeq \mathbf{Sing}_{X^{\mathrm{an}}}(\mathcal{E}^{\mathrm{an}}).$$

Idea: Property $(\mathbb{L}_Z)^{\mathrm{an}} \to \mathbb{L}^{\mathrm{an}}_{Z^{\mathrm{an}}}$ an equiv. in $\mathsf{Perf}(Z^{\mathrm{an}})$ implies:

- $\left(\mathrm{Sym}^*_{\mathcal{O}_Z}(\mathbb{T}_Z)\right)^{\mathrm{an}} \xrightarrow{\simeq} \mathrm{Sym}^*_{\mathcal{O}_{Z^{\mathrm{an}}}}(\mathbb{T}^{\mathrm{an}}_{Z^{\mathrm{an}}})$.
- $\left(\mathrm{Sym}^*_{\mathcal{O}_{cl\,Z}}(H^1(\mathbb{L}_Z))\right)^{\mathrm{an}} \xrightarrow{\simeq} \mathrm{Sym}^*_{\mathcal{O}_{cl\,Z^{\mathrm{an}}}}(H^1(\mathbb{L}^{\mathrm{an}}_{Z^{\mathrm{an}}}))$.

So we may pass freely between the algebraic and analytic pictures when computing singular schemes.

Outline

1 Introduction

2 Reminders: homological algebra

3 Quasi-smooth PDEs

4 Prolongation and formal integrability

5 Singularity categories

6 Summary

Recollections: prolongation/projection

Forgetting derivatives gives a jet-tower:

$$\cdots \to J^{k+\ell}_X E \xrightarrow{\pi^{k+\ell}_{k+\ell-1}} J^{k+\ell-1}_X E \to \cdots \to J^k_X E \to J^{k-1}_X E \to \cdots \to E \to X.$$

Considering $Z_k = \{F(x, u(x), \partial^\sigma_x u(x) = 0\} \hookrightarrow J^k_X E$, two main operations:

- Projection: The r-fold projection, $\rho_r(Z_k) := \pi^k_{k-r}(Z_k) \subseteq J^{k-r}_X E$.
- Prolongation: $\mathrm{pr}_1(Z_k) := J^1_X(Z_k) \times_{J^1_X(J^k_X E)} J^{k+1}_X E$,

$$\text{Put } p^k := \pi^l|_{Z_k} : Z_k \to X \Rightarrow Z_{k+r} := \mathrm{pr}_r(Z_k) = J^r_X(p_k) \cap J^{q+r}_X E \subseteq J^{q+r}_X.$$

Remark 1. Neither $\rho_r(Z_{k-r})$ nor Z_{k+r} are smooth in general (not relevant for us).
Remark 2. More simply $\rho_k Z_k \subseteq Z_{k-1}$ and $\mathrm{pr}_1(Z_k) \supseteq Z_{k+1}$(not equality!).
Remark 3. They are not inverse operations: take $Z_k \subset J^k_X E$ prolong $Z_{k+r} \subseteq J^{k+r}_X$ and project back to $J^k_X E$, we only get $\rho_r Z_k \subseteq Z_k$ (not equality!).

Homotopy-prolongation

The homotopy-prolongation of $\iota : \mathcal{E} \hookrightarrow J^k_X(E)$ is the homotopy Cartesian diagram

$$\begin{array}{ccc} \mathbb{R}\mathrm{Pr}_1(\mathcal{E}) & \longrightarrow & LJ^{k+1}(E) \\ \downarrow & & \downarrow \\ LJ^1(\mathcal{E}) & \xrightarrow{LJ^1(\iota)} & LJ^1\big(LJ^k(E)\big), \end{array}$$

using natural transformation $L\mathcal{J}^{k+1}(-) \to L\mathcal{J}^1\big(L\mathcal{J}^k(-)\big)$.

Proposition

If $\mathcal{E} \hookrightarrow J^k_X(E)$ is a quasi-smooth PDE of order $\leq k$, then

$$\mathbb{R}\mathrm{Pr}_1(\mathcal{E}) \hookrightarrow J^{k+1}(E)$$

is a quasi-smooth closed immersion of order $\leq k+1$. Moreover,

$$t_0(\mathbb{R}\mathrm{Pr}_1(\mathcal{E})) \simeq LJ^1 t_0(\mathcal{E}) \times_{J^1 J^k_X(E)} J^{k+1}_X(E).$$

$\mathbb{T}_{\mathbb{R}\mathrm{Pr}_1(\mathcal{E})} \simeq \mathrm{cone}\big(\mathbb{T}_{\mathcal{E}} \oplus \mathbb{T}_{LJ^1(LJ^k E)} \to \mathbb{T}_{J^k E}\big)[-1]$.

Ex. $\mathbb{R}\mathbf{Z}\big(g_1, \ldots, g_c, D_1 g_1, \ldots, D_n g_c\big) \subseteq L\mathcal{J}^{k+1}_X(A_\bullet)$ where D_i are the i-th total derivatives.

Derived 1-formal integrability

Definition (Tentative, (after Laumon, Moret-Bailly' 2000))

Let $\iota : \mathcal{E} \hookrightarrow J^k_X(E)$ be a quasi-smooth PDE of order $\leq k$. It is 1-formally integrable if it is an effective epimorphism (surjection on π_0).

Roughly, $t_0(p) : t_0(\mathbb{R}\mathrm{Pr}_1(\mathcal{E})) \to t_0(\mathcal{E})$ is a surjective submersion of analytic manifolds.
For every classical point $\widetilde{\theta}$ of $\mathbb{R}\mathrm{Pr}_1(\mathcal{E})$ with $p(\widetilde{\theta}) = \theta$:

1. $dt_0(p)_{\widetilde{\theta}} : T_{\widetilde{\theta}} t_0(\mathbb{R}\mathrm{Pr}_1(\mathcal{E})) \to T_\theta t_0(\mathcal{E})$ is surjective;

- H^0-surjectivity: no classical infinitesimal deformation of a solution jet is lost when passing to the prolongation.

For $\ell \geq 0$, define the ℓ-th homotopy-prolongation by

$$\mathbb{R}\mathrm{Pr}_\ell(\mathcal{E}) := \mathbb{R}\mathrm{Pr}_1\big(\mathbb{R}\mathrm{Pr}_{\ell-1}(\mathcal{E})\big).$$

Derived 1-formal integrability

Definition (Tentative, (after Laumon, Moret-Bailly' 2000))

Let $\iota : \mathcal{E} \hookrightarrow J^k_X(E)$ be a quasi-smooth PDE of order $\leq k$. It is 1-formally integrable if it is an effective epimorphism (surjection on π_0).

Roughly, $t_0(p) : t_0(\mathbb{R}\mathrm{Pr}_1(\mathcal{E})) \to t_0(\mathcal{E})$ is a surjective submersion of analytic manifolds.
For every classical point $\widetilde{\theta}$ of $\mathbb{R}\mathrm{Pr}_1(\mathcal{E})$ with $p(\widetilde{\theta}) = \theta$:

1. $dt_0(p)_{\widetilde{\theta}} : T_{\widetilde{\theta}} t_0(\mathbb{R}\mathrm{Pr}_1(\mathcal{E})) \to T_\theta t_0(\mathcal{E})$ is surjective;
   - H^0-surjectivity: no classical infinitesimal deformation of a solution jet is lost when passing to the prolongation.

For $\ell \geq 0$, define the ℓ-th homotopy-prolongation by

$$\mathbb{R}\mathrm{Pr}_\ell(\mathcal{E}) := \mathbb{R}\mathrm{Pr}_1\big(\mathbb{R}\mathrm{Pr}_{\ell-1}(\mathcal{E})\big).$$

Questions

- What are the obstructions to homotopy-prolongation?
- Is there a quasi-smooth Cartan–Kähler theorem?

Recollections: the codifferential

Let $f : X \to Y$ be a morphism of complex analytic manifolds.
Denote by $\pi : T^*X \to X$ the cotangent bundle with coordinates $(x;\xi)$.
Recall the 'microlocal' correspondence:

$$T^*X \xleftarrow{f_d} X \times_Y T^*Y \xrightarrow{f_\pi} T^*Y. \tag{6}$$

Explicitly, $f_d\big(x;(f(x);\xi)\big) = (x; f^*\xi)$ and $f_\pi\big(x;(f(x);\xi)\big) = \big(f(x);\xi\big)$.
Arinkin–Gaitsgory gave an 'H^1-version', in terms of singularity schemes.

Relations and the singular codifferential

There is a canonical map

$$p : \mathbb{R}\mathrm{Pr}_1(\mathcal{E}) \to \mathcal{E}.$$

We have a correspondence

$$\mathsf{Sing}(\mathbb{R}\mathrm{Pr}_1(\mathcal{E})) \xleftarrow{p_d} X \times_{\mathcal{E}} \mathsf{Sing}(\mathcal{E}) \xrightarrow{p_\pi} \mathsf{Sing}(\mathcal{E}). \tag{7}$$

This may be used to relate support conditions.
We have the fundamental triangle

$$\mathbb{L}_{\mathcal{E}/X} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{E}}} \mathcal{O}_{\mathbb{R}\mathrm{Pr}_1(\mathcal{E})} \to \mathbb{L}_{\mathbb{R}\mathrm{Pr}_1(\mathcal{E})/X} \to \mathbb{L}_{\mathbb{R}\mathrm{Pr}_1(\mathcal{E})/\mathcal{E}} \to \big(\mathbb{L}_{\mathcal{E}/X} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{E}}} \mathcal{O}_{\mathbb{R}\mathrm{Pr}_1(\mathcal{E})}\big)[1].$$

The singular codifferential is

$$\mathrm{Sing}(p) : \mathbf{Sing}(\mathcal{E}) \times_{\mathcal{E}} \mathbb{R}\mathrm{Pr}_1(\mathcal{E}) \to \mathbf{Sing}\big(\mathbb{R}\mathrm{Pr}_1(\mathcal{E})\big).$$

Lemma

The singular codifferential $\mathrm{Sing}(p)$ is a closed embedding.

Idea of the proof

For each point θ of $\mathbb{R}\mathrm{Pr}_1(\mathcal{E})$ the map

$$(dp_\theta)^* : H^{-1}\left(\mathbb{L}_{\mathcal{E},p(\theta)}\right) \to H^{-1}\left(\mathbb{L}_{\mathbb{R}\mathrm{Pr}_1(\mathcal{E}),\theta}\right)$$

is injective. This follows from the cotangent fiber sequence above and the quasi-smoothness of $\mathcal{E}$.
Dually, this means that

$$H^1(\mathbb{T}_{\mathbb{R}\mathrm{Pr}_1(\mathcal{E})}) \to H^1(p^*\mathbb{T}_{\mathcal{E}})$$

is surjective.

Outline

1 Introduction

2 Reminders: homological algebra

3 Quasi-smooth PDEs

4 Prolongation and formal integrability

5 Singularity categories

6 Summary

Ind-coherent sheaves and support

For a quasi-smooth space $\mathbf{X}$, set

$$\mathrm{IndCoh}(\mathbf{X}) := \mathrm{Ind}\big(\mathrm{Coh}(\mathbf{X})\big).$$

If $\mathbf{X} \simeq X$ is smooth, then

$$\mathrm{IndCoh}(X) \simeq \mathrm{QCoh}(X).$$

For a closed $\mathbb{C}^\times$-conic subset $\Lambda \subseteq \mathbf{Sing}(\mathbf{X})$, define

$$\mathrm{IndCoh}_\Lambda(\mathbf{X}) := \big\{F \in \mathrm{IndCoh}(\mathbf{X}) : \mathrm{SingSupp}(F) \subseteq \Lambda\big\}.$$

Singularity category of a PDE

Let $\mathcal{E}$ be a quasi-smooth PDE of order $\leq k$. There are well-defined $\mathrm{IndCoh}(\mathcal{E})$ and $\mathbf{Sing}(\mathcal{E})$.

Definition

Fix a conical subset $\Lambda \subseteq \mathbf{Sing}(\mathcal{E})$. Let $\mathcal{E}^{\mathrm{an}}$ be the analytification. The singularity category of the differential equation is

$$\mathcal{S}\mathrm{ing}_{\Lambda}(\mathcal{E}) := \mathrm{IndCoh}_{\Lambda}(\mathcal{E}^{\mathrm{an}}).$$

Singularity category of a PDE

Let $\mathcal{E}$ be a quasi-smooth PDE of order $\leq k$. There are well-defined $\mathrm{IndCoh}(\mathcal{E})$ and $\mathbf{Sing}(\mathcal{E})$.

Definition

Fix a conical subset $\Lambda \subseteq \mathbf{Sing}(\mathcal{E})$. Let $\mathcal{E}^{\mathrm{an}}$ be the analytification. The singularity category of the differential equation is

$$\mathcal{S}\mathrm{ing}_{\Lambda}(\mathcal{E}) := \mathrm{IndCoh}_{\Lambda}(\mathcal{E}^{\mathrm{an}}).$$

* In reality, we form a quotient by QCoh, but for our purposes this is enough!

There is a canonical functor

$$p^* : \mathcal{S}\mathrm{ing}_{\Lambda}(\mathcal{E}) \to \mathcal{S}\mathrm{ing}_{\Lambda^{(1)}}\left(\mathbb{R}\mathrm{Pr}_1(\mathcal{E})\right).$$

Outline

1 Introduction

2 Reminders: homological algebra

3 Quasi-smooth PDEs

4 Prolongation and formal integrability

5 Singularity categories

6 Summary

Two notions of characteristic directions

Two different (but related) geometric constructions.

Classical PDE theory For a PDE $\mathcal{E} \subset J^k(E)$, linearisation and the principal symbol determine its characteristic variety:

$$\mathrm{Char}(\ell_{\mathcal{E}}) \subseteq T^*X.$$

It describes characteristic covectors governing the propagation of singularities of solutions of the linearised equation.

Singularity theory of quasi-smooth spaces
For a quasi-smooth derived equation $\mathcal{E}$ (refining the usual zero-locus) we had $\mathbf{Sing}(\mathcal{E})$.
We can study a varying family of support conditions $\{\Lambda \subseteq \mathbf{Sing}(\mathcal{E})\}$ which describe the directions in which coherent objects fail to be perfect.

Question

Can the singularity geometry of a PDE be related to its classical characteristic geometry?
Can prolongation induce a natural transformation between these two types of characteristic data?

Extracting invariants from singularity categories

Compute categorical invariants.
One useful invariant is periodic cyclic homology:

$$HP_\bullet\big(\mathrm{IndCoh}_\Lambda(Z)\big).$$

Microlocal Feigin–Tsygan principle (Schefers). In the appropriate quasi-smooth setting,

$$\boxed{HP_\bullet\big(\mathrm{IndCoh}_\Lambda(Z)\big) \simeq H_\Lambda(Z),}$$

microlocal homology.

Work in progress

Compute $HP_\bullet\big(\mathrm{IndCoh}_\Lambda(\mathcal{E}^{\mathrm{an}})\big)$ and interpret the homology theory in terms of horizontal cohomologies of $t_0(\mathcal{E})$.

All done

Thank you for your attention!