On structures of the first order linear differential operators

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The problem

- Given differential operators $\Delta, \Delta' \in \mathbf{Diff}_1(\pi, \pi)$, acting in sections of a vector bundle $\pi : E(\pi) \to M$ over manifold M. Here $\dim \pi \geq 2$, $\dim M \geq 2$.

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- The problem: is there a bundle automorphism $A \in \mathbf{Aut}(\pi)$ such that

$$A_* (\Delta) = \Delta'?$$

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- Natural coordinates, models and equivalence

Artin conjecture & Procesi theorem

- Consider set $\mathfrak{A}$ of n-tuples of operators $A = \{A_1, ..., A_n\} \in \mathrm{End}\,(E)$, $\dim E = m$, equipped with the natural $\mathrm{GL}\,(E)$ $-$action: $c\,(A) = \{c\,(A_1)\,, ..., c\,(A_n)\}$, where $c\,(A_i) = c \circ A_i \circ c^{-1}$.

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- Artin conjecture & Procesi theorem: The algebra of polynomial invariants of the above action is generated by polynomials (we'll call them AP-invariants)

$$\mathcal{P}_I(A) = \mathrm{Tr}(A_{i_1} \cdots A_{i_k}),$$

for all multi indices $I = (i_1, .., i_k)$ of the length $\leq 2^m - 1$.

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for all multi indices $I = (i_1, .., i_k)$ of the length $\leq 2^m - 1$.
- Rosenlicht Theorem $\Rightarrow$ The field of rational $\mathrm{GL}(E)-$ invariants has transcendence degree $\nu = (n-1)m^2 + 1$ and therefore generated by ν algebraically independent AP-invariants.

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- We call $\mathrm{End}(E) \otimes T$ -*symbol space* and its elements -*symbols*.

Symbols for coordinate lovers

- Operators

$$\Delta = \sum_{i=1}^{n} A_i\left(x\right)\partial_i + A_0\left(x\right),$$

where $A_i\left(x\right)$ are $m \times m$ matrices, $x = \left(x_1, ..., x_n\right)$, $\partial_i = \partial/\partial x_i$.

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- The value of the symbol at covector $\theta = (p_1, .., p_n)$ (or $\theta = p_1 dx_1 + \cdots p_n dx_n$) is the matrix

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$$\sigma_\theta = \sum_{i=1}^{n} p_i A_i(x) : E \to E.$$
- Remark: $\{p_i = \partial_i\}$ are coordinates in T^*.

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- Consider linear functions $A_{\sigma,I} : (T^*)^{\otimes k} \to \mathbb{R}$, as follows

$$A_{\sigma,I}\,(\theta_1 \otimes \cdots \otimes \theta_k) = \mathrm{Tr}\left(\sigma_{\theta_{i_1}} \cdots \sigma_{\theta_{i_k}}\right),$$

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- Tensors $A_{\sigma,I} \in \left((T^*)^{\otimes k}\right)^* = T^{\otimes k}$ are G-invariants. We call them *Artin-Procesi* or *AP- tensors*.

First AP-tensors

- $\mathbf{k} = \mathbf{1}$. We have G- invariant vector $\chi_\sigma \in T$, where
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- $\mathbf{k} = \mathbf{2}$. We have G- invariant symmetric 2-vector $g_\sigma \in S^2 T$, where
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$$g_\sigma\left(\theta_1, \theta_2\right) = \mathrm{Tr}\left(\sigma_{\theta_1}\sigma_{\theta_2}\right).$$
- $\mathbf{k}=\mathbf{3}$. We have two G- invariant tensors $h_i \in T^{\otimes 3}$:
$$\begin{aligned} h_{1,\sigma}\left(\theta_1, \theta_2, \theta_3\right) &= \mathrm{Tr}\left(\sigma_{\theta_1}\sigma_{\theta_2}\sigma_{\theta_3}\right), \\ h_{2,\sigma}\left(\theta_1, \theta_2, \theta_3\right) &= \mathrm{Tr}\left(\sigma_{\theta_2}\sigma_{\theta_1}\sigma_{\theta_3}\right), \end{aligned}$$
or equivalently two tensors
$$\begin{aligned} h_{\sigma,s} &= \frac{1}{2}\left(h_{\sigma,1} + h_{\sigma,2}\right) \in S^2 T \otimes T, \\ h_{\sigma,a} &= \frac{1}{2}\left(h_{\sigma,1} - h_{\sigma,2}\right) \in \Lambda^2 T \otimes T. \end{aligned}$$

Example: The Euler system (R, 27.04.20)

- The system:

$$\begin{aligned} \rho_t + (\rho u)_x &= 0, \\ u_t + u\ u_x + a(\rho)\rho_x &= 0, \\ p &= p(\rho), a = \rho^{-1} p'. \end{aligned}$$

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- The symbol of the linearization on a solution $\rho = \rho_0, u = u_0$ equals

$$\sigma_\theta = p \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + q \begin{bmatrix} u_0 & \rho_0 \\ a(\rho_0) & u_0 \end{bmatrix},$$

where $\theta = p \ dt + q \ dx. \left(p \Leftrightarrow \partial_t, p \Leftrightarrow \partial_x\right).$

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- $$g_\sigma = \partial_t^2 + 2u\partial_t\partial_x + \left(a\rho + u^2\right)\partial_x^2,$$
and
$$g_\sigma^{-1} = \frac{\left(\rho + u^2\right)\ dt^2 - 2u\ dtdx + dx^2}{2a\rho}.$$

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- The symmetrized 3rd tensor

$$h = \chi_\sigma g_\sigma.$$

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  3. Let $\widehat{\chi_\sigma} = \chi_\sigma \rfloor g_\sigma^{-1} \in T^*$ and

$$\widetilde{h_\sigma} (\theta_1, \theta_2) = h_{\sigma,s} (\theta_1, \theta_2, \widehat{\chi_\sigma},) .$$

Let $\widehat{h_\sigma} : T^* \to T^*$ be the operator, associated with the pair quadratic forms g_σ and $\widetilde{g_\sigma}$ on T^*. We require that covectors

$$e_1^* = \widehat{\chi_\sigma}, e_2^* = \widehat{h_\sigma} e_1^*, ..., e_n^* = \widehat{h_\sigma} e_{n-1}^*,$$

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     are linear independent and form invariant coframe $e^* (\sigma)$
  4. Each covector $\theta \in T^* \setminus 0$ defines also bivector
     $$\zeta_{\theta,\sigma} (\theta_1, \theta_2) = h_{\sigma,a} (\theta_1, \theta_2, \theta) \in \Lambda^2 T.$$
     The last regularity condition requires that $\zeta_{\theta,\sigma} \neq 0$ and σ_θ has distinct eigenvalues on a Zariski open subset in T^*.

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2. Coefficients $A_I^J(\sigma)$ of Artin-Procesi tensors $A_{\sigma,I}$ are rational functions on the symbol space $\mathrm{End}(E) \otimes T$ and are $G-$ invariants (AP-invariants).
3. .Due to Rosenlicht theorem any $a_1, ..., a_\nu$ algebraically independent AP-invariants generate the field of rational $G-$invariants of symbols.

Quantizations and connections

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- Any connection ∇ in the vector bundle π gives us a splitting

$$\begin{aligned} \mathbf{Q}_\nabla &: \mathrm{End}\,(\pi) \otimes \Sigma_1\,(M) \to \mathbf{Diff}_1\,(\pi,\pi)\,, \\ \mathbf{Q}_\nabla &: A \otimes X \mapsto A \circ \nabla_X\,. \end{aligned}$$

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$$\mathbf{Q}_\nabla : A \otimes X \mapsto A \circ \nabla_X.$$

- Any $\Delta \in \mathbf{Diff}_1(\pi, \pi)$ is uniquely represented in the following form

$$\Delta = \mathbf{Q}_\nabla(\sigma) + \Delta_0(\nabla),$$

where $\sigma = \mathrm{smbl}(\Delta)$, $\Delta_0(\nabla) \in \mathrm{End}(\pi)$.

Let $\widetilde{\nabla}$ be another connection and

$$\widetilde{\nabla} = \nabla + \alpha,$$

where $\alpha \in \mathrm{End}\left(\pi\right) \otimes \Omega^{1}\left(M\right)$ is an $\mathrm{End}\left(\pi\right)$-valued 1-form, and $\widetilde{\nabla}_X = \nabla_X + \alpha\left(X\right).$
Then

$$\begin{aligned} \mathbf{Q}_{\widetilde{\nabla}} &= \mathbf{Q}_{\nabla} + \left\langle \cdot, \alpha \right\rangle, \\ \Delta_0\left(\widetilde{\nabla}\right) &= \Delta_0\left(\nabla\right) - \left\langle \sigma, \alpha \right\rangle \end{aligned}$$

where

$$\left\langle , \right\rangle : \left(\mathrm{End}\left(\pi\right) \otimes \Sigma_1\left(M\right)\right) \otimes \left(\mathrm{End}\left(\pi\right) \otimes \Omega^{1}\left(M\right)\right) \to \mathrm{End}\left(\pi\right)$$

is the natural pairing:

$$\left\langle A \otimes X, B \otimes \omega \right\rangle = \left\langle \omega, X \right\rangle \ A \circ B.$$

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- Then

$$d_\nabla : \mathrm{End}(\pi) \otimes \Sigma_1(M) \to \mathrm{End}(\pi) \otimes \Sigma_1(M) \otimes \Omega^1(M) = \mathrm{End}(\pi) \otimes \mathrm{End}(\tau),$$

i.e

$$d_\nabla(\sigma) \in \mathrm{End}(\pi) \otimes \mathrm{End}(\tau)$$

and

$$d_{\tilde{\nabla}}\sigma - d_\nabla\sigma = [\alpha, \sigma],$$

where $[A \otimes \omega, B \otimes X] = [A, B] \otimes (\omega \otimes X) \in \mathrm{End}(\pi) \otimes \mathrm{End}(\tau)$.

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$$d_\nabla(\sigma) \perp W(\sigma),$$
with respect to the standard metric structure.
- The minimal connections do exist and any two of them differ on tensors of the form
$$\alpha = \mathrm{id} \otimes \lambda,$$
where $\lambda \in \Omega^1(M)$.

Invariants, produced by minimal connections

- Operator $D(\sigma) = (\mathrm{Tr} \otimes \mathrm{id})(d_{\nabla}\sigma) \in \mathrm{End}(\tau)$ does not depend on choice of minimal connection, and therefore is an invariant of the symbol.

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- Tensor

$$R^0(\sigma) = R_\nabla(\sigma) - \frac{1}{m}\mathrm{id} \otimes \mathrm{ch}(\nabla) \in \mathrm{End}(\pi) \otimes \Omega^2(M),$$

is invariant of the symbol if ∇ is a minimal connection, $R_\nabla(\sigma)$ -its curvature tensor and $\mathrm{ch}(\nabla) = \mathrm{Tr}(R_\nabla(\sigma)) \in \Omega^2(M)$ is the first *Chern form.*

Connections, associated with differential operators

- For subsymbols $\sigma_0\left(\widetilde{\nabla}\right)$, $\sigma_0\left(\nabla\right)$, computing for minimal connections ∇ and $\widetilde{\nabla}$, we have $\sigma_0\left(\widetilde{\nabla}\right) = \sigma_0\left(\nabla\right) - \sigma_\lambda$.

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- Let $W_0(\Delta) \subset \mathrm{End}(\pi)$ generates by subsymbols $\sigma_0(\nabla)$, taking for all minimal connections ∇. We say that a minimal connection ∇ is *associated with operator* Δ if the subsymbol $\sigma_0(\nabla)$ is orthogonal to $W_0(\Delta)$.

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- Let $W_0\left(\Delta\right) \subset \mathrm{End}\left(\pi\right)$ generates by subsymbols $\sigma_0\left(\nabla\right)$, taking for all minimal connections ∇. We say that a minimal connection ∇ is *associated with operator* Δ if the subsymbol $\sigma_0\left(\nabla\right)$ is orthogonal to $W_0\left(\Delta\right)$.
- There exists and unique associated connection ∇^Δ and the subsymbol $\sigma_0\left(\nabla^\Delta\right) \in \mathrm{End}\left(\pi\right)$ in the decomposition

$$\Delta = Q_{\nabla^\Delta}\left(\sigma\right) + \sigma_0\left(\nabla^\Delta\right)$$

satisfies the following conditions:

$$\mathrm{Tr}\left(\sigma_0\left(\nabla^\Delta\right) \cdot \sigma_\theta\right) = 0,$$

for all differential 1-forms $\theta \in \Omega^1\left(M\right)$.

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Natural coordinates and Models

- A differential operator Δ is said to be *regular* at a point $a \in M$ if its symbol σ is regular at the point and among extended Artin-Procesi invariants $a_1, ..., a_{\nu_0}$, defining $\mathrm{GL}(\pi) \times \mathrm{GL}(T)$ $-$orbits of the pairs (σ_0, σ), there are $n = \dim M$ invariants, say $a_1, .., a_n$, such that functions $a_1(\Delta), .., a_n(\Delta)$ are local coordinates in a neighborhood U of the point (we call them *natural coordinates*).

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- The above data $(\phi_U, \mathbf{D}_U, F_U, \mathrm{ch}_U(\Delta))$ we call *model of the differential operator* in neighborhood U.

Equivalence of operators

- **Local.** Let differential operators Δ and Δ' has the same model in a simply connected open set U. Then there is and unique automorphism $A_U \in \mathrm{GL}\,(\pi_U)$ such that $A_{U*}\,(\Delta) = \Delta'$.

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- **Global.** Two linear differential operators Δ, Δ' are $\mathrm{Aut}(\pi)$-equivalent if and only if a natural atlas for operator Δ is the natural atlas for Δ', i.e. they have the same models.

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Thank you for your attention and patience.