On structures of the first order linear differential operators

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The problem

- Given differential operators $\Delta, \Delta' \in \text{Diff}_1(\pi, \pi)$, acting in sections of a vector bundle $\pi : E(\pi) \to M$ over manifold $M$. Here $\dim \pi \geq 2$, $\dim M \geq 2$. 

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- The problem: is there a bundle automorphism $A \in \text{Aut}(\pi)$ such that $A_*(\Delta) = \Delta'$?
Steps to solution

- Artin conjecture, Procesi theorem and invariants of symbols
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- Connections, naturally associated with operators, and quantizations
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- Connections, naturally associated with operators, and quantizations
- Invariants of differential operators
- Natural coordinates, models and equivalence
Consider set \( \mathcal{A} \) of \( n \)-tuples of operators
\[
A = \{A_1, ..., A_n\} \in \text{End} (E), \quad \text{dim} \, E = m,
\]
equipped with the natural\( \text{GL} (E) \)–action: \( c(A) = \{c(A_1), ..., c(A_n)\} \), where
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Artin conjecture & Procesi theorem: The algebra of polynomial invariants of the above action is generated by polynomials (we’ll call them AP-invariants)
\[ \mathcal{P}_I (A) = \text{Tr} (A_{i_1} \cdots A_{i_k}), \]
for all multi indices \( I = (i_1, \ldots, i_k) \) of the length \( \leq 2^m - 1 \).
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**Rosenlicht Theorem** ⇒ The field of rational $\text{GL} (E)$—invariants has transcendence degree $\nu = (n - 1) m^2 + 1$ and therefore generated by $\nu$ algebraically independent AP-invariants.
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$$0 \to \text{End} (\pi) \to \textbf{Diff}_1 (\pi, \pi)^{\text{smbl}} \to \text{End} (\pi) \otimes \Sigma_1 (M) \to 0.$$
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- We call $\text{End} (E) \otimes T$ -symbol space and its elements -symbols.
Symbols for coordinate lovers

- Operators

\[
\Delta = \sum_{i=1}^{n} A_i(x) \partial_i + A_0(x),
\]

where \( A_i(x) \) are \( m \times m \) matrices, \( x = (x_1, \ldots, x_n) \), \( \partial_i = \partial / \partial x_i \).
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\[ \sigma(\Delta) = \sum_{i=1}^{n} A_i(x) \otimes \partial_i, \]

where \( \partial_i \) are basic vectors in \( T \).
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- The value of the symbol at covector \( \theta = (p_1, ..., p_n) \) (or \( \theta = p_1 dx_1 + \cdots + p_n dx_n \)) is the matrix

\[ \sigma_\theta = \sum_{i=1}^{n} p_i A_i(x) : E \rightarrow E. \]
Symbols for coordinate lovers

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- **Symbol**

\[ \sigma(\Delta) = \sum_{i=1}^{n} A_i(x) \otimes \partial_i, \]

where \( \partial_i \) are basic vectors in \( T \).

- **Remark:** \( \{ p_i = \partial_i \} \) are coordinates in \( T^* \).
Let $\sigma_\theta \in \text{End} (E)$ be the value of symbol $\sigma$ at covector $\theta \in T^*$. 
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Consider linear functions $A_{\sigma, l} : (T^*)^k \rightarrow \mathbb{R}$, as follows

$$A_{\sigma, l} (\theta_1 \otimes \cdots \otimes \theta_k) = \text{Tr} \left( \sigma_{\theta_{i_1}} \cdots \sigma_{\theta_{i_k}} \right),$$

where $l = (i_1, \ldots, i_k) \in S_k$ is a permutation of $k$-letters.
Artin-Procesi tensors

- Let $\sigma_\theta \in \text{End}(E)$ be the value of symbol $\sigma$ at covector $\theta \in T^*$.
- Consider linear functions $A_{\sigma, I} : (T^*)^k \to \mathbb{R}$, as follows
  \[ A_{\sigma, I} (\theta_1 \otimes \cdots \otimes \theta_k) = \text{Tr} \left( \sigma_{\theta_{i_1}} \cdots \sigma_{\theta_{i_k}} \right), \]
  where $I = (i_1, \ldots, i_k) \in S_k$ is a permutation of $k$-letters.
- Tensors $A_{\sigma, I} \in \left( (T^*)^k \right)^* = T^\otimes k$ are $G$-invariants. We call them Artin-Procesi or AP- tensors.
**First AP-tensors**

- **k = 1.** We have $G$-invariant vector $\chi_\sigma \in T$, where
  \[
  \langle \theta, \chi_\sigma \rangle = \text{Tr} (\sigma_\theta),
  \]
  for $\theta \in T^*$. 


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  for \( \theta \in T^* \).
- \( k = 2 \). We have \( G \)-invariant symmetric 2-vector \( g_\sigma \in S^2 T \), where
  \[
  g_\sigma (\theta_1, \theta_2) = \text{Tr} (\sigma_{\theta_1} \sigma_{\theta_2}).
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First AP-tensors

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- **k = 2.** We have $G$- invariant symmetric 2-vector $g_\sigma \in S^2 T$, where
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  g_\sigma (\theta_1, \theta_2) = \text{Tr} \left( \sigma_{\theta_1} \sigma_{\theta_2} \right).
  \]

- **k = 3.** We have two $G$- invariant tensors $h_i \in T \otimes^3$:
  \[
  h_{1,\sigma} (\theta_1, \theta_2, \theta_3) = \text{Tr} \left( \sigma_{\theta_1} \sigma_{\theta_2} \sigma_{\theta_3} \right),
  \]
  \[
  h_{2,\sigma} (\theta_1, \theta_2, \theta_3) = \text{Tr} \left( \sigma_{\theta_2} \sigma_{\theta_1} \sigma_{\theta_3} \right),
  \]
  or equivalently two tensors
  \[
  h_{\sigma,s} = \frac{1}{2} (h_{\sigma,1} + h_{\sigma,2}) \in S^2 T \otimes T,
  \]
  \[
  h_{\sigma,a} = \frac{1}{2} (h_{\sigma,1} - h_{\sigma,2}) \in \Lambda^2 T \otimes T.
  \]
Example: The Euler system (R, 27.04.20)

The system:

\[ \rho_t + (\rho u)_x = 0, \]
\[ u_t + uu_x + a(\rho) \rho_x = 0, \]
\[ p = p(\rho), \quad a = \rho^{-1} p'. \]
The system:

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\end{align*}
\]

The symbol of the linearization on a solution \( \rho = \rho_0, u = u_0 \) equals

\[
\sigma_\theta = p \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + q \begin{bmatrix} u_0 & \rho_0 \\ a(\rho_0) & u_0 \end{bmatrix},
\]

where \( \theta = p \, dt + q \, dx \). (\( p \leftrightarrow \partial_t, p \leftrightarrow \partial_x \)).
Invariant vector

\[ \frac{1}{2} \chi_\sigma = \partial_t + u \partial_x. \]

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\[ g_\sigma = \partial_t^2 + 2u \partial_t \partial_x + (a \rho + u^2) \partial_x^2, \]

and

\[ g_\sigma^{-1} = \frac{(\rho + u^2) \ dt^2 - 2u \ dtdx + dx^2}{2a \rho}. \]
The Euler system-2

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- The symmetrized 3rd tensor

\[ h = \chi_\sigma g_\sigma. \]
We use the only symbols in general position (general or regular symbols), i.e. symbols where the following regularity conditions hold.

1. $\chi_\sigma = 0$.
2. Quadratic form $g_\sigma$ on $T$ is non-degenerated.
3. Let $c_\chi_\sigma = \chi_\sigma c_\sigma g_1$, and $e_\theta_\sigma(\theta_1, \theta_2) = h_\theta_\sigma, s(\theta_1, \theta_2, \chi_\sigma)$.
4. Let $b_\theta_\sigma: T \rightarrow T$ be the operator, associated with the pair quadratic forms $g_\sigma$ and $e_\sigma$ on $T$.
   - Covectors $e_1 = c_\chi_\sigma$, $e_2 = b_\theta_\sigma e_1$, ..., $e_n = b_\theta_\sigma e_n$ are linear independent and form invariant coframe $e_\sigma(\sigma)$.
5. Each covector $\theta_2$ defines also bivector $\zeta_{\theta_1, \sigma}(\theta_1, \theta_2) = h_\theta_\sigma, a(\theta_1, \theta_2)$. The last regularity condition requires that $\zeta_{\theta_1, \sigma} = 0$ and $\sigma_{\theta_2}$ has distinct eigenvalues on a Zariski open subset in $T$. 

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3. Let $\tilde{\chi}_\sigma = \chi_\sigma | g_\sigma^{-1} \in T^*$ and
   \[ \tilde{h}_\sigma (\theta_1, \theta_2) = h_\sigma, s (\theta_1, \theta_2, \tilde{\chi}_\sigma, ) . \]

Let $\tilde{h}_\sigma : T^* \rightarrow T^*$ be the operator, associated with the pair quadratic forms $g_\sigma$ and $\tilde{g}_\sigma$ on $T^*$. We require that covectors

\[ e_1^* = \tilde{\chi}_\sigma, e_2^* = \tilde{h}_\sigma e_1^*, ..., e_n^* = \tilde{h}_\sigma e_{n-1}^* , \]

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- Each covector $\theta \in T^* \setminus 0$ defines also bivector

\[ \zeta_{\theta,\sigma} (\theta_1, \theta_2) = h_{\sigma,a} (\theta_1, \theta_2, \theta) \in \Lambda^2 T. \]

The last regularity condition requires that $\zeta_{\theta,\sigma} \neq 0$ and $\sigma_\theta$ has distinct eigenvalues on a Zariski open subset in $T^*$. 
A symbol tensor $\sigma \in \text{End} (E) \otimes T$, when $n \geq 3$, is said to be in general position if the above conditions hold.
Rational invariants of symbols

- A symbol tensor \( \sigma \in \text{End} (E) \otimes T \), when \( n \geq 3 \), is said to be in general position if the above conditions hold.
- Let \( \sigma \) be a general symbol then, using the invariant frame \( e(\sigma) \), we represent \( \sigma \) in the form
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  \sigma = \sum_{i=1}^{n} \sigma_i \otimes e_i, \sigma_i \in \text{End} (E)
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$$\nu = (n - 1) \left( m^2 - n - 1 \right).$$
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2. Coefficients $A_{I}^{J}(\sigma)$ of Artin-Procesi tensors $A_{\sigma,I}$ are rational functions on the symbol space $\text{End}(E) \otimes T$ and are $G$—invariants (AP-invariants).
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3. Due to Rosenlicht theorem any $a_1, \ldots, a_\nu$ algebraically independent AP-invariants generate the field of rational $G$—invariants of symbols.
Quantizations and connections

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- Quantization = a splitting of the exact sequence

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- Any connection $\nabla$ in the vector bundle $\pi$ gives us a splitting
  \[ Q_\nabla : \text{End} (\pi) \otimes \Sigma_1 (M) \to \text{Diff}_1 (\pi, \pi), \]
  \[ Q_\nabla : A \otimes X \mapsto A \circ \nabla X. \]
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  \[
  Q_\nabla : A \otimes X \mapsto A \circ \nabla_X.
  \]
- Any \( \Delta \in \text{Diff}_1 (\pi, \pi) \) is uniquely represented in the following form
  \[
  \Delta = Q_\nabla (\sigma) + \Delta_0 (\nabla),
  \]
  where \( \sigma = \text{smbl} (\Delta), \Delta_0 (\nabla) \in \text{End} (\pi) \).
Let $\nabla$ be another connection and

$$\nabla = \nabla + \alpha,$$

where $\alpha \in \text{End}(\pi) \otimes \Omega^1(M)$ is an $\text{End}(\pi)$-valued 1-form, and $\nabla_X = \nabla_X + \alpha(X)$.

Then

$$Q_{\nabla} = Q_{\nabla} + \langle \cdot , \alpha \rangle,$$

$$\Delta_0 (\nabla) = \Delta_0 (\nabla) - \langle \sigma , \alpha \rangle$$

where

$$\langle , \rangle : (\text{End}(\pi) \otimes \Sigma_1(M)) \otimes (\text{End}(\pi) \otimes \Omega^1(M)) \rightarrow \text{End}(\pi)$$

is the natural pairing:

$$\langle A \otimes X , B \otimes \omega \rangle = \langle \omega , X \rangle \ A \circ B.$$
Connections, associated with symbols

- Invariant connection $\nabla_c = \text{the Levi-Civita connection, associated with } g_\sigma$. 

\[ \text{d}r : \text{End}(\pi) \rightarrow \text{End}(\tau), \] 
\[ \text{d}r(\sigma_2) \text{End}(\pi) \text{End}(\tau), \] 
\[ \text{d}e r \sigma = [\alpha, \sigma], \] 
where 
\[ [A \omega_X, B X] = [A, B](\omega X). \]
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- For given connection $\nabla$ in the bundle we’ll denote the connections generated in $\text{End}(\pi)$ as well as in the symbol bundle $\text{End}(\pi) \otimes \Sigma_1$ by $\nabla$. 
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- Then

$$d_\nabla : \text{End}(\pi) \otimes \Sigma_1(M) \rightarrow \text{End}(\pi) \otimes \Sigma_1(M) \otimes \Omega^1(M) = \text{End}(\pi) \otimes \text{End}(\tau),$$

i.e

$$d_\nabla(\sigma) \in \text{End}(\pi) \otimes \text{End}(\tau)$$

and

$$d_\nabla \sigma - d_\nabla \sigma = [\alpha, \sigma],$$

where $[A \otimes \omega, B \otimes X] = [A, B] \otimes (\omega \otimes X) \in \text{End}(\pi) \otimes \text{End}(\tau)$. 
Let

\[ W(\sigma) = \{ d\nabla(\sigma) \} \subset \text{End}(\pi) \otimes \text{End}(\tau). \]
Minimal connections

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- We say that a connection \( \nabla \) is *minimal* (or symbol preserving in the best way) if
  \[ d_{\nabla}(\sigma) \perp W(\sigma), \]
  with respect to the standard metric structure.
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- The minimal connections do exist and any two of them differ on tensors of the form

\[ \alpha = \text{id} \otimes \lambda, \]

where \( \lambda \in \Omega^1(M) \).
Invariants, produced by minimal connections

- Operator $D(\sigma) = (\text{Tr} \otimes \text{id})(d\nabla \sigma) \in \text{End}(\tau)$ does not depend on choice of minimal connection, and therefore is an invariant of the symbol.
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- Tensor

$$R^0(\sigma) = R_\nabla(\sigma) - \frac{1}{m} \text{id} \otimes \text{ch}(\nabla) \in \text{End}(\pi) \otimes \Omega^2(M),$$

is invariant of the symbol if $\nabla$ is a minimal connection, $R_\nabla(\sigma)$ -its curvature tensor and $\text{ch}(\nabla) = \text{Tr}(R_\nabla(\sigma)) \in \Omega^2(M)$ is the first Chern form.
Connections, associated with differential operators

For subsymbols $\sigma_0 \left( \tilde{\nabla} \right)$, $\sigma_0 \left( \nabla \right)$, computing for minimal connections $\nabla$ and $\tilde{\nabla}$, we have $\sigma_0 \left( \tilde{\nabla} \right) = \sigma_0 \left( \nabla \right) - \sigma_\lambda$. 
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- Let $\mathcal{W}_0 (\Delta) \subset \operatorname{End} (\pi)$ generates by subsymbols $\sigma_0 \left( \nabla \right)$, taking for all minimal connections $\nabla$. We say that a minimal connection $\nabla$ is associated with operator $\Delta$ if the subsymbol $\sigma_0 \left( \nabla \right)$ is orthogonal to $\mathcal{W}_0 (\Delta)$.
Connections, associated with differential operators

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- There exists and unique associated connection $\nabla^\Delta$ and the subsymbol $\sigma_0 (\nabla^\Delta) \in \text{End} (\pi)$ in the decomposition

$$\Delta = Q_{\nabla^\Delta} (\sigma) + \sigma_0 (\nabla^\Delta)$$

satisfies the following conditions:

$$\text{Tr} \left( \sigma_0 (\nabla^\Delta) \cdot \sigma_\theta \right) = 0,$$

for all differential 1-forms $\theta \in \Omega^1 (M)$. 
Extended AP-invariants for the symbol set \( \{\sigma_i\} \) and the subsymbol \( \sigma_0 \left( \nabla^\Delta \right) \)
Invariants of differential operators

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- The curvature tensor

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The operator

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D (\sigma) = (\text{Tr} \otimes \text{id}) (d_{\nabla^c} \sigma) \in \text{End} (\tau).
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The Zoo of invariants

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- The curvature tensor $R_{\nabla^\Delta} \in \text{End}(\pi) \otimes \Omega^2(M)$. 
A differential operator $\Delta$ is said to be regular at a point $a \in M$ if its symbol $\sigma$ is regular at the point and among extended Artin-Procesi invariants $a_1, ..., a_{\nu_0}$, defining $\text{GL}(\pi) \times \text{GL}(T)$—orbits of the pairs $(\sigma_0, \sigma)$, there are $n = \dim M$ invariants, say $a_1, ..., a_n$, such that functions $a_1(\Delta), ..., a_n(\Delta)$ are local coordinates in a neighborhood $U$ of the point (we call them natural coordinates).

Remark: $\nu_0 = \nu + \frac{m^2 - n}{n}$ if $m^2 + 2$. The values $a_j(\Delta)$ are functions of the natural coordinates: $a_j(\Delta) = F_j(a_1(\Delta), ..., a_n(\Delta))$, $n + 1$. $j = \nu_0$. Let $\phi_U: U \rightarrow D_U \in \mathbb{R}^n$ be the natural local chart, $\phi_U(b) = (a_1(\Delta)(b), ..., a_n(\Delta)(b))$, and let $F_U: D_U \rightarrow \mathbb{R}^{\nu_0}$ be the above function and $\chi_U(\Delta) = \phi_U(\chi(\Delta))^2 \Omega^2(D_U)$. The above data $(\phi_U, D_U, F_U, \chi_U(\Delta))$ we call model of the differential operator in neighborhood $U$.
A differential operator $\Delta$ is said to be \emph{regular} at a point $a \in M$ if its symbol $\sigma$ is regular at the point and among extended Artin-Procesi invariants $a_1, \ldots, a_{\nu_0}$, defining $GL(\pi) \times GL(T)$-orbits of the pairs $(\sigma_0, \sigma)$, there are $n = \dim M$ invariants, say $a_1, \ldots, a_n$, such that functions $a_1(\Delta), \ldots, a_n(\Delta)$ are local coordinates in a neighborhood $U$ of the point (we call them \emph{natural coordinates}).

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Equivalence of operators

- **Local.** Let differential operators $\Delta$ and $\Delta'$ have the same model in a simply connected open set $U$. Then there is a unique automorphism $A_U \in \text{GL}(\pi_U)$ such that $A_U^* (\Delta) = \Delta'$.
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- **Natural atlas.** We say that an atlas $\{(\phi_{U^\alpha}, D_{U^\alpha})\}$ given by models $\{(\phi_{U^\alpha}, D_{U^\alpha}, F_{U^\alpha}, \text{ch}_{U^\alpha}(\Delta))\}$ is *natural* if the sets of basic invariants $(a_1^\alpha, \ldots, a_n^\alpha)$ are different for different $\alpha$.
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- **Global.** Two linear differential operators $\Delta, \Delta'$ are $\text{Aut}(\pi)$-equivalent if and only if a natural atlas for operator $\Delta$ is the natural atlas for $\Delta'$, i.e. they have the same models.
Thank you for your attention and patience.