Dispersionless integrable systems in 3D and Einstein-Weyl geometry

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Plan:

- Formal linearisation: applications
- Einstein-Weyl geometry
- Integrability in 3D and Einstein-Weyl geometry
- Integrability in 4D and self-duality

Based on:

E.V. Ferapontov and B. Kruglikov, Dispersionless integrable systems in 3D and Einstein-Weyl geometry, arXiv:1208.2728v3, J. Diff. Geom. **97** *(2014) 215-254.*

Formal linearisation

Given a PDE

$$F(x^i, u, u_{x^i}, u_{x^i x^j}, \dots) = 0,$$

its formal linearisation results upon setting $u \to u + \epsilon v$, and keeping terms of the order ϵ. This leads to a linear PDE for v,

$$\ell_F(v) = 0,$$

where ℓ_F is the operator of formal linearisation,

$$\ell_F = F_u + F_{u_{x^i}} \mathcal{D}_{x^i} + F_{u_{x^i x^j}} \mathcal{D}_{x^i} \mathcal{D}_{x^j} + \dots.$$

Example

dKP equation: $u_{xt} - (uu_x)_x - u_{yy} = 0.$

linearised dKP: $v_{xt} - (uv)_{xx} - v_{yy} = 0.$

Applications of formal linearisation

- *Stability analysis*
- *Symmetries*
- *Contact invariants of ODEs, generalised Laplace invariants of Monge-Ampère equations*
- *Integrability of ODEs can be seen from the monodromy group of linearised equations*

Can one read the integrability of a given PDE off the geometry of its formal linearisation?

Yes, for broad classes of 3D dispersionless PDEs.

Types of PDEs studied:

Quasilinear wave equations:

$$f_{11}u_{xx} + f_{22}u_{yy} + f_{33}u_{tt} + 2f_{12}u_{xy} + 2f_{13}u_{xt} + 2f_{23}u_{yt} = 0.$$

Hirota-type equations:

$$F(u_{xx}, u_{xy}, u_{yy}, u_{xt}, u_{yt}, u_{tt}) = 0.$$

Equations possessing the 'central quadric ansatz':

$$(a(u))_{xx} + (b(u))_{yy} + (c(u))_{tt} + 2(p(u))_{xy} + 2(q(u))_{xt} + 2(r(u))_{yt} = 0.$$

The corresponding formal linearisations are second-order linear PDEs. On every solution, their symbols define conformal structures (that depend on a solution). Which conformal geometries correspond to integrable PDEs? Which conformal geometries should be regarded as 'integrable'?

Einstein-Weyl geometry

This is a triple $(\mathbb{D},\ g,\ \omega)$ where $\mathbb{D}$ is a symmetric connection, g is a conformal structure and ω is a covector such that

$$\mathbb{D}_k g_{ij} = \omega_k g_{ij}, \quad R_{(ij)} = \Lambda g_{ij}.$$

Here $R_{(ij)}$ is the symmetrised Ricci tensor of $\mathbb{D}$, and Λ is some function (the first set of equations defines $\mathbb{D}$ uniquely, so it is sufficient to specify g and ω only).

Conformal invariance: $g \to \lambda g,\ \omega \to \omega + d\ln\lambda$.

Theorem (E. Cartan, 1941): The triple $(\mathbb{D},\ g,\ \omega)$ satisfies the Einstein-Weyl equations if and only if there exists a two-parameter family of surfaces which are totally geodesic with respect to $\mathbb{D}$, and null with respect to g.

Generic Einstein-Weyl structures depend on 4 arbitrary functions of 2 variables.

Einstein-Weyl equations are integrable (Hitchin, 1980).

Einstein-Weyl structures via the Manakov-Santini system

The Manakov-Santini system (2006) is a generalization of dKP:

$$u_{xt}-u_{yy}+(uu_x)_x+v_xu_{xy}-v_yu_{xx}=0,\;\; v_{xt}-v_{yy}+uv_{xx}+v_xv_{xy}-v_yv_{xx}=0.$$

Its solutions give rise to Einstein-Weyl structures (Dunajski):

$$g=(dy-v_xdt)^2-4(dx-(u-v_y)dt)dt,$$

$$\omega=-v_{xx}dy+(4u_x-2v_{xy}+v_xv_{xx})dt.$$

One can show that all Einstein-Weyl structures arise in this way.

Lax pair $[X,Y]=0$:

$$X=\partial_y+(\lambda-v_x)\partial_x+u_x\partial_\lambda,\quad Y=\partial_t-(\lambda^2-\lambda v_x+v_y-u)\partial_x-(\lambda u_x-u_y)\partial_\lambda.$$

Projecting integral surfaces of the distribution spanned by X,Y from (x,y,t,λ) to (x,y,t) one obtains a two-parameter family of null totally geodesic surfaces.

Einstein-Weyl structures via the Bogdanov system

The Bogdanov system (2010) is a generalization of 2D Toda:

$$(e^{-\phi})_{tt} = m_t\phi_{xy} - m_x\phi_{yt}, \quad m_{tt}e^{-\phi} = m_x m_{yt} - m_t m_{xy}.$$

Its solutions give rise to Einstein-Weyl structures:

$$g = (m_x dx + m_t dt)^2 + 4e^{-\phi} m_t dx dy,$$

$$\omega = \left(\frac{m_{tt}}{m_t^2} - 2\frac{\phi_t}{m_t}\right)(m_x\, dx + m_t\, dt) + 2\frac{m_{yt}}{m_t}\, dy.$$

It is likely that all Einstein-Weyl structures can be parametrised in this form.

Lax pair $[X, Y] = 0$:

$$X = \partial_x - \left(\lambda + \frac{m_x}{m_t}\right)\partial_t + \lambda\left(\phi_t\frac{m_x}{m_t} - \phi_x\right)\partial_\lambda, \quad Y = \partial_y + \frac{1}{\lambda}\frac{e^{-\phi}}{m_t}\partial_t + \frac{(e^{-\phi})_t}{m_t}\partial_\lambda.$$

Equivalence of the Manakov-Santini and Bogdanov systems?

Main results

Theorem 1. A second-order PDE is linearisable (by a transformation from the natural equivalence group) if and only if the conformal structure g corresponding to the symbol of formal linearisation is conformally flat (has vanishing Cotton tensor) on every solution.

Theorem 2. A second-order PDE is integrable if and only if, on every solution, the conformal structure g corresponding to the symbol of formal linearisation satisfies the Einstein-Weyl equations, with the covector $\omega = \omega_s dx^s$ given by the formula

$$\omega_s = 2g_{sj}\mathcal{D}_{x^k}(g^{jk}) + \mathcal{D}_{x^s}(\ln \det g_{ij}).$$

The corresponding two-parameter family of null totally geodesic surfaces is provided by the dispersionless Lax pair.

Example of dKP

As an illustration let us consider the dKP equation,

$$u_{xt} - (uu_x)_x - u_{yy} = 0.$$

The corresponding Einstein-Weyl structure is provided by the conformal metric $g = 4dxdt - dy^2 + 4udt^2$ and the covector $\omega = -4u_x dt$ (Dunajski, Mason, Tod). One can verify that they satisfy the Einstein-Weyl conditions if and only if u satisfies dKP. The dispersionless Lax pair is given by vector fields

$$X = \partial_y - \lambda\partial_x + u_x\partial_\lambda, \quad Y = \partial_t - (\lambda^2 + u)\partial_x + (u_x\lambda + u_y)\partial_\lambda,$$

such that the commutativity condition, $[X, Y] = 0$, is equivalent to dKP. Projecting integral surfaces of the distribution spanned by X, Y in the extended space (x, y, t, λ) to the space of independent variables (x, y, t), one obtains a two-parameter family of surfaces that are null with respect to g, and totally geodesic in the Weyl connection $\mathbb{D}$ specified by g and ω.

Equations possessing the central quadric ansatz

Based on

E.V. Ferapontov, B. Huard and A. Zhang, On the central quadric ansatz: integrable models and Painlevé reductions, J. Phys. A: Math. Theor. **45** *(2012) 195204; arXiv:1201.5061.*

Let us consider PDEs of the form

$$(a(u))_{xx} + (b(u))_{yy} + (c(u))_{tt} + 2(p(u))_{xy} + 2(q(u))_{xt} + 2(r(u))_{yt} = 0.$$

Equations of this type possess solutions $u(x, y, t)$ in implicit form,

$$(x, y, t)M(u)(x, y, t)^T = 1.$$

Level surfaces $u = const$ are central quadrics in the space of independent variables x, y, t. The 3×3 matrix $M(u)$ satisfies an ODE which, in integrable cases, reduces to one of the Painlevé equations.

Which PDEs of the above form are integrable?

Classification result

Given a PDE of the form

$$(a(u))_{xx} + (b(u))_{yy} + (c(u))_{tt} + 2(p(u))_{xy} + 2(q(u))_{xt} + 2(r(u))_{yt} = 0,$$

we introduce the 3×3 symmetric matrix of coefficients,

$$V(u) = \begin{pmatrix} a' & p' & q' \\ p' & b' & r' \\ q' & r' & c' \end{pmatrix}.$$

The method of hydrodynamic reductions consists of seeking solutions in the form $u = u(R^1, \ldots, R^N)$ where the phases $R^i(x, y, t)$ are required to satisfy a pair of compatible systems of hydrodynamic type,

$$R^i_t = \lambda^i(R) R^i_x, \quad R^i_y = \mu^i(R) R^i_x.$$

The requirement of the existence of such solutions imposes strong constraints on V.

Five canonical forms

Theorem The integrability by the method of hydrodynamic reductions, which is equivalent to the Einstein-Weyl property of the symbol of formal linearisation, implies that $V(u)$ satisfies the constraint

$$V'' = (\ln \det V)' V' + kV,$$

for some scalar function k. This gives five canonical forms of integrable PDEs possessing the central quadric ansatz:

$$u_{xx} + u_{yy} - [\ln(1 - e^u)]_{yy} - [\ln(1 - e^u)]_{tt} = 0,$$

$$u_{xx} + u_{yy} - (e^u)_{tt} = 0,$$

$$(e^u - u)_{xx} + 2u_{xy} + (e^u)_{tt} = 0,$$

$$u_{xt} - (uu_x)_x - u_{yy} = 0,$$

$$(u^2)_{xy} + u_{yy} + 2u_{xt} = 0.$$

Einstein-Weyl structures

Equation 1: $u_{xx} + u_{yy} - (\ln(e^u - 1))_{yy} - (\ln(e^u - 1))_{tt} = 0$ (gauge-invariant dispersionless Hirota equation).

Conformal structure: $g = dx^2 + (1 - e^u)dy^2 + (e^{-u} - 1)dt^2$.

Covector: $\omega = \frac{e^u+1}{e^u-1}u_x dx - u_y dy + u_t dt$.

Equation 2: $u_{xx} + u_{yy} - (e^u)_{tt} = 0$ (BF equation).

Conformal structure: $g = dx^2 + dy^2 - e^{-u}dt^2$.

Covector: $\omega = -u_x dx - u_y dy + u_t dt$.

This Einstein-Weyl structure was obtained by Ward.

Equation 3: $(e^u - u)_{xx} + 2u_{xy} + (e^u)_{tt} = 0$.

Conformal structure: $g = 2dxdy + (1 - e^u)dy^2 + e^{-u}dt^2$.

Covector: $\omega = -u_x dx + (2e^u u_x - u_y)dy + u_t dt$.

Equation 4: $u_{xt} - (uu_x)_x - u_{yy} = 0$ (dKP equation).

Conformal structure: $g = 4dxdt - dy^2 + 4udt^2$.

Covector: $\omega = -4u_x dt$.

This Einstein-Weyl structure was obtained by Dunajski, Mason and Tod.

Equation 5: $(u^2)_{xy} + u_{yy} + 2u_{xt} = 0$.

Conformal structure: $g = 2dxdt + dy^2 - 2udydt + u^2dt^2$.

Covector: $\omega = 2u_x dy + 2(u_y - uu_x)dt$.

Integrability in 4D and self-duality

Integrable equations of Monge-Ampére type in 4D were classified in

B. Doubrov and E.V. Ferapontov, On the integrability of symplectic Monge-Ampère equations, J. Geom. Phys. **60** *(2010) 1604-1616.*

- $u_{11} - u_{22} - u_{33} - u_{44} = 0$ (linear wave equation)
- $u_{13} + u_{24} + u_{11}u_{22} - u_{12}^2 = 0$ (second heavenly equation)
- $u_{13} = u_{12}u_{44} - u_{14}u_{24}$ (modified heavenly equation)
- $u_{13}u_{24} - u_{14}u_{23} = 1$ (first heavenly equation)
- $u_{11} + u_{22} + u_{13}u_{24} - u_{14}u_{23} = 0$ (Husain equation)
- $\alpha u_{12}u_{34} + \beta u_{13}u_{24} + \gamma u_{14}u_{23} = 0$ (general heavenly equation), $\alpha + \beta + \gamma = 0$.

Conjecture: A 4D second-order dispersionless PDE is integrable if and only if the corresponding conformal structure is self-dual on every solution.

Lax pairs in 4D and totally null surfaces

Second heavenly equation, $u_{13} + u_{24} + u_{11}u_{22} - u_{12}^2 = 0$, possesses the Lax pair $[X, Y] = 0$ where

$$X = \partial_4 + u_{11}\partial_2 - u_{12}\partial_1 + \lambda\partial_1, \quad Y = \partial_3 - u_{12}\partial_2 + u_{22}\partial_1 - \lambda\partial_2.$$

Conformal structure corresponding to the principal symbol has the form

$$g = dx^1dx^3 + dx^2dx^4 - u_{22}(dx^3)^2 - u_{11}(dx^4)^2 + 2u_{12}dx^3dx^4.$$

Projecting integral surfaces of the distribution spanned by X, Y in the extended space (x^i, λ) to the space of independent variables x^i, one obtains a three-parameter family of surfaces which are totally null with respect to g. These are also known as α-surfaces. According to Penrose, the existence of such surfaces is necessary and sufficient for self-duality.

Questions:

- Contact-invariant approach to dispersionless integrability?
- Higher-order PDEs and higher Einstein-Weyl geometry?