Differential invariants of Kundt spacetimes

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Introduction

The equivalence problem of Lorentzian manifolds under the Lie pseudogroup action of local diffeomorphisms is an important problem in mathematics and physics, and there are several approaches for solving it:

Scalar polynomial curvature invariants are obtained by complete contractions of the Riemann tensor, its covariant derivatives and their tensor products.

Cartan invariants are obtained from the structure functions of the absolute parallelism on the reduced frame bundle, and their derivatives.

Rational scalar differential invariants are rational invariants of the Lie pseudogroup of local diffeomorphisms acting on the space of jets of metrics. The invariants are generated by a finite number of differential invariants and invariant derivations.

We will use the last approach to solve the equivalence problem for the degenerate Kundt spacetimes, a class of metrics that can not be separated by the first approach.

Separating generic metrics

Lychagin and Yumaguzhin (2015) showed that the rational invariants for Lorentzian metrics on an n-dimensional manifold can be generated in the following way.

Let g be a Lorentzian metric, and $\mathrm{Ric}\colon TM \to TM$ be the corresponding Ricci operator. We construct n invariants

$$I^i = \mathrm{Tr}(\mathrm{Ric}^i), \qquad i = 1, ..., n.$$

For a generic metric g, the invariants $I^1, \ldots, I^n$ are independent functions and can be chosen as coordinates. We can write g in these coordinates:

$$g = G_{ij} dI^i dI^j.$$

Two generic metrics can be distinguished by simply comparing the corresponding functions $G_{ij}(I^1, \ldots, I^n)$.

The functions I^i, G_{ij} can be thought of as restrictions of functions on $J^3(S^2T^*M)$ to the (prolongation of) a particular section g. From these we can generate all other invariants by algebraic operations and differentiations with respect to $I^1, \ldots, I^n$.

Kundt spacetimes

A Lorentzian metric g on an n-dimensional manifold M is a Kundt metric if there exists a vector field ℓ such that

$$\|\ell\|_g^2 = 0, \quad \nabla^g_\ell \ell = 0, \quad \mathrm{Tr}(\nabla^g \ell) = 0, \quad \|\nabla^g \ell^{\mathsf{sym}}\|_g^2 = 0, \quad \|\nabla^g \ell^{\mathsf{alt}}\|_g^2 = 0,$$

where ∇^g is the Levi-Civita connection given by g. We call g a degenerate Kundt metric if, in addition,

- The Riemann tensor Riem is aligned and of algebraically special type II, and
- $\nabla^g(\mathrm{Riem})$ is aligned and of algebraically special type II.

For any Kundt metric, there exist local coordinates $u, x^1, ..., x^{n-2}, v$ in which g takes the form

$$g = du\left(dv + H(u,x,v)\,du + W_i(u,x,v)\,dx^i\right) + h_{ij}(u,x)\,dx^i dx^j.$$

In these coordinates, $\ell = \partial_v$, and g is a degenerate Kundt metric if and only if $(W_i)_{vv} = 0$ and $H_{vvv} = 0$.

The equivalence problem

We say that two Kundt metrics g and $\tilde{g}$ on M are (locally) equivalent if there exists a local diffeomorphism $\varphi\colon U \subset M \to M$ such that $\varphi^*(\tilde{g}) = g$.

Important task: To recognize equivalent Kundt metrics and distinguish inequivalent ones.

One of the standard ways is to use polynomial curvature invariants, i.e. total contractions of the curvature tensor and its covariant derivatives. However, not all spacetimes are separated by such invariants. In particular, in dimension $n = 4$ the degenerate Kundt spacetimes are exactly those that can not separated by polynomial curvature invariants (Coley, Hervik, Pelavas 2009).

Therefore we must use other invariants!

Shape-preserving transformations

To simplify the problem, we will use the coordinates in which g takes the form

$$g = du\left(dv + H(u,x,v)\,du + W_i(u,x,v)\,dx^i\right) + h_{ij}(u,x)\,dx^i dx^j. \tag{1}$$

Then we must also restrict to diffeomorphisms preserving this form. For $n = 4$, this pseudogroup of diffeomorphisms was found by Pravda, Pravdova, Coley and Milson (2002).

Theorem

The transformations preserving the form of (1) *are given by*

$$(u, x^i, v) \mapsto \left(C(u), A^i(u,x), \frac{v}{C'(u)} + B(u,x)\right), \quad \det[A^i_{x^j}] \neq 0, C'(u) \neq 0.$$

The Lie algebra $\mathfrak{g}$ *corresponding to this Lie pseudogroup consists of the vector fields*

$$c(u)\partial_u + a^i(u,x)\partial_{x^i} + \left(b(u,x) - c'(u)v\right)\partial_v.$$

The normalized Kundt spacetimes as sections of a bundle

The Kundt metrics of form (1) can be considered as sections of a bundle

$$\pi\colon M \times F \to M$$

where $F \subset \mathbb{R}^N$ with $N = \binom{n}{2}$. Let $u, x^1, \ldots, x^{n-2}, v$ be coordinates on M and h_{ij}, W_i, H be coordinates on $\mathbb{R}^N$, with $1 \leq i \leq j \leq n-2$. The domain $F \subset \mathbb{R}^N$ is defined by the requirement that the matrix $[h_{ij}]$ is positive definite.

The vector fields of $\mathfrak{g}$ can be lifted to $F \times M$ by requiring the lifts to preserve the horizontal symmetric form

$$G = du \left(dv + H\, du + W_i\, dx^i\right) + h_{ij}\, dx^i dx^j.$$

The lift $\hat{X}$ of the vector field $X = c(u)\partial_u + a^i(u,x)\partial_{x^i} + \big(b(u,x) - c'(u)v\big)\partial_v$ is found by setting $\hat{X} = X + A_{ij}\partial_{h_{ij}} + B_i \partial_{W_i} + C\partial_H$, and determining the coefficients from the equation $L_{\hat{X}} G = 0$:

$$\begin{aligned}\hat{X} =& c\partial_u + a^i\partial_{x^i} + (b - c'v)\partial_v - (a^l_i h_{lj}\partial_{h_{ij}} + a^l_i h_{li}\partial_{h_{ii}}) \\ &- (c'W_i + a^j_i W_j + b_i + 2a^j_u h_{ij})\partial_{W_i} - (2c'H - c''v + b_u + a^j_u W_j)\partial_H.\end{aligned}$$

The space of jets

Let $J^k\pi$ denote the space of k-jets of sections of π. The choice of coordinates on $F \times M$ gives a natural set of coordinates on $J^k\pi$. For example, on $J^1\pi$ we use the following coordinates:

$$u,\ x^i,\ v,\ h_{ij},\ W_i,\ H,\ (h_{ij})_u,\ (h_{ij})_{x^k},\ (h_{ij})_v,\ (W_i)_u,\ (W_i)_{x^k},\ (W_i)_v,\ H_u,\ H_{x^k},\ H_v.$$

If g is a section of π given by $h_{ij} = \tilde{h}_{ij}(u,x,v), W_i = \tilde{W}_i(u,x,v), H = \tilde{H}(u,x,v)$, then it prolongs naturally to a section j^1g of the bundle $J^1\pi \to M$:

$$(h_{ij})_u = \frac{\partial \tilde{h}_{ij}}{\partial u}(u,x,v), \quad (h_{ij})_{x^k} = \frac{\partial \tilde{h}_{ij}}{\partial x^k}(u,x,v), \quad \cdots, \quad H_v = \frac{\partial \tilde{H}}{\partial v}(u,x,v).$$

In a similar way g prolongs to a section j^kg of the bundle $J^k\pi \to M$.

We are not interested in arbitrary sections, but in sections satisfying certain differential equations.

The PDEs

- For Kundt spacetimes we have

$$\mathcal{E}^1 = \{(h_{ij})_v = 0\} \subset J^1\pi.$$

  Its prolongation to $J^2\pi$ is given by

$$\mathcal{E}^2 = \{(h_{ij})_v = 0, (h_{ij})_{uv} = 0, (h_{ij})_{x^k v} = 0, (h_{ij})_{vv} = 0\}.$$

  In a similar way we define $\mathcal{E}^k \subset J^k\pi$.
- For degenerate Kundt spacetimes we have a sub-PDE $\tilde{\mathcal{E}}^k \subset \mathcal{E}^k \subset J^k\pi$ defined by

$$\tilde{\mathcal{E}}^1 = \mathcal{E}^1, \qquad \tilde{\mathcal{E}}^2 = \mathcal{E}^2 \cap \{(W_i)_{vv} = 0\},$$
$$\tilde{\mathcal{E}}^3 = \mathcal{E}^3 \cap \{(W_i)_{vv} = 0, (W_i)_{vvv} = 0, H_{vvv} = 0\}.$$

  For $k > 3$, we define $\tilde{\mathcal{E}}^k \subset J^k\pi$ as the prolongation of $\tilde{\mathcal{E}}^3$.

The PDEs

If g is a section of π, then the section $j^k g$ of $J^k\pi \to M$ is contained in $\mathcal{E}^k$ if and only if g is a Kundt metric of the form (1). (The same statement holds for $\tilde{\mathcal{E}}^k$ and degenerate Kundt metrics.)

As explained above, any vector field X in $\mathfrak{g}$ prolongs to a vector field $\hat{X}^{(k)}$ on $J^k\pi$. The prolonged vector field $\hat{X}^{(k)}$ is tangent to $\mathcal{E}^k$ and $\tilde{\mathcal{E}}^k$ for every k. In other words, the Lie algebra $\mathfrak{g}$ consists of symmetries $\mathcal{E}$ and $\tilde{\mathcal{E}}$.

We would like to distinguish sections of π satisfying $\mathcal{E}$ or $\tilde{\mathcal{E}}$ under the equivalence relation given by the Lie algebra $\mathfrak{g}$ (or the corresponding Lie pseudogroup). Both of these are of infinite dimension. However, each manifold $\mathcal{E}^k, \tilde{\mathcal{E}}^k \subset J^k\pi$ is of finite dimension, and so are $\mathfrak{g}$-orbits on these.

Differential invariants

Definition

A differential invariant of order k is a function I on $\mathcal{E}^k$ (or $\tilde{\mathcal{E}}^k$) which is constant on $\mathfrak{g}$-orbits.

- The differential invariants are solutions to the system

$$\hat{X}^{(k)}(I) = 0, \qquad X \in \mathfrak{g}.$$

The rational differential invariants of order k form a field whose transcendence degree is equal to the codimension of a generic orbit in $\mathcal{E}^k$ (or $\tilde{\mathcal{E}}^k$). The following statement follows from the global version of the Lie-Tresse theorem (Kruglikov, Lychagin 2016).

Theorem

The algebra of rational differential invariants separates orbits in general position in $\mathcal{E}^\infty$ and $\tilde{\mathcal{E}}^\infty$. It is generated by a finite number of differential invariants and invariant derivations.

Distinguishing metrics

A differential invariant I is a function on $\mathcal{E}^k$. By restricting it to a section g of π, we obtain a function on M:

$$I_g = I \circ j^k g.$$

Now assume that we have n invariants $I^1, ..., I^n$ that are independent when restricted to g, i.e.

$$dI_g^1 \wedge \cdots \wedge dI_g^n \neq 0.$$

Then $I_g^1, ..., I_g^n$ can be used as coordinates on M, and we can write g in these coordinates:

$$g = K_{ij}(I_g^1, ..., I_g^n) dI_g^i dI_g^j.$$

Two metrics that are written in these invariant coordinates can be compared directly. They are equivalent if and only if the functions $K_{ij}(I_g^1, ..., I_g^n)$ are equal.

Invariants for general Kundt metrics

For generic Kundt metrics we may take the invariants

$$I^i = \mathrm{Tr}(\mathrm{Ric}^i)$$

for $i = 1, \ldots, n$, where Ric is the Ricci operator. We have

$$\hat{d}I^1 \wedge \cdots \wedge \hat{d}I^n \neq 0$$

on a Zariski open set in $\mathcal{E}^3$. Here $\hat{d}$ is the horizontal differential. It is defined on a function $f \in C^\infty(\mathcal{E}^k)$ by $\hat{d}f \circ j^k g = d(f \circ j^k g)$, or in coordinates by

$$\hat{d}f = D_u(f)du + D_{x^i}(f)dx^i + D_v(f)dv.$$

On this Zariski open set, $\hat{d}I^1, \ldots, \hat{d}I^n$ form an invariant horizontal coframe. There is also a dual frame of invariant derivations $\hat{\partial}_i = AD_u + B_i^j D_{x^j} + CD_v$, defined by $\hat{d}I^i(\hat{\partial}_j) = \delta^i_j$.

Invariants for general Kundt metrics

We can express the horizontal symmetric form

$$G = du\left(dv + H\,du + W_i\,dx^i\right) + h_{ij}\,dx^i dx^j.$$

as

$$G = K_{ij}\hat{d}I^i\hat{d}I^j,$$

where $K_{ij} = G(\hat{\partial}_i, \hat{\partial}_j)$ are rational differential invariants of order 3.

Theorem

The algebra of rational differential invariants is generated by I^i, K_{ij} and the invariant derivations $\hat{\partial}_i$.

Invariants for degenerate Kundt metrics

The above approach does not work for degenerate Kundt metrics, because for these metrics we have

$$\hat{d}I^1 \wedge \cdots \wedge \hat{d}I^n = 0.$$

In general, only $n-1$ of these invariants are horizontally independent (in our adapted coordinates, we have $D_v(I^i) = 0$ for all of them).

For $n-1$ horizontally independent invariants $J^1, ..., J^{n-1}$ from the above set, let $\nabla_1, ..., \nabla_{n-1}$ be the G-duals to $\hat{d}J^1, ..., \hat{d}J^{n-1}$. The invariants can be chosen such that $\nabla_2, ..., \nabla_{n-1}$ are spacelike vector fields (after restriction to a degenerate Kundt metric), making the matrix $[G(\nabla_i, \nabla_j)]_{i=2}^{n-1}$ positive definite.

Define the nth derivation by

$$G(\nabla_1, \nabla_n) = 1, \qquad G(\nabla_i, \nabla_n) = 0 \text{ for } i = 2, ..., n.$$

Invariants for degenerate Kundt metrics

Let $\omega^1, ..., \omega^n$ denote the horizontal coframe dual to $\nabla_1, ..., \nabla_n$, defined by $\omega^i(\nabla_j) = \delta^i_j$. Then we have

$$G = L_{ij}\omega^i\omega^j$$

where $L_{ij} = G(\nabla_i, \nabla_j)$ are differential invariants of order 3. In this case, we have nontrivial commutation relations $[\nabla_i, \nabla_j] = c^k_{ij}\nabla_k$, where c^k_{ij} are differential invariants of order 3.

Theorem

The algebra of differential invariants is generated by the differential invariants L_{ij}, c^k_{ij} and the invariant derivations ∇_i.

Other choices of generators of invariants

The above approach is very flexible regarding the choice of n invariants or n invariant derivations. All we require is that $\hat{d}I^1 \wedge \cdots \wedge \hat{d}I^n$ is nonvanishing, or that $\nabla_1, ..., \nabla_n$ are independent, on generic points.

For any choice of dimension n, exactly one differential invariant of order 1:

Theorem

There is one algebraically independent invariant of first order, and it is given by

$$I_1 = (W_i)_v (W_j)_v h^{ij}.$$

Note that the matrix $[h^{ij}]$ is the inverse to $[h_{ij}]$. In particular, we have the determinant of the latter in the denominator of I_1.

We will now focus on the low dimensions, $n = 3$ and $n = 4$, and find a different set of generators.

Three-dimensional Kundt spacetimes

We simplify our notation and use coordinates u, x, v, h, W, H on $F \times M$. Let us start by giving an invariant horizontal frame.

Theorem

The derivations

$$\nabla_1 = \frac{W_v}{W_{vv}} D_v, \qquad \nabla_2 = \frac{2}{W_v} D_x + \frac{h_x W_v - 2hW_{xv}}{hW_v W_{vv}} D_v,$$
$$\nabla_3 = \frac{1}{W_v} \left(H_{vv} D_x - W_{vv} D_u + (W_{uv} - H_{xv}) D_v \right)$$

are invariant, and they are independent on a Zariski open subset of $\mathcal{E}^2$.

The derivations satisfy $[\hat{X}^{(\infty)}, \nabla_i] = 0$ for each $X \in \mathfrak{g}$, and they were found by solving this system of PDEs.

Three-dimensional Kundt spacetimes

If we let α^j denote the elements of the dual horizontal coframe ($\alpha^j(\nabla_i) = \delta_i^j$), then the horizontal symmetric 2-form G written in terms of this coframe will have coefficients given by $G(\nabla_i, \nabla_j)$. It takes the form

$$G = I_1^{-1}\left((J_1\alpha^3 + J_2\alpha^2 - I_1\alpha^1)\alpha^3 + 4(\alpha^2)^2\right)$$

where $I_1 = \frac{W_v^2}{h}$ is the first-order differential invariant from the previous slide and

$$J_1 = \frac{HW_{vv}^2 + (-H_{vv}W + H_{xv} - W_{uv})W_{vv} + H_{vv}^2 h}{h},$$
$$J_2 = \frac{4H_{vv}h^2 + 2(W_{xv} - WW_{vv})h - W_v h_x}{h^2}$$

are second-order differential invariants. We have $\hat{d}I_1 \wedge \hat{d}J_1 \wedge \hat{d}J_2 \neq 0$.

Theorem

For $n = 3$ the algebra of differential invariants on $\mathcal{E}$ is generated by the differential invariants I_1, J_1, J_2 and the invariant derivations $\nabla_1, \nabla_2, \nabla_3$.

Three-dimensional Kundt spacetimes

We also have the second-order differential invariants

$$\begin{aligned}
\nabla_3(I_1) &= 2\,\frac{H_{vv}W_{xv} - H_{xv}W_{vv}}{h} - \frac{W_v(H_{vv}h_x - W_{vv}h_u)}{h^2},\\
J_3 &= \frac{W_{vv}^2(h_u^2 - 2hh_{uu})}{h^3} - \frac{2W_{vv}(H_vW_{vv} - H_{vv}W_v)h_u}{h^2}\\
&\quad - \frac{((H_vW - H_x + W_u)W_{vv}^2 - W_v(H_{vv}W - H_{xv} + W_{uv})W_{vv} + 2H_{vv}^2hW_v)h_x}{h^3}\\
&\quad + \frac{(-2H_xW_v + 2H_vW_x + 2H_{xv}W - 2H_{xx} + 2W_{ux})W_{vv}^2}{h^2}\\
&\quad + \frac{((-2H_{vv}W + 2H_{xv} - 2W_{uv})W_{xv} - 4H_{xv}H_{vv}h)W_{vv} + 4H_{vv}^2hW_{xv}}{h^2}
\end{aligned}$$

which, together with I_1, J_1, J_2, constitute a transcendence basis for the field of second-order differential invariants on $\mathcal{E}^2$.

Three-dimensional degenerate Kundt spacetimes

Let us now consider degenerate kundt metrics. We still have the invariant $I_1 = \frac{W_v^2}{h}$.
We define the following functions:

$$I_{2a} = H_{vv}, \quad I_{2b} = \frac{W_v h_x - 2hW_{xv}}{h^2}, \quad K_{2a} = \frac{H_{xv} - W_{uv}}{W}, \quad K_{2b} = \frac{W_v h_u - 2hW_{uv}}{Wh}.$$

The functions I_{2a} and I_{2b} are second-order differential invariants on $\tilde{\mathcal{E}}^2$. We also define

$$Q = \frac{(2I_{2a}K_{2b} + I_{2b}K_{2a} - I_{2a}I_{2b})W}{I_1},$$
$$R = \frac{I_{2b}HW_v^2}{I_1} - \frac{(I_{2b}I_{2a}^2 - 2K_{2a}(I_{2b} - 2K_{2b})I_{2a} + I_{2b}K_{2a}^2)W^2}{4I_{2a}^2}.$$

A fourth second-order differential invariant is given by

$$I_{2c} = \frac{1}{Q^2}\Big(\frac{(I_1^2 h_u(WW_v + h_u) - (h_x(H_vW - H_x + W_u)I_1^2 - W_v^4 I_{2b}H))I_{2a}I_{2b}}{W_v^2}$$
$$-2\left(W_vH_x + (h_u - W_x)H_v + H_{xx} - W_{ux} + h_{uu}\right)I_1I_{2a}I_{2b}$$
$$-W^2I_1(K_{2b}(I_{2b} - K_{2b})I_{2a} - 2I_{2b}K_{2a}^2)\Big).$$

Three-dimensional degenerate Kundt spacetimes

We have $\hat{d}I_1 \wedge \hat{d}I_{2a} \wedge \hat{d}I_{2c} \neq 0$ on a Zariski open set in $\tilde{\mathcal{E}}^3$. It is possible to express G in terms of $\hat{d}I_1, \hat{d}I_{2a}, \hat{d}I_{2c}$ as before, and in this way find a generating set of invariants.

Alternatively, we may use the following invariant derivations:

$$\nabla_1 = \frac{I_1}{I_{2a}I_{2b}} \cdot \frac{Q}{W_v} D_v, \qquad \nabla_2 = \frac{1}{W_v}\left(D_x - \frac{K_{2a}}{I_{2a}} W D_v\right),$$
$$\nabla_3 = \frac{2I_{2a}}{I_1} \cdot \frac{1}{QW_v}\left(K_{2b} W D_x - I_{2b} h D_u + R D_v\right)$$

Theorem

The algebra of differential invariants on $\tilde{\mathcal{E}}$ is generated by the differential invariants I_1, I_{2a}, I_{2c} and the invariant derivations $\nabla_1, \nabla_2, \nabla_3$.

Four-dimensional degenerate Kundt spacetimes

We have a single first-order invariant:

$$I_1 = \frac{(W_1)_v^2 h_{22} - 2(W_1)_v (W_2)_v h_{12} + (W_2)_v^2 h_{11}}{h_{11}h_{22} - h_{12}^2}.$$

There are 12 additional algebraically independent invariants of order 2. Here are two of them:

$$I_{2a} = H_{vv}, \qquad I_{2b} = \frac{((W_1)_{x^2 v} - (W_2)_{x^1 v})^2}{h_{11}h_{22} - h_{12}^2}.$$

These three invariants are horizontally independent:

$$\hat{d}I_1 \wedge \hat{d}I_{2a} \wedge \hat{d}I_{2b} \neq 0.$$

We also have $D_v(I_1) = D_v(I_{2a}) = D_v(I_{2b}) = 0$. We need one invariant I with $D_v(I) \neq 0$. In order to find this, we construct a horizontal frame.

Four-dimensional degenerate Kundt spacetimes

The horizontal 1-forms $\hat{d}I_1, \hat{d}I_{2a}, \hat{d}I_{2b}$ are independent. By taking an appropriate linear combination

$$\hat{d}I_1 + a_1\hat{d}I_{2a} + a_2\hat{d}I_{2b}$$

where a_i are rational functions on $\tilde{\mathcal{E}}^3$, we obtain an invariant horizontal form which is proportional to du. We turn it into a horizontal vector field by using G and denote the resulting invariant derivation, which is proportional to D_v, by ∇_1. We have three derivations:

$$\nabla_1, \qquad \nabla_2 = G^{-1}\hat{d}I_{2a}, \qquad \nabla_3 = G^{-1}\hat{d}I_{2b}.$$

We complete the horizontal frame by requiring ∇_4 to satisfy

$$G(\nabla_1, \nabla_4) = 1, \qquad G(\nabla_2, \nabla_4) = 0, \qquad G(\nabla_3, \nabla_4) = 0, \qquad G(\nabla_4, \nabla_4) = 0.$$

Theorem

The derivations $\nabla_1, \nabla_2, \nabla_3, \nabla_4$ are invariant, and they are independent on a Zariski open subset of $\tilde{\mathcal{E}}^3$.

Four-dimensional degenerate Kundt spacetimes

Let us end by finding a fourth absolute invariant which is independent from I_1, I_{2a}, I_{2b}. The invariants $G(\nabla_i, \nabla_j)$ satisfy $D_v(G(\nabla_i, \nabla_j)) = 0$. We have commutation relations

$$[\nabla_i, \nabla_j] = c_{ij}^k \nabla_k$$

where c_{ij}^k are rational differential invariants. Some of these depend on v for generic degenerate Kundt spacetimes. We have, for example $D_v(c_{23}^1) \neq 0$.

Theorem

The differential invariants $I_1, I_{2a}, I_{2b}, c_{23}^1$ are horizontally independent.

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