# The Navier-Stokes system on a space curve


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# The Navier-Stokes system

Consider flows of an viscid medium

$$\begin{cases} \rho(u_t + uu_a) + p_a - \zeta u_{aa} - \rho g h' = 0, \\ \rho_t + (\rho u)_a = 0, \\ \rho T(s_t + us_a) - kT_{aa} - \zeta u_a^2 = 0. \end{cases} \tag{1}$$

on a naturally-parametrised space curve in the three-dimensional Euclidean space

$$M = \{x = f(a),\ y = g(a),\ z = h(a)\}.$$

in a field of constant gravitational force.

# Thermodynamics

A thermodynamic state is a two-dimensional Legendrian manifold $L \subset \mathbb{R}^5(p, \rho, s, T, \epsilon)$, a maximal integral manifold of the differential 1-form

$$\theta = d\epsilon - T ds - p\rho^{-2} d\rho,$$

i.e. a manifold such that the first law of thermodynamics $\theta\big|_L = 0$ holds.

Following [1], we require that the quadratic differential form

$$\kappa = d(T^{-1}) \cdot d\epsilon - \rho^{-2} d(pT^{-1}) \cdot d\rho$$

on the surface $L$ be negative definite,

$$\kappa\big|_L < 0,$$

and the entropy $s$ satisfies the inequality $s \leq s_0$, where the constant $s_0$ depends on the nature of a process under consideration.

# Thermodynamics

Consider the projection $\pi\colon\ (p, \rho, s, T, \epsilon) \longmapsto (p, \rho, s, T)$. The restriction of this map on the state surface $L$ is a diffeomorphism $\bar{L} = \pi(L)$, and the surface $\bar{L} \subset \mathbb{R}^4$ is a Lagrangian manifold in the 4-dimensional symplectic space $\mathbb{R}^4$ equipped with the structure form

$$\Omega = d\theta = ds \wedge dT + \rho^{-2} d\rho \wedge dp.$$

Thus, the *thermodynamic state* is the Lagrangian submanifold $\bar{L}$ in the symplectic space $(\mathbb{R}^4, \Omega)$:

$$\begin{cases} f(p, \rho, s, T) = 0, \\ g(p, \rho, s, T) = 0, \end{cases} \quad \text{and} \quad [f, g]|_{\bar{L}} = 0, \tag{2}$$

where $[f, g]$ is the Poisson bracket, and the symmetric differential form $\kappa$ is negative definite on this surface.

# Symmetry Lie algebra

We consider a Lie algebra $\mathfrak{g}$ of point symmetries of the Navier-Stokes system (1).
Let $\vartheta\colon \mathfrak{g} \to \mathfrak{h}$ be the following Lie algebras homomorphism

$$\vartheta\colon X \mapsto X(\rho)\partial_\rho + X(s)\partial_s + X(p)\partial_p + X(T)\partial_T,$$

where $\mathfrak{h}$ is a Lie algebra generated by vector fields that act on the thermodynamic variables $p$, $\rho$, $s$ and $T$.
The kernel of the homomorphism $\vartheta$ is an ideal $\mathfrak{g}_{\mathfrak{m}} \subset \mathfrak{g}$ (geometric symmetries).
Let also $\mathfrak{h}_{\mathfrak{t}}$ be the Lie subalgebra of the algebra $\mathfrak{h}$ that preserves thermodynamic state (2).

## Theorem

*A Lie algebra $\mathfrak{g}_{\mathfrak{sym}}$ of symmetries of the Navier-Stokes system $\mathcal{E}$ coincides with*

$$\vartheta^{-1}(\mathfrak{h}_{\mathfrak{t}}).$$

# Symmetry Lie algebra

- $h(a)$ is arbitrary

$$X_1 = \partial_t, \qquad X_2 = \partial_p, \qquad X_3 = \partial_s$$

- $h(a) = const$

$$\begin{aligned} &X_1 = \partial_t, \qquad X_2 = \partial_p, \qquad X_3 = \partial_s, \\ &X_4 = \partial_a, \qquad X_5 = t\,\partial_a + \partial_u, \\ &X_6 = t\,\partial_t + a\,\partial_a - p\,\partial_p - \rho\,\partial_\rho, \\ &X_7 = a\,\partial_a + u\,\partial_u - 2\rho\,\partial_\rho + 2T\,\partial_T \end{aligned}$$

- $h(a) = \lambda a$, $\lambda \neq 0$

$$\begin{aligned} &X_1 = \partial_t, \qquad X_2 = \partial_p, \qquad X_3 = \partial_s, \\ &X_4 = \partial_a, \qquad X_5 = t\,\partial_a + \partial_u, \\ &X_6 = t\,\partial_t + 2a\,\partial_a + u\,\partial_u - p\,\partial_p - 3\rho\,\partial_\rho + 2T\partial_T, \\ &X_7 = t\,\partial_t + \left(\frac{\lambda g t^2}{2} + a\right)\partial_a + \lambda g t\,\partial_u - p\,\partial_p - \rho\,\partial_\rho \end{aligned}$$

# Symmetry Lie algebra

- $h(a) = \lambda a^2,\ \lambda \neq 0$
$$X_1 = \partial_t, \qquad X_2 = \partial_p, \qquad X_3 = \partial_s,$$
$$X_4 = \sin(\sqrt{2\lambda g}\, t)\,\partial_a + \sqrt{2\lambda g}\cos(\sqrt{2\lambda g}\, t)\,\partial_u,$$
$$X_5 = \cos(\sqrt{2\lambda g}\, t)\,\partial_a - \sqrt{2\lambda g}\sin(\sqrt{2\lambda g}\, t)\,\partial_u,$$
$$X_6 = a\,\partial_a + u\,\partial_u - 2\rho\,\partial_\rho + 2T\partial_T$$
- $h(a) = \lambda_1 a^{\lambda_2},\ \lambda_2 \neq 0, 1, 2$
$$X_1 = \partial_t, \qquad X_2 = \partial_p, \qquad X_3 = \partial_s,$$
$$X_4 = t\,\partial_t - \frac{2a}{\lambda_2 - 2}\partial_a - \frac{\lambda_2 u}{\lambda_2 - 2}\partial_u - p\,\partial_p + \frac{\lambda_2 + 2}{\lambda_2 - 2}\rho\,\partial_\rho - \frac{2\lambda_2}{\lambda_2 - 2}\,T\,\partial_T$$
- $h(a) = \lambda_1 e^{\lambda_2 a},\ \lambda_2 \neq 0$
$$X_1 = \partial_t, \qquad X_2 = \partial_p, \qquad X_3 = \partial_s,$$
$$X_4 = t\,\partial_t - \frac{2}{\lambda_2}\,\partial_a - u\,\partial_u - p\,\partial_p + \rho\,\partial_\rho - 2T\partial_T$$
- $h(a) = \ln a$
$$X_1 = \partial_t, \qquad X_2 = \partial_p, \qquad X_3 = \partial_s,$$
$$X_4 = t\,\partial_t + a\,\partial_a - p\,\partial_p - \rho\,\partial_\rho$$

# Symmetry Lie algebra

Let $h(a)$ be an arbitrary function.

The Lie algebra $\mathfrak{g}$ of point symmetries of the system (1) is generated by the vector fields

$$X_1 = \partial_t, \qquad X_2 = \partial_p, \qquad X_3 = \partial_s.$$

The pure thermodynamic part $\mathfrak{h}$ of the system symmetry algebra is

$$Y_1 = \partial_p, \qquad Y_2 = \partial_s.$$

Thus, in this case the system of differential equations $\mathcal{E}$ has the smallest Lie algebra of point symmetries $\vartheta^{-1}(\mathfrak{h}_\mathfrak{t})$.

# Lifting curves from the plane

Let a curve in the space be defined as a pair of a plane curve $(x(\tau), y(\tau))$ and a 'lifting' function $z(\tau)$.
Let $l(\tau) = \int\limits_0^\tau \sqrt{x_\theta^2 + y_\theta^2}\, d\theta$ – the length of the plane curve.
Then the following relation between natural parameter $a$ and the parameter $\tau$ is valid

$$h_a = \frac{z_\tau}{\sqrt{x_\tau^2 + y_\tau^2 + z_\tau^2}}. \tag{3}$$

**1.** $h(a) = const$
The first way of lifting a plane curve is to translate the whole curve along $z$-axis, i.e. if $h(a) = const$ then $z(\tau) = const$.

# Lifting curves from the plane

2. $h(a) = \lambda a$, $\lambda \neq 0$

The second way to lift curve is lifting proportional to the length of the plane part, i.e. if $h(a) = \lambda a$ then we have the following differential equitation on the 'lifting' function $z(\tau)$

$$\left(1 - \lambda^2\right) z_\tau^2 = \lambda^2 \left(x_\tau^2 + y_\tau^2\right),$$

solving which given $1 - \lambda^2 > 0$, we get

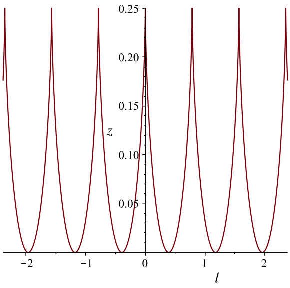
$$z(\tau) = \pm \frac{\lambda}{\sqrt{1 - \lambda^2}}\, l(\tau) + C,$$

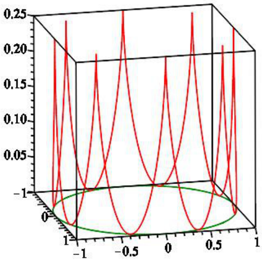
where $l(\tau)$ – is length of plane projection of curve and $C = const$. If $\lambda = \pm 1$, then $x(t) = y(t) = const$ and we have a vertical line.

# Lifting curves from the plane

**3.** $h(a) = \lambda a^2$, $\lambda \neq 0$
The relation between the 'lifting' function $z(\tau)$ and the length $l(\tau)$ of the plane curve is

$$\sqrt{4\lambda z(1 - 4\lambda z)} - \arccos(\sqrt{4\lambda z}) = \pm 4\lambda l(\tau).$$



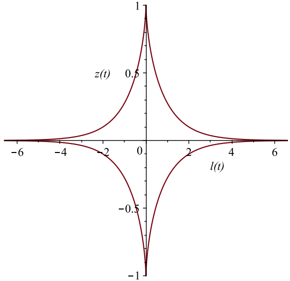



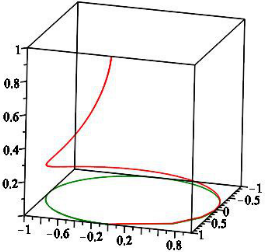
# Lifting curves from the plane

**5.** $h(a) = \lambda_1 e^{\lambda_2 a}$

The relation between the 'lifting' function $z(\tau)$ and the length $l(\tau)$ of the plane curve is

$$\sqrt{1-\lambda_2^2 z^2} - \frac{1}{2}\ln\frac{1+\sqrt{1-\lambda_2^2 z^2}}{1-\sqrt{1-\lambda_2^2 z^2}} = \pm\lambda_2 l(\tau).$$

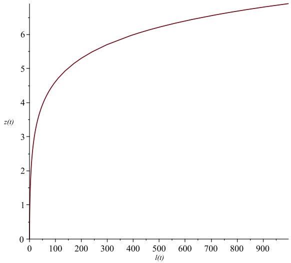


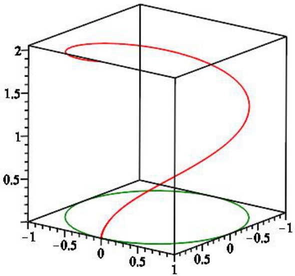



# Lifting curves from the plane

**6.** $h(a) = \ln a$
The relation between the 'lifting' function $z(\tau)$ and the length $l(\tau)$ of the plane curve is

$$\sqrt{e^{2z} - 1} - \arctan\sqrt{e^{2z} - 1} = \pm l(\tau).$$



# Thermodynamic states

Let $h(a)$ be a quadratic function $h(a) = \lambda a^2$, and let the thermodynamic state admit a one-dimensional symmetry algebra

$$Z = \gamma_1 \partial_p + \gamma_2 \partial_s + \gamma_3 (\rho\, \partial_\rho - T\, \partial_T),$$

then the Lagrangian surface $\bar{L}$ can be found from the conditions

$$\begin{cases} \Omega|_{\bar{L}} = 0, \\ (\iota_Z \Omega)|_{\bar{L}} = 0, \end{cases} \tag{4}$$

which lead to the following PDE system on the internal energy

$$\begin{cases} \gamma_4 \rho\, \epsilon_{\rho\rho} + (\gamma_2 - \gamma_4 s)\epsilon_{\rho s} + \gamma_4 \epsilon_\rho - \gamma_1 \rho^{-2} = 0, \\ (\gamma_2 - \gamma_4 s)\epsilon_{ss} + \gamma_4 \rho\, \epsilon_{\rho s} - \gamma_3\, \epsilon_s = 0. \end{cases}$$

# Thermodynamic states

**Theorem**

*The thermodynamic states admitting a one-dimensional symmetry algebra for the case $h(a) = \lambda a^2$ have the form*

$$p = \frac{\gamma_2}{\gamma_3}F' - F - \frac{\gamma_1}{\gamma_3}(\ln\rho - 1), \quad T = \frac{F'}{\rho}, \quad F = F\left(s + \frac{\gamma_2}{\gamma_3}\ln\rho\right),$$

*where $F$ is a function that satisfies the following inequalities:*

$$F' > 0, \quad F'' > 0, \quad \frac{(\gamma_2 F' - \gamma_1)F''}{\gamma_3} - F'^2 > 0.$$

# Thermodynamic states

Let $h(a) = const$ or $h(a) = \lambda a$. The thermodynamic states admitting a one-dimensional symmetry algebra have the form

$$T = \rho^{\frac{\lambda_4}{\lambda_3}-1}F', \quad p = \rho^{\frac{\lambda_4}{\lambda_3}}\left(\left(\frac{\lambda_4}{\lambda_3}-1\right)F - \frac{\lambda_1}{\lambda_3}F'\right) - \frac{\lambda_2}{\lambda_4},$$

where $F = F\left(s - \frac{\lambda_1}{\lambda_3}\ln\rho\right)$ is a smooth function, $F'$ is positive and

$$\lambda_1^2 F'' + \lambda_1(\lambda_3 - 2\lambda_4)F' + \lambda_4(\lambda_4 - \lambda_3)F > 0,$$
$$F''(\lambda_4(\lambda_4 - \lambda_3)F - \lambda_1\lambda_3 F') - (F')^2(\lambda_4 - \lambda_3)^2 > 0.$$

The thermodynamic states admitting a two-dimensional commutative symmetry algebra have the form

$$p = C(\beta - 1)e^{\alpha s}\rho^{\beta} - \frac{\beta_2}{\beta_4}, \quad T = C\alpha e^{\alpha s}\rho^{\beta-1},$$

where

$$\alpha = \frac{\alpha_4\beta_3 - \alpha_3\beta_4}{\alpha_1\beta_3 - \alpha_3\beta_1} > 0, \quad \beta = \frac{\alpha_1\beta_4 - \beta_1\alpha_4}{\alpha_1\beta_3 - \alpha_3\beta_1} > 1, \quad C > 0, \quad \frac{\beta_2}{\beta_4} < 0.$$

# Differential invariants

We consider two group actions on the Navier–Stokes system $\mathcal{E}$ – the prolonged actions of the groups generated by actions of the Lie algebras $\mathfrak{g}_{\mathfrak{m}}$ and $\mathfrak{g}_{\mathfrak{sym}}$.
A function $J$ on the manifold $\mathcal{E}_k$ is a *kinematic differential invariant of order* $\leq k$ if

1. $J$ is a rational function along fibers of the projection $\pi_{k,0} : \mathcal{E}_k \to \mathcal{E}_0$,
2. $J$ is invariant with respect to the prolonged action of the Lie algebra $\mathfrak{g}_{\mathfrak{m}}$, i.e., for all $X \in \mathfrak{g}_{\mathfrak{m}}$,

$$X^{(k)}(J) = 0, \tag{5}$$

where $\mathcal{E}_k$ is the prolongation of the system $\mathcal{E}$ to $k$-jets, and $X^{(k)}$ is the $k$-th prolongation of a vector field $X \in \mathfrak{g}_{\mathfrak{m}}$.
A kinematic invariant is *a Navier–Stokes invariant* if condition (5) holds for all $X \in \mathfrak{g}_{\mathfrak{sym}}$.

# Kinematic invariants

## Theorem

- *The kinematic invariants field is generated by first-order basis differential invariants and by basis invariant derivations. This field separates regular orbits.*
- *The number of independent invariants of pure order $k$ is equal to 5 for $k \geq 1$.*
- *For the general cases of $h(a)$, as well as for $h(a) = \lambda_1 a^{\lambda_2}$, $h(a) = \lambda_1 e^{\lambda_2 a}$ and $h(a) = \ln a$, the basis differential invariants are*

$$a, \quad u, \quad \rho, \quad s, \quad u_t, \quad u_a, \quad \rho_a, \quad s_t, \quad s_a,$$

*and the basis invariant derivatives are*

$$\frac{\mathrm{d}}{\mathrm{d}t}, \quad \frac{\mathrm{d}}{\mathrm{d}a}.$$

# Kinematic invariants

## Theorem

- *For the cases $h(a) = const$, $h(a) = \lambda a$ the basis differential invariants are*

$$\rho, \quad s, \quad u_a, \quad u_t + u u_a, \quad \rho_a, \quad s_a, \quad s_t + u s_a,$$

*and basis invariant derivatives are*

$$\frac{\mathrm{d}}{\mathrm{d}t} + u \frac{\mathrm{d}}{\mathrm{d}a}, \quad \frac{\mathrm{d}}{\mathrm{d}a}.$$

- *For the case $h(a) = \lambda a^2$ the basis differential invariants are*

$$\rho, \quad s, \quad u_a, \quad u_t + u u_a - 2\lambda g a, \quad \rho_a, \quad s_a, \quad s_t + u s_a,$$

*and basis invariant derivatives are*

$$\frac{\mathrm{d}}{\mathrm{d}t} + u \frac{\mathrm{d}}{\mathrm{d}a}, \quad \frac{\mathrm{d}}{\mathrm{d}a}.$$

# Navier–Stokes invariants

Let $h(a) = \lambda a$, $\lambda \neq 0$. If the thermodynamic state admits a one-dimensional symmetry algebra generated by the vector field

$$\xi_1\partial_p + \xi_2\partial_s + \xi_3(t\,\partial_t + 2a\,\partial_a + u\,\partial_u - p\,\partial_p - 3\rho\,\partial_\rho + 2T\partial_T) + \xi_4\left(t\,\partial_t + \left(\frac{\lambda g t^2}{2} + a\right)\partial_a + \lambda g t\,\partial_u - p\,\partial_p - \rho\,\partial_\rho\right),$$

then the field of Navier–Stokes differential invariants is generated by the differential invariants

$$s + \frac{\xi_2}{3\xi_3 + \xi_4}\ln\rho, \quad u_a\rho^{-\frac{\xi_3+\xi_4}{3\xi_3+\xi_4}}, \quad \rho_a\rho^{\frac{\xi_3}{3\xi_3+\xi_4}-2},$$
$$\frac{\rho^2(u_t + uu_a - \lambda g)}{\rho_a u_a}, \quad \frac{\rho s_a}{\rho_a}, \quad \frac{s_t + us_a}{u_a}$$

of the first order and by the invariant derivations

$$\rho^{-\frac{\xi_3+\xi_4}{3\xi_3+\xi_4}}\left(\frac{\mathrm{d}}{\mathrm{d}t} + u\frac{\mathrm{d}}{\mathrm{d}a}\right), \quad \rho^{-\frac{2\xi_3+\xi_4}{3\xi_3+\xi_4}}\frac{\mathrm{d}}{\mathrm{d}a}.$$

# Navier–Stokes invariants

Let $h(a) = \lambda a^2$, $\lambda \neq 0$.
If the thermodynamic state admits a one-dimensional symmetry algebra generated by the vector field

$$\xi_1 \partial_p + \xi_2 \partial_s + \xi_3 (a\,\partial_a + u\,\partial_u - 2\rho\,\partial_\rho + 2T\partial_T),$$

then the field of Navier–Stokes differential invariants is generated by the differential invariants

$$s + \frac{\xi_2}{2\xi_3}\ln\rho, \quad u_a, \quad \rho(u_t + uu_a - 2\lambda g a)^2, \quad \frac{\rho_a^2}{\rho^3}, \quad \frac{s_a^2}{\rho}, \quad s_t + us_a$$

of the first order and by the invariant derivations

$$\frac{\mathrm{d}}{\mathrm{d}t} + u\frac{\mathrm{d}}{\mathrm{d}a}, \quad \rho^{-\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}a}.$$

# Literature

V. Lychagin, *Contact Geometry, Measurement and Thermodynamics*, in: *Nonlinear PDEs, Their Geometry and Applications. Proceedings of the Wisla 18 Summer School*, Springer Nature, Switzerland, 3-54 (2019).

Thank you for attention.