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- Felipe Contatto, MD, \texttt{arXiv:1510.01906}.
Given an affine connection $\nabla$ on a surface $\Sigma$, determine necessary/sufficient local conditions (explicit curvature invariants) for the existence of first integrals.
Results

- Given an affine connection $\nabla$ on a surface $\Sigma$, determine necessary/sufficient local conditions (explicit curvature invariants) for the existence of first integrals.
- If $\nabla$ is a Levi–Civita connection, then there can exist 0, 1 or 3 linear first integrals. Understand the non-metric case with exactly two local linear first integrals.

Application (unexpected!): Given a one–dimensional system of hydrodynamic type in Riemann invariants, determine necessary/sufficient conditions for the existence of a Hamiltonian (bi–Hamiltonian, tri–Hamiltonian) formulation of Dubrovin-Novikov type. Examples: Zoll connections. Hamiltonian systems from two–dimensional Froebenius manifolds, . . .
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A simply-connected surface with a torsion–free affine connection \((\Sigma, \nabla)\) of differentiability class \(C^4\).
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Affinely parametrised geodesic $\gamma : \mathbb{R} \to \Sigma$, $\nabla \dot{\gamma} \gamma = 0$. Or in local coordinates $X^a$ on $U \subset \Sigma$

$$\ddot{X}^a + \Gamma^a_{bc} \dot{X}^b \dot{X}^c = 0, \quad a, b, c = 1, 2.$$
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Linear first integral: \(\kappa \equiv K_a(X) \dot{X}^a\) s.t. \(d\kappa/d\tau = 0\) along the geodesics. Equivalently

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\nabla_a K_b + \nabla_b K_a = 0. \quad (K).
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Prolong this system to a connection on a rank–3 vector bundle \(E \to \Sigma\). Find the integrability conditions for the existence of one/two/three parallel sections. Express them in terms of the curvature of \(\nabla\) and its covariant derivatives (of order up to 3).
**Prolongation connection**

- Curvature decomposition

\[
R_{ab}^{\hspace{1em}c} \hspace{1em} d = \delta_a^{\hspace{1em} c} P_{bd} - \delta_b^{\hspace{1em} c} P_{ad} + B_{ab} \delta_d^{\hspace{1em} c}.
\]

Schouten tensor \( P_{ab} = (2/3) R_{ab} + (1/3) R_{ba} \), and \( B_{ab} = -2 P_{[ab]} \). Set \( \beta = B_{ab} \epsilon^{ab} \) for an arbitrary volume form \( \epsilon \).
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**Proposition.** There is a one-to-one correspondence between solutions to the Killing equations \((K)\), and parallel sections of the prolongation connection \(D\) on a rank–3 vector bundle \( E = \Lambda^1(\Sigma) \oplus \Lambda^2(\Sigma) \rightarrow \Sigma \)

\[
D_a \left( \begin{array}{c} K_b \\ \mu \end{array} \right) = \left( \begin{array}{c} \nabla_a K_b - \epsilon_{ab} \mu \\ \nabla_a \mu - \left( P_b{}^a + \frac{1}{2} \beta \delta^b{}_a \right) K_b + \mu \theta_a \end{array} \right).
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**Prolongation connection**

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\]

- Compute the curvature of \( D \), restrict its holonomy so that parallel sections \( \Psi = (K_a, \mu) \) exist. Find obstructions.
Integrability conditions for $D\Psi = 0$: $\mathcal{F}\Psi = 0$ where $\mathcal{F} = [D, D]$. 

If $\mathcal{F} = 0$ then $\nabla$ is projectively flat. Otherwise differentiate:

$$(DF)_{\Psi} = 0,$$  
$$(DDF)_{\Psi} = 0,$$ ...

After $K$ steps $\mathcal{F}^K_{\Psi} = 0$, where $\mathcal{F}^K$ is a matrix of linear blue eqn.

Stop when $\text{rank}(\mathcal{F}^K) = \text{rank}(\mathcal{F}^{K+1})$.

The space of parallel sections has dimension $(3 - \text{rank}(\mathcal{F}^K))$.

Set $L_b \equiv \epsilon_{cd} \nabla^c P_{db}$ and define $F_a = \frac{1}{3} \epsilon_{ab} (L_b - \epsilon_{cd} \nabla^b B_{cd})$, $N_a = -F_a + \epsilon_{bc} \nabla^a B_{bc}$, $M_{ab} = \frac{1}{3} \epsilon_{bc} \epsilon_{de} (\nabla^a Y_{dec} - \nabla^a \nabla^c B_{de}) + \beta P_{ba} + \frac{1}{2} \beta^2 \delta_{ba}$, $I_{N} = \epsilon_{cd} \epsilon_{be} M_{ec} (N_b F_d - \frac{1}{2} \beta M_{bd})$. 

Dunajski (DAMTP, Cambridge) 
Affine connections, hydrodynamic integrability 
October 2015
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$$M_a^b = \frac{1}{3} \epsilon^{bc} \epsilon^{de}(\nabla_a Y_{dec} - \nabla_a \nabla_c B_{de}) + \beta P^b_a + \frac{1}{2} \beta^2 \delta^b_a,$$

$$I_N = \epsilon_{cd} \epsilon^{be} M_e^c \left( N_b F^d - \frac{1}{2} \beta M_b^d \right).$$
Theorem (Contatto, MD) The necessary condition for a $C^4$ torsion–free affine connection $\nabla$ on a surface $\Sigma$ to admit a linear first integral is the vanishing, on $\Sigma$, of invariants $I_N$ and $I_S$ respectively. For any point $p \in \Sigma$ there exists a neighbourhood $U \subset \Sigma$ of $p$ such that conditions $I_N = I_S = 0$ on $U$ are sufficient for the existence of a first integral on $U$. There exist precisely two independent linear first integrals on $U$ if and only if the tensor

$$T_a^b \equiv N_a F^b - \beta M_a^b.$$ 

vanishes and the skew part of the Ricci tensor of $\nabla$ is non–zero on $U$. There exist three independent first integrals on $U$ if and only if the connection is projectively flat and its Ricci tensor is symmetric.
If $\nabla$ is a Levi–Civita connection of some metric on $\Sigma$ with scalar curvature $R$, then (Darboux 1887)

$$I_N := \star \frac{1}{432} dR \wedge d(|\nabla R|^2), \quad I_S := \star dR \wedge d(\triangle R).$$
Connections with two first integrals

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- A Levi–Civita connection can not admit precisely two local first integrals. A non–metric connection can:

\textbf{Theorem (Contatto, MD).} Let $\nabla$ be an affine connection on a surface $\Sigma$ which admits exactly two non–proportional linear first integrals which are independent at some point $p \in \Sigma$. Coordinates $X^a = (X, Y)$ can be chosen on an open set $U \subset \Sigma$ containing $p$ such that

$$\Gamma_{12}^1 = \Gamma_{21}^1 = \frac{c}{2}, \quad \Gamma_{11}^2 = \frac{P_X}{Q}, \quad \Gamma_{12}^2 = \Gamma_{21}^2 = \frac{P_Y + Q_X - cP}{2Q}, \quad \Gamma_{22}^2 = \frac{Q_Y}{Q},$$

and all other components vanish, where $c$ is a constant equal to 0 or 1, and $(P, Q)$ are arbitrary functions of $(X, Y)$. 
One–dimensional systems of hydrodynamic type

\[ \frac{\partial X^1}{\partial t} = \lambda^1(X^1, X^2) \frac{\partial X^1}{\partial x}, \quad \frac{\partial X^2}{\partial t} = \lambda^2(X^1, X^2) \frac{\partial X^2}{\partial x}. \]  \((HT)\)
Hamiltonian Systems of Hydrodynamic Type

- One-dimensional systems of hydrodynamic type
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  \( (HT) \)

- Local hydrodynamic Hamiltonian formulation
  \[ \frac{\partial X^a}{\partial t} = \Omega^{ab} \frac{\delta H}{\delta X^b}, \]
  where
  \[ H[X^1, X^2] = \int_{\mathbb{R}} \mathcal{H}(X^1, X^2) dx, \quad \Omega^{ab} = g^{ab} \frac{\partial}{\partial x} + b^a_c \frac{\partial X^c}{\partial x}. \]
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- Poisson bracket \( \{ F, G \} = \int_{\mathbb{R}} \frac{\delta F}{\delta X^a} \left( g^{ab} \frac{\partial}{\partial x} + b^{ab}_c \frac{\partial X^c}{\partial x} \right) \frac{\delta G}{\delta X^b} \, dx \)
  - Skew-symmetry+Jacobi identity: \( g^{ab} \) is a flat metric with Christoffel symbols \( \gamma^c_{ab} \) defined by \( b^{ab}_c = -g^{ad} \gamma^b_{dc} \).
Theorem 3 (Contatto, MD). The hydrodynamic type system \((HT)\) admits one, two or three Hamiltonian formulations with hydrodynamic Hamiltonians if and only if the affine torsion–free connection \(\nabla\) defined by its non–zero components

\[
\begin{align*}
\Gamma_{11}^1 &= \partial_1 \ln A - 2B, \\
\Gamma_{22}^2 &= \partial_2 \ln B - 2A, \\
\Gamma_{12}^1 &= -\left(\frac{1}{2} \partial_2 \ln A + A\right), \\
\Gamma_{12}^2 &= -\left(\frac{1}{2} \partial_1 \ln B + B\right),
\end{align*}
\]

where \(A = \frac{\partial_2 \lambda^1}{\lambda^2 - \lambda^1}, \quad B = \frac{\partial_1 \lambda^2}{\lambda^1 - \lambda^2}, \quad \text{and} \quad \partial_a = \partial/\partial X^a\)

admits one, two or three independent linear first integrals respectively.
The connection from Theorem 3 is generically not metric but is metrisable by the metric

\[ h = AB \, dX \otimes dY, \quad X^a = (X, Y). \]

The unparametrised geodesics of \( h \) and of \( \nabla \) conicide, and are integral curves of a 2nd order ODE

\[ Y''' = (\partial_X Z)Y' - (\partial_Y Z)(Y')^2, \quad \text{where} \quad Z \equiv \ln (AB), \]
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In the tri–Hamiltonian case (Ferapontov (1991)) the connection from Theorem 3 has symmetric Ricci tensor, and is projectively flat. Equivalently, the metric \( h \) has constant Gaussian curvature i.e.

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Example: HT system with \( \lambda^1 = -\lambda^2 = (X - Y)^n(X + Y)^m \). Always bi-Hamiltonian. Tri-Hamiltonian iff \( nm(n^2 - m^2) = 0 \).
Two–dimensional Frobenius Manifolds

- Two–dimensional Frobenius manifolds: coordinates $u^a = (u, v)$, a function $F: U \to \mathbb{R}$, associative structure constants $C_{abc}^a := \eta^{ad}C_{bcd}$

$$C = \frac{\partial^3 F}{\partial u^a \partial u^b \partial u^c} du^a du^b du^c, \quad e = \frac{\partial}{\partial u^1}, \quad \eta = \frac{\partial^3 F}{\partial u^1 \partial u^a \partial u^b} du^a du^b.$$
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- (Dubrovin (1996), Hitchin (1997)) $F(u, v) = \frac{1}{2} u^2 v + f(v)$, where

$$f = v^k, \quad k \neq 0, 2, \quad f = v^2 \ln v, \quad f = \ln v, \quad f = e^{2v}.$$
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- Hydrodynamic type system with Riemann invariants

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- Theorem 3: Tri–hamiltonian with 3-parameter family of flat metrics

$$g(c_1, c_2, c_3) = \lambda^{-1} \left( \frac{dX^2}{c_1 + c_2 X + c_3 X^2} - \frac{dY^2}{c_1 + c_2 Y + c_3 Y^2} \right).$$
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- Two–dimensional Frobenius manifolds: coordinates $u^{\alpha} = (u, v)$, a function $F : U \rightarrow \mathbb{R}$, associative structure constants $C_{\alpha \beta \gamma} : = \eta^{\alpha \delta} C_{\delta \beta \gamma}$

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\]

- $\eta \equiv g(1, 0, 0)$, $I \equiv g(0, 1, 0)$ (intersection form), $J \equiv g(0, 0, 1)$ s. t. $J_{ab} = I_{ac} I_{bd} \eta^{cd}$ (Romano 2014).
A connection $\nabla$ on a compact surface $\Sigma$ is *Zoll* if its unparametrised geodesics are simple closed curves.
Zoll Connections

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- Axisymmetric Zoll metrics on $\Sigma = S^2$

$$h = (F - 1)^2 dX^2 + \sin^2 X dY^2, \quad F = F(X), \quad F : [0, \pi] \to [0, 1]$$

where $F(0) = F(\pi) = 0$ and $F(\pi - X) = -F(X)$.
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- Non-metric Zoll connection with a linear first integral
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  \Gamma^1_{11} = \frac{F'}{F - 1} - 2 \cot X, \quad \Gamma^1_{22} = -\frac{(H^2 + 1) \sin X \cos X}{(F - 1)^2}
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  \[
  \Gamma^1_{12} = \Gamma^1_{21} = \frac{1}{2} \frac{H' \sin X \cos X - 2H}{\cos X (F - 1)}, \quad H = H(X)
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where $H(0) = H(\pi) = H(\pi/2) = 0$, and $H(\pi - X) = H(X)$.
- Exactly two linear first integrals? Find that $T^a_b = 0$ if

$$F = 1 + c(H^2 + 1) \cot X, \quad c \in \mathbb{R}$$

but the boundary conditions do not hold ... (open problem).
Thank You!