First integrals of affine connections and Hamiltonian systems of hydrodynamic type

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- Felipe Contatto, MD, arXiv:1510.01906.

RESULTS

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- Application (unexpected!): Given a one–dimensional system of hydrodynamic type in Riemann invariants, determine necessary/sufficient conditions for the existence of a Hamiltonian (bi–Hamiltonian, tri–Hamiltonian) formulation of Dubrovin-Novikov type.
- Examples: Zoll connections. Hamiltonian systems from two–dimensional Frobebnius manifolds, . . .

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- Prolong this system to a connection on a rank–3 vector bundle $E \to \Sigma$. Find the integrablility conditions for the existence of one/two/three parallel sections. Express them in terms of the curvature of ∇ and its covariant derivatives (of order up to 3).

Prolongation connection

- Curvature decomposition

$$R_{ab}{}^{c}{}_{d} = \delta_a{}^c \mathrm{P}_{bd} - \delta_b{}^c \mathrm{P}_{ad} + B_{ab}\delta_d{}^c.$$

Schouten tensor $\mathrm{P}_{ab} = (2/3)R_{ab} + (1/3)R_{ba}$, and $B_{ab} = -2\mathrm{P}_{[ab]}$. Set $\beta = B_{ab}\epsilon^{ab}$ for an arbitrary volume form ϵ.

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- Proposition. There is a one–to–one correspondence between solutions to the Killing equations (K), and parallel sections of the prolongation connection D on a rank–3 vector bundle $E = \Lambda^1(\Sigma) \oplus \Lambda^2(\Sigma) \to \Sigma$

$$D_a \begin{pmatrix} K_b \\ \mu \end{pmatrix} = \begin{pmatrix} \nabla_a K_b - \epsilon_{ab}\mu \\ \nabla_a \mu - \left(\mathrm{P}^b{}_a + \frac{1}{2}\beta\delta^b{}_a\right)K_b + \mu\theta_a \end{pmatrix}.$$

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- Compute the curvature of D, restric its holonomy so that parallel sections $\Psi = (K_a, \mu)$ exist. Find obstructions.

Prolongation Curvature

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- Set $L_b \equiv \epsilon^{cd}\nabla_c \mathrm{P}_{db}$ and define

$$
\begin{aligned}
F^a &= \frac{1}{3}\epsilon^{ab}(L_b - \epsilon^{cd}\nabla_b B_{cd}), \quad N_a = -F_a + \epsilon^{bc}\nabla_a B_{bc} \\
M_a{}^b &= \frac{1}{3}\epsilon^{bc}\epsilon^{de}(\nabla_a Y_{dec} - \nabla_a\nabla_c B_{de}) + \beta\, \mathrm{P}^b{}_a + \frac{1}{2}\beta^2\delta^b{}_a, \\
I_N &= \epsilon_{cd}\epsilon^{be}M_e{}^c\Big(N_b F^d - \frac{1}{2}\beta M_b{}^d\Big).
\end{aligned}
$$

Main Theorem

Theorem (Contatto, MD) The necessary condition for a C^4 torsion–free affine connection ∇ on a surface Σ to admit a linear first integral is the vanishing, on Σ, of invariants I_N and I_S respectively. For any point $p \in \Sigma$ there exists a neighbourhood $U \subset \Sigma$ of p such that conditions $I_N = I_S = 0$ on U are sufficient for the existence of a first integral on U. There exist precisely two independent linear first integrals on U if and only if the tensor

$$T_a{}^b \equiv N_a F^b - \beta M_a{}^b.$$

vanishes and the skew part of the Ricci tensor of ∇ is non–zero on U. There exist three independent first integrals on U if and only if the connection is projectively flat and its Ricci tensor is symmetric.

Connections with two first integrals

- If ∇ is a Levi–Civita connection of some metric on Σ with scalar curvature R, then (Darboux 1887)

$$I_N := *\frac{1}{432} dR \wedge d(|\nabla R|^2), \quad I_S := * dR \wedge d(\triangle R).$$

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- A Levi–Civita connection can not admit precisely two local first integrals. A non–metric connection can:
  Theorem (Contatto, MD). Let ∇ be an affine connection on a surface Σ which admits exactly two non–proportional linear first integrals which are independent at some point $p \in \Sigma$. Coordinates $X^a = (X, Y)$ can be chosen on an open set $U \subset \Sigma$ containing p such that

$$\Gamma^1_{12} = \Gamma^1_{21} = \frac{c}{2}, \Gamma^2_{11} = \frac{P_X}{Q}, \Gamma^2_{12} = \Gamma^2_{21} = \frac{P_Y + Q_X - cP}{2Q}, \Gamma^2_{22} = \frac{Q_Y}{Q}$$

  and all other components vanish, where c is a constant equal to 0 or 1, and (P, Q) are arbitrary functions of (X, Y).

Hamiltonian Systems of Hydrodynamic Type

- One–dimensional systems of hydrodynamic type

$$\frac{\partial X^1}{\partial t} = \lambda^1(X^1, X^2)\frac{\partial X^1}{\partial x}, \quad \frac{\partial X^2}{\partial t} = \lambda^2(X^1, X^2)\frac{\partial X^2}{\partial x}. \quad (HT)$$

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- Local hydrodynamic Hamiltonian formulation

$$\frac{\partial X^a}{\partial t} = \Omega^{ab}\frac{\delta H}{\delta X^b},$$

where

$$H[X^1, X^2] = \int_{\mathbb{R}} \mathcal{H}(X^1, X^2)dx, \quad \Omega^{ab} = g^{ab}\frac{\partial}{\partial x} + b^{ab}_c\frac{\partial X^c}{\partial x}.$$

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- Poisson bracket (Dubrovin+Novikov (1983), Tsarev (1985))

$$\{F, G\} = \int_{\mathbb{R}} \frac{\delta F}{\delta X^a}\Big(g^{ab}\frac{\partial}{\partial x} + b^{ab}_c\frac{\partial X^c}{\partial x}\Big)\frac{\delta G}{\delta X^b}dx$$

Skew-symmetry+Jacobi identity: g^{ab} is a flat metric with Christoffel symbols γ^c_{ab} defined by $b^{ab}_c = -g^{ad}\gamma^b{}_{dc}$.

Obstructions to the hydrodynamic Hamiltonian formulation

Theorem 3 (Contatto, MD). The hydrodynamic type system (HT) admits one, two or three Hamiltonian formulations with hydrodynamic Hamiltonians if and only if the affine torsion–free connection ∇ defined by its non–zero components

$$\Gamma^1_{11} = \partial_1 \ln A - 2B, \quad \Gamma^2_{22} = \partial_2 \ln B - 2A, \quad \Gamma^1_{12} = -\Big(\frac{1}{2}\partial_2 \ln A + A\Big),$$

where $\quad A = \dfrac{\partial_2 \lambda^1}{\lambda^2 - \lambda^1}, \quad B = \dfrac{\partial_1 \lambda^2}{\lambda^1 - \lambda^2}, \quad$ and $\quad \partial_a = \partial / \partial X^a$

admits one, two or three independent linear first integrals respectively.

Remarks

- The connection from Theorem 3 is generically not metric but is metrisable by the metric

$$h = AB\, dX \odot dY, \quad X^a = (X, Y).$$

The unparametrised geodesics of h and of ∇ conicide, and are integral curves of a 2nd order ODE

$$Y'' = (\partial_X Z)Y' - (\partial_Y Z)(Y')^2, \quad \text{where} \quad Z \equiv \ln{(AB)},$$

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- Example: HT system with $\lambda^1 = -\lambda^2 = (X-Y)^n(X+Y)^m$. Always bi-Hamiltonian. Tri-Hamiltonian iff $nm(n^2 - m^2) = 0$.

Two–dimensional Frobenius Manifolds

- Two–dimensional Frobenius manifolds: coordinates $u^a = (u, v)$, a function $F : U \to \mathbb{R}$, associative sructure constants $C^a{}_{bc} := \eta^{ad} C_{bcd}$

$$C = \frac{\partial^3 F}{\partial u^a \partial u^b \partial u^c} du^a du^b du^c, \quad \mathbf{e} = \frac{\partial}{\partial u^1}, \quad \eta = \frac{\partial^3 F}{\partial u^1 \partial u^a \partial u^b} du^a du^b.$$

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$$X = u + \int \sqrt{f'''(v)} dv, \quad Y = u - \int \sqrt{f'''(v)} dv.$$

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- Theorem 3: Tri–hamiltonian with 3-parameter family of flat metrics

$$g(c_1, c_2, c_3) = \lambda^{-1} \Big(\frac{dX^2}{c_1 + c_2 X + c_3 X^2} - \frac{dY^2}{c_1 + c_2 Y + c_3 Y^2} \Big),$$

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- $\eta \equiv g(1, 0, 0)$, $I \equiv g(0, 1, 0)$ (intersection form), $J \equiv g(0, 0, 1)$ s. t. $J_{ab} = I_{ac} I_{bd} \eta^{cd}$ (Romano 2014).

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- Non–metric Zoll connection with a linear first integral

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where $H(0) = H(\pi) = H(\pi/2) = 0$, and $H(\pi - X) = H(X)$.
- Exactly two linear first integrals? Find that $T^a_b = 0$ if

$$F = 1 + c(H^2+1)\cot X, \quad c \in \mathbb{R}$$

but the boundary conditions do not hold . . . (open problem).

Thank You!