Solitons on Wormholes

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Sine–Gordon kink

- Sine–Gordon equation

$$\phi_{xx} - \phi_{tt} = \sin(2\phi), \quad \phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}.$$

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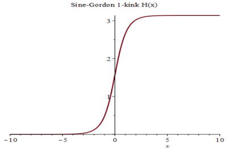

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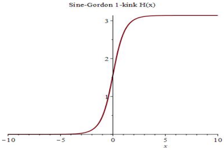

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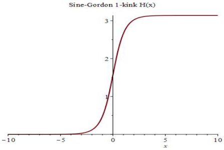


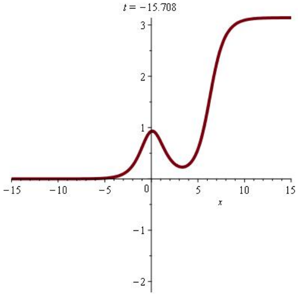
- Topologically stable: $N \equiv \pi^{-1}(H(\infty) - H(-\infty)) = 1$.
- Linear stability

$$\begin{aligned} \phi(t,x) &= H(x) + \epsilon u(x,t), \quad u(x,t) = e^{i\omega t}\psi(x) \\ L(\psi) &= \omega^2\psi, \quad L = -\partial_x{}^2 - 4\text{sech}^2(\sqrt{2}x) + 2. \end{aligned}$$

$\text{Spec}(L) = \{0\} \cup [2, \infty)$.

Wobbling kink

SG kink is not asymptotically stable: time dependent wobbling kinks (time–dependent solitons resulting from integrability of Sine–Gordon)



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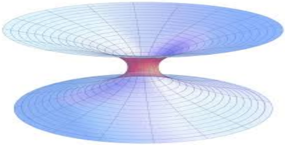
- Linear stablity: $\exists$ internal oscillation modes.
- A step towards the soliton resolution conjecture: Solutions of dispersive wave equations asymptotically resolve into a superposition of localised structures (solitons, black holes, ...) and radiation. (e.g. Terence Tao's blog *Structure and randomness in PDE*).

Ellis–Bronnikov wormhole

- $(t, r) \in \mathbb{R} \times \mathbb{R}, (\theta, \phi) \in S^2$. Wormhole with area $4\pi a^2$

$$g = -dt^2 + dr^2 + (r^2 + a^2)(d\theta^2 + \sin^2\theta d\phi^2)$$



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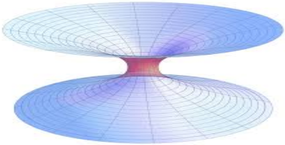

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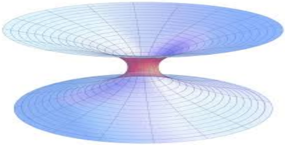

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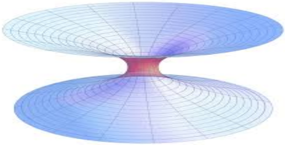

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- Sine–Gordon $\Box_g(\phi) = \sin(2\phi)$. Finite energy solutions s.t

$$\phi \sim N_{\pm}\pi \text{ as } r \to \pm\infty, \quad N \equiv \pi^{-1}(N_+ - N_-) \neq 0.$$

$$\phi_{rr} - \phi_{tt} + \frac{2r}{r^2 + a^2}\phi_r = \sin(2\phi)$$

- Not integrable (e.g. fails the Painleve test), not scale invariant if $a \neq 0$, dispersive (a shadow of higher dimensions).

Radial Sine–Gordon on a wormhole

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- Finite energy solutions: $\phi(t, -\infty) = 0, \phi(t, \infty) = N\pi$.

$$H'' + \frac{2r}{r^2 + a^2} H' = \sin(2H), \quad H = H(r) \quad (*)$$

- For any $a \neq 0$ and any $N \in \mathbb{N}$ $\exists$ a unique solution $H_N(r)$ with kink number N.

Static kinks

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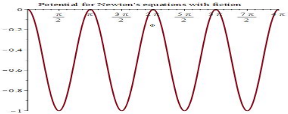
- For any $a \neq 0$ and any $N \in \mathbb{N}$ $\exists$ a unique solution $H_N(r)$ with kink number N.
- Proof: $(*)$ =the Newton equation (with time r) of a particle in a reversed potential $-\sin^2 H$ with time–dependent friction.



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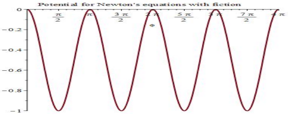
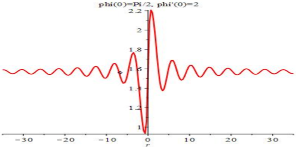


- Generic initial velocity $\phi'(0)$ with $\phi(0) = N\pi/2$.



Multi kink approximation

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\begin{aligned}
H_N(r) \quad &\sim \quad H_{SG}(r-(2k-1)R) + \cdots + H_{SG}(r-R) + H_{SG}(r+R) \\
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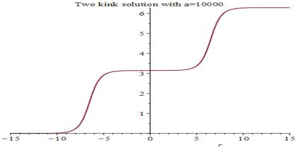
where $R = \sqrt{2}/4W(32\coth(3\sqrt{2})a^2/N^2)$, and $W(x)e^{W(x)} = x$.



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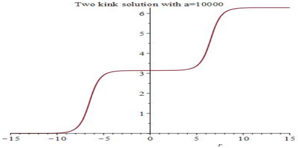

- Small a

$$H_N(r) \sim N\Big(\frac{\pi}{2} + \arctan(r/a)\Big).$$

- Let $\phi(r,t) = H_N(r) + \epsilon(r^2+a^2)^{-1/2}u(r,t)$. Then $u_{tt} + L_N(u) = 0$

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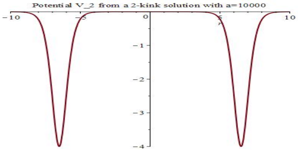

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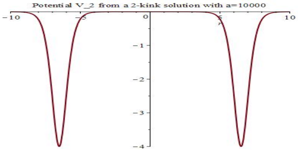

- As a decreases the eigenvalues disappear one by one into the continuous spectrum. For $N=1$ the continuous spectrum for $a < a_c \sim 0.53$.

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- Mechanism: resonant interaction of internal modes in the discrete spectrum with radiation (Soffer–Weinstein 1999).