On the relation between symplectic structures and variational principles in continuum mechanics

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Outline

1. Main problems
   - How to get all stationary-action principles for a given equation of continuum mechanics?
   - How stationary-action principles in the Eulerian description are related to ones in the Lagrangian description?
2. Main results (spoiler)
   - Local symplectic structures of equations of continuum mechanics can be regarded as nontrivial stationary-action principles.
   - Lagrangian variables are nonlocal variables in a differential covering, hence one can lift symplectic structures from the Eulerian description to the Lagrangian one.

Equations in an extended Kovalevskaya form

Most systems of equations in continuum mechanics can be written in an extended Kovalevskaya form. By a system of equations in an extended Kovalevskaya form we shall mean a system of the following kind

$$F = 0\,, \qquad \text{where} \qquad F = u_{bt} - f\,.$$

Here

- $u = (u^1, \ldots, u^m)^T$ is a vector of dependent variables;
- $x = (x^1, \ldots, x^{n-1}, x^n = t)^T$ is a vector of independent variables;
- u_{bt} is a vector of derivatives $(u^1_{b_1 t}, \ldots, u^m_{b_m t})^T$, where b_i are positive integers (orders of derivatives);
- vector-function $f = (f^1, \ldots, f^m)^T$ depends on x, u and derivatives up to some finite order. We also assume that f does not depend on $u^i_{b_i t}$ and their derivatives.

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Geometrical point of wiev on an extended Kovalevskaya form

From geometrical point of wiev:

- independent variables $(x^1, \dots, x^{n-1})$ are local coordinates on a smooth manifold X, $\dim X = n - 1$;
- $t = x^n$ is a global coordinate on a real line $\mathbb{R}$;
- dependent variables $(u^1, \dots, u^m)$ are local coordinates along the fibres of a vector bundle
$$\eta \colon E \to X\,.$$
We assume that either $b_1 = \ldots = b_m$, or η has a trivial structure group;

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u_{bt} and f are sections of some bundle, which can be described as follows:

- denote by M and N products $M = X \times \mathbb{R}$, $N = E \times \mathbb{R}$;
- consider a bundle $\pi = (\eta, 1_{\mathbb{R}})$

$$\pi \colon N \to M\,;$$

- denote natural projection from the infinite jets to M by π_∞:

$$\pi_\infty \colon J^\infty(\pi) \to M;$$

- Let $\varkappa(\pi)$ be the module of sections of the pullback:

$$\varkappa(\pi) = \Gamma(\pi^*_\infty(\pi))\,.$$

Then $u_{bt}, f, F \in \varkappa(\pi)$.

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Infinite prolongation

- Let α be a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, where all α_i are non-negative integers.
  We shall use the following multi-index notations:

$$\alpha x = \alpha_1 x^1 + \ldots + \alpha_n x^n = \alpha_i x^i \,, \qquad |\alpha| = \sum_i \alpha_i \,,$$

$$D_{\alpha x} = {D_{x^1}}^{\alpha_1} \circ \ldots \circ {D_{x^n}}^{\alpha_n} \,, \qquad u^i_{\alpha x} = D_{\alpha x}(u^i),$$

  where D_{x^i} are total derivatives.

- Let $\mathcal{E} \subset J^\infty(\pi)$ be the infinite prolongation of the system of equations $F = 0$, i.e.

$$\mathcal{E}: \;\; D_{\alpha x}(u^i_{b_i t} - f^i) = 0 \qquad \text{for all} \;\; \alpha \text{ and } i = 1, \ldots, m \,. \tag{1}$$

  Further we consider only regular systems of equations of the form (1) with trivial de Rham cohomology group $H^{n+1}_{dR}(\mathcal{E})$.

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Admissible stationary-action principles

Denote

- an algebra of smooth functions on $J^\infty(\pi)$ by $\mathcal{F}(\pi)$,
- an algebra of horizontal n-forms on $J^\infty(\pi)$ by $\Lambda_h^n(\pi)$.

Also we shall use a notation $\widehat{\varkappa}(\pi)$ for the following module

$$\widehat{\varkappa}(\pi) = Hom_{\mathcal{F}(\pi)}(\varkappa(\pi), \Lambda_h^n(\pi)) .$$

Admissible (global) stationary-action principles for $\mathcal{E}$ amount to admissible variarional operators, i.e. differential operators in total derivatives

$$A\colon \varkappa(\pi) \to \widehat{\varkappa}(\pi) ,$$

such that for some Lagrangians $L \in \Lambda_h^n(\pi)$ the following relation

$$A(F) = \mathbf{E}(L)$$

holds. Here $\mathbf{E}$ is the Euler operator (variational derivative).

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Basic notations and definitions

Denote the universal linearization operator for F by l_F. Here

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For an element $\varphi \in \varkappa(\pi)$ we have $l_F(\varphi) = E_\varphi(F)$, where E_φ is the corresponding evolutionary vector field on $J^\infty(\pi)$.

In adapted local coordinates

- $\varphi = (\varphi^1, \ldots, \varphi^m)^T$
- $E_\varphi = \varphi^i \partial_{u^i} + D_{x^j}(\varphi^i)\partial_{u^i_{x^j}} + \ldots = D_{\alpha x}(\varphi^i)\partial_{u^i_{\alpha x}}$
- $$l_F(\varphi)^k = l_F{}^k_i(\varphi^i) = \frac{\partial F^k}{\partial u^i_{\alpha x}} D_{\alpha x}(\varphi^i)\,.$$

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Symplectic structures as closed variational 2-forms

Motivation

As we will see later, the problem of classification of all admissible variational operators for $\mathcal{E}$ is (almost) precisely the problem of classification of all admissible (local) symplectic structures for $\mathcal{E}$.

Let us recall the definition of a symplectic structure of a system of equations $\mathcal{E}$.

- A symplectic structure of a system of equations $\mathcal{E}$ is a closed variational 2-form on $\mathcal{E}$, i.e., an element of the kernel of the variational differential

$$\delta\colon E_1^{2,\,n-1}(\mathcal{E}) \to E_1^{3,\,n-1}(\mathcal{E})\,.$$

Here $E_1^{p,\,n-1}(\mathcal{E})$ are groups of variational p-forms from the A. Vinogradov $\mathcal{C}$-spectral sequence.

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Symplectic structures as classes of operators

- Variational 2-forms of a system $\mathcal{E}$ can also be described as operators in total derivatives $\Delta\colon \varkappa(\mathcal{E}) \to \widehat{\varkappa}(\mathcal{E})$ which satisfy the relation

$$\Delta^* \circ l_{\mathcal{E}} = l_{\mathcal{E}}^* \circ \Delta\,, \tag{2}$$

modulo the operators of the form $\nabla \circ l_{\mathcal{E}}$, where $\nabla = \nabla^*$. Here the operator Δ^* is formally adjoint to the operator Δ.

Let us also recall

Noether theorem

Every symplectic structure of $\mathcal{E}$ determines a mapping from symmetries of $\mathcal{E}$ to its variational 1-forms $E_1^{1,\,n-1}(\mathcal{E})$.

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Variational operators and symplectic structures

The relation between symplectic structures and variational operators is based on two results

- Every admissible variational operator for $\mathcal{E}$ determines a symplectic structure (possibly trivial). Namely, $\Delta = A^*|_{\mathcal{E}}$ satisfy the desired relation
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The corresponding variational 2-form is closed.
- Every symplectic structure for $\mathcal{E}$ can be obtained from a variational operator at least locally (it follows from I. Khavkine results and quite a simple analysis of A. Vinogradov two-line theorem).

On trivial variational operators

If a variational operator determines a trivial symplectic structure, then the corresponding Noether mapping is also trivial. We will not be interested in such variational operators.

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Restoring of an action functional from a symplectic structure

So, nontrivial symplectic structures for $\mathcal{E}$ can be regarded (at least locally) as its nontrivial variational principles. Now we only need for a canonical way to restore a local action functional from a symplectic structure.

Specificity of systems in extended Kovalevskaya form

It turns out that there is a way to restore a global variational operator from a symplectic structure of $\mathcal{E}$. If extended Kovalevskaya form is a canonical form of $\mathcal{E}$, then this way is also canonical.

The key idea is to identify symplectic structures for $\mathcal{E}$ with conservation laws of a special form for another system of equations.

To this end we introduce another equation $F' = 0$. Informally speaking, the difference between F and F' amounts to our assumption that there is another one independent variable $a \in \mathbb{R}$ for F'.

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Construction of another one bundle π'

Lets consider a bundle $\pi' = (\pi, 1_{\mathbb{R}})$,

$$\pi' \colon N \times \mathbb{R} \to M \times \mathbb{R}$$

and denote a global coordinate on this real line $\mathbb{R}$ by a.

Consider a projection $pr_N \colon N \times \mathbb{R} \to N$, which forgets about a. Assume that a pair (P', L') determines a 1-jet of a section of π'. Here $P' \in N \times \mathbb{R}$, $L' \subset T_{P'}(N \times \mathbb{R})$. Then a pair $(pr_N(P'), pr_{N*}(L'))$ determines 1-jet of a section of π. Thus, we have a mapping

$$h^1 \colon J^1(\pi') \to J^1(\pi)\,.$$

Mappings $h^k \colon J^k(\pi') \to J^k(\pi)$ are defined similarly. Hence we have a mapping

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A pair (h, pr_N) also determines a mapping

$$g \colon {\pi'_\infty}^*(N \times \mathbb{R}) \to {\pi_\infty}^*(N)\,.$$

Construction of another one bundle π'

Lets consider a bundle $\pi' = (\pi, 1_{\mathbb{R}})$,

$$\pi' \colon N \times \mathbb{R} \to M \times \mathbb{R}$$

and denote a global coordinate on this real line $\mathbb{R}$ by a.

Consider a projection $pr_N \colon N \times \mathbb{R} \to N$, which forgets about a. Assume that a pair (P', L') determines a 1-jet of a section of π'. Here $P' \in N \times \mathbb{R}$, $L' \subset T_{P'}(N \times \mathbb{R})$. Then a pair $(pr_N(P'), pr_{N*}(L'))$ determines 1-jet of a section of π. Thus, we have a mapping

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System of equations $\mathcal{E}'$

For a section $\varphi \in \varkappa(\pi)$ we can define $\varphi' \in \varkappa(\pi')$ as a unique section, such that the square commutes

$$\begin{tikzcd} {\pi'_\infty}^*(N \times \mathbb{R}) \arrow[r, "g"] & {\pi_\infty}^*(N) \\ J^\infty(\pi') \arrow[u, "\varphi'"'] \arrow[r, "h"] & J^\infty(\pi) \arrow[u, "\varphi"'] \end{tikzcd}$$

We denote the infinite prolongation of a system $F' = 0$ by $\mathcal{E}'$.

Dimension of the base of π' is $n+1$, hence

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Variational principles for $\mathcal{E}$ and conservation laws of $\mathcal{E}'$

Using these two natural mappings $\varkappa(\pi) \to \varkappa(\pi')$, $\widehat{\varkappa}(\pi) \to \widehat{\varkappa}(\pi')$ and an operator $A\colon \varkappa(\pi) \to \widehat{\varkappa}(\pi)$, one can define a unique operator $A'\colon \varkappa(\pi') \to \widehat{\varkappa}(\pi')$, such that the square commutes

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Now we can formulate the following important lemma

Lemma

$\psi \in \mathrm{Im}\,\mathbf{E}$ if and only if $\langle \psi', u_a \rangle \in \mathrm{Im}\, d_h$

This lemma and the Green identity imply the following corollary

Corollary

$A(F) \in \mathrm{Im}\,\mathbf{E}$ if and only if $\langle A'^*(u_a), F' \rangle \in \mathrm{Im}\, d_h$

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$A(F) \in \mathrm{Im}\, \mathbf{E}$ if and only if $\langle A'^{*}(u_a), F' \rangle \in \mathrm{Im}\, d_h$

Therefore, A is a nontrivial variational operator for $\mathcal{E}$ if and only if $A'^{*}(u_a)$ is a characteristic of a nontrivial conservation law for $\mathcal{E}'$. Notice that $F' = 0$ is also a system in the extended Kovalevskaya form.

According to L. Martinez Alonso lemma each conservation law of a system in an extended Kovalevskaya form has a characteristic of a simplest form (i.e. components of such a characteristic are independent of variables u_{bt} and their derivatives).

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Main theorem

Kovalevskaya form of a system $F = 0$ also allows us to extend an operator (in total derivatives) of the form $\Delta\colon \varkappa(\mathcal{E}) \to \widehat{\varkappa}(\mathcal{E})$ to an operator of the form $\widetilde{\Delta}\colon \varkappa(\pi) \to \widehat{\varkappa}(\pi)$ in a coordinate-naive way. Here coordinate-naive way is a way to identify local coordinates on $J^\infty(\pi)$, except for coordinates of the form $D_{\alpha x}(u^i_{b_i t})$, with local coordinates on $\mathcal{E}$.

Using nontation $\overline{D}_{\alpha x} = D_{\alpha x}|_{\mathcal{E}}$ and applying L. Martinez Alonso lemma we obtain the following result

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A symplectic structure of a system of equations $\mathcal{E}$ in an extended Kovalevskaya form (with $H^{n+1}_{dR}(\mathcal{E}) = 0$) can be represented by a unique operator $\Delta\colon \varkappa(\mathcal{E}) \to \widehat{\varkappa}(\mathcal{E})$, such that in local coordinates Δ has the form

$$\Delta(\varphi)_i = \Delta^{\alpha}_{ik}\, \overline{D}_{\alpha x}(\varphi^k)\,, \qquad \Delta^{\alpha}_{ik} = 0 \quad \text{only if} \quad \alpha_n < b_k\,. \tag{4}$$

An operator $\widetilde{\Delta}^*$ (adjoint to the corresponding coordinate-naive extension) is a variational operator for a system $\mathcal{E}$.

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Examples

Consider the stationary boundary layer equations

$$\begin{aligned} u_{yy} - uu_x - vu_y &= 0\,, \\ v_y + u_x &= 0\,. \end{aligned} \tag{5}$$

Then the linearization operator is

$$l_{\mathcal{E}} = \begin{pmatrix} \overline{D}_y^2 - u\overline{D}_x - u_x - v\overline{D}_y & -u_y \\ \overline{D}_x & \overline{D}_y \end{pmatrix}.$$

Every symplectic sturucture of (5) can be represented by a unique operator of the form (4):

$$\Delta = \begin{pmatrix} \Delta_{11}^{(j,0)}\overline{D}_{jx} + \Delta_{11}^{(j,1)}\overline{D}_{jx+y} & \Delta_{12}^{(j,0)}\overline{D}_{jx} \\ \Delta_{21}^{(j,0)}\overline{D}_{jx} + \Delta_{21}^{(j,1)}\overline{D}_{jx+y} & \Delta_{22}^{(j,0)}\overline{D}_{jx} \end{pmatrix}.$$

However it is possible to show that in this case there is no nonzero solutions to the equation

$$\Delta^* \circ l_{\mathcal{E}} = l_{\mathcal{E}}^* \circ \Delta\,.$$

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Examples

Consider the Potential Korteweg-de Vries equation

$$u_t = 3u_x^2 + u_{xxx}\,. \tag{6}$$

It admits symplectic structure [Dorfman] generated by the operator

$$\nabla = 2u_{xx} + 4u_x\bar{D}_x + \bar{D}_x^3\,.$$

There are two different ways to represent equation (6) in an extended Kovalevskaya form. Lets consider an unusual representation:

$$u_{xxx} = u_t - 3u_x^2\,,$$

i.e. let local coordinates on $J^\infty(2,1)$ be local coordinates on (6), except u_{xxx} and its derivatives. Then ∇ is equivalent to the operator

$$\Delta = \bar{D}_t - 2u_x\bar{D}_x + 2u_{xx}\,.$$

According to the previous theorem, such choise of local coordinates on (6) leads to the following first-order variational operator

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Part II. Eulerian and Lagrangian descriptions

This part of investigation is joint with A.V. Aksenov.

Lets consider the mass conservation law in continuum mechanics,

$$\rho_t + (u\rho)_x + (v\rho)_y + (w\rho)_z = 0\,. \tag{7}$$

Here

- $\mathbf{v} = u\partial_x + v\partial_y + w\partial_z$ is the velocity field;
- ρ is the mass density. Below we assume that $\rho > 0$.

Choosing suitable nonlocal variables, one can introduce a potential for the mass conservation law, which satisfies the following relation:

$$\begin{gathered}\rho\, dx \wedge dy \wedge dz - u\rho\, dt \wedge dy \wedge dz + v\rho\, dt \wedge dx \wedge dz - w\rho\, dt \wedge dx \wedge dy = \\ = d_h(\xi^1 d_h\xi^2 \wedge d_h\xi^3) = d_h\xi^1 \wedge d_h\xi^2 \wedge d_h\xi^3\,.\end{gathered} \tag{8}$$

Relation (8) is equivalent to the following system of equations:

$$\rho = \det\left(\frac{D\xi}{Dx}\right), \qquad \xi^i_t + u\xi^i_x + v\xi^i_y + w\xi^i_z = 0, \qquad i = 1, 2, 3\,. \tag{9}$$

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Lagrangian description as differential covering

Conditions (9) show that functions $\xi^i(t, x, y, z)$ are Lagrangian variables.

Remark

Transformations that preserve the volume $d\xi^1 \wedge d\xi^2 \wedge d\xi^3$ in the (ξ^1, ξ^2, ξ^3)-space form the symmetry group for equations in the Lagrangian variables.

- Since the only consistency condition for the system (9) is the mass conservation law, it follows that, for any system of equations in the Eulerian variables, the potential (8) determines its differential covering.
- One can choose x, y, z as new dependent variables for a covering system and obtain its usual Lagrangian representation.
- The description of Lagrangian variables as nonlocal variables in a differential covering allows us to lift symplectic structures from the Eulerian description to the Lagrangian one.

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- Since the only consistency condition for the system (9) is the mass conservation law, it follows that, for any system of equations in the Eulerian variables, the potential (8) determines its differential covering.
- One can choose x, y, z as new dependent variables for a covering system and obtain its usual Lagrangian representation.
- The description of Lagrangian variables as nonlocal variables in a differential covering allows us to lift symplectic structures from the Eulerian description to the Lagrangian one.

Lagrangian description as differential covering

Conditions (9) show that functions $\xi^i(t,x,y,z)$ are Lagrangian variables.

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Also main theorem

Recall that a system of equations $\mathcal{E}$ is l-normal if the condition

$$\nabla \circ l_{\mathcal{E}} = 0 \tag{10}$$

implies $\nabla = 0$. All systems in an extended Kovalevskaya form are l-normal. We shall say that a system of equations $F = 0$ is variational if for some Lagrangian L the relation

$$F = \mathbf{E}(L) \tag{11}$$

holds. Let

- $F = 0$ be a system of differential equations in Eulerian variables;
- $\widetilde{F} = 0$ be the corresponding system in Lagrangian variables.

Denote the covering from $\widetilde{\mathcal{E}}$ to $\mathcal{E}$ by τ.

Also main theorem

If a system of equations $\widetilde{\mathcal{E}}$ is an l-normal variational system, then the corresponding symplectic structure of $\widetilde{\mathcal{E}}$ is not a lift of a symplectic structure of $\mathcal{E}$.

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Proof.

- Consider the algebra $\tau_*\text{-}\mathrm{sym}\,\widetilde{\mathcal{E}}$ of τ-projectable symmetries of $\widetilde{\mathcal{E}}$. Then, for any variational 2-form $\omega \in E_1^{2,\,n-1}(\mathcal{E})$, the following diagram is commutative:

$$\begin{tikzcd} \tau_*\text{-}\mathrm{sym}\,\widetilde{\mathcal{E}} \arrow[r, "\tau^*(\omega)"] \arrow[d, "\tau_*"'] & E_1^{1,\,n-1}(\widetilde{\mathcal{E}}) \\ \mathrm{sym}\,\mathcal{E} \arrow[r, "\omega"] & E_1^{1,\,n-1}(\mathcal{E}) \arrow[u, "\tau^*"'] \end{tikzcd}$$

- The relation

$$\tau^*(\omega)(\varphi) = 0 \tag{12}$$

holds for any symmetry $\varphi \in \tau_*\text{-}\mathrm{sym}\,\widetilde{\mathcal{E}}$, which acts in a fiber of τ.

Then $\tau^*(\omega)$ can not contain the identity operator (as the equivalence class of operators).

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This theorem shows that, if an l-normal system of equations in Lagrangian variables is variational, then the corresponding variational principle has no analogs in the Eulerian variables.

Remark

A similar result holds for any covering from an l-normal system of differential equations such that the fiber symmetry algebra is nontrivial.

In particular, it holds true for coverings from l-normal systems of differential equations which are based on the introduction of potentials for conservation laws.

Finally, let's look at last example.

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Example

Equations of motion of a polytropic gas ($p = C\rho^\gamma$) in the Lagrangian variables are Euler-Lagrange equations for the Lagrangian

$$L = \Big(\frac{x_t^2 + y_t^2 + z_t^2}{2} - V - U\Big) dt \wedge d\xi^1 \wedge d\xi^2 \wedge d\xi^3 \,, \tag{13}$$

where V is the potential energy density and U is the internal energy density. The corresponding system of equations has the following (canonical) extended Kovalevskaya form:

$$x_{tt} = -\frac{\delta(V+U)}{\delta x}\,, \qquad y_{tt} = -\frac{\delta(V+U)}{\delta y}\,, \qquad z_{tt} = -\frac{\delta(V+U)}{\delta z}\,.$$

Thus, this system of equations is l-normal, and there is no corresponding variational principle for the equations of motion of a polytropic gas in the Eulerian variables.

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Thank you very much for your attention!