Equivalence of scalar ODEs under contact, point and fiber-preserving transformations

Boris Doubrov
(joint work with Tohru Morimoto and Igor Zelenko)

Belarusian State University

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Outline

Setup

- We study scalar ODEs of arbitrary order k:
$$y^{(k)} = F(x, y, y', \ldots, y^{(k-1)})$$
up to one of the three pseudogroups:
- Pseudogroup of contact transformations:
$$(x, y, y') \mapsto (A(x, y, y'), B(x, y, y'), C(x, y, y')),$$
$$dB - C\,dA = \lambda(dy - y'\,dx).$$
- Pseudogroup of point transformations:
$$(x, y) \mapsto (A(x, y), B(x, y)),$$
- Pseudogroup of fiber-preserving transformations:
$$(x, y) \mapsto (A(x), B(x, y)).$$
- Equations are viewed as submanifolds $\mathcal{E} \subset J^k(\mathbb{R}, \mathbb{R})$. The above pseudogroups are naturally prolonged to $J^k(\mathbb{R}, \mathbb{R})$.

Questions

- When two ODEs are (locally) equivalent under one of the pseudogroups?
- Explicitly characterize the class of *trivializable* ODEs, that is the orbit of the trivial equation $y^{(k)} = 0$ under the action of one of the pseudogroups.
- Out of scope: we do not discuss today the question of *constructive equivalence*, that is how to find the equivalence transformation, if we know that two equations are equivalent.
- Similar, but not covered today for brevity: equivalence of systems of ODEs under point and fiber-preserving transformations.

Fundamental invariants

- A (relative differential) invariant of order ℓ of a scalar k-th order ODE $\mathcal{E} \subset J^k(\mathbb{R}, \mathbb{R})$ under the action of the pseudogroup G is a function:

$$I \colon J^\ell(J^k(\mathbb{R}, \mathbb{R}), k+1) \to \mathbb{R}$$

  such that $g^* I = \lambda I$ for any $g \in G$. Here $J^\ell(J^k(\mathbb{R}, \mathbb{R}), k+1)$ is the space of ℓ-jets of codimension 1 (= dimension $k+1$) submanifolds in $J^k(\mathbb{R}, \mathbb{R})$.
- For any such invariant I the condition $I = 0$ defines a family of scalar ODEs stable under the action of the pseudogroup G.
- We say that $\{I_1, \dots, I_r\}$ is a set of fundamental invariants of k-th order ODEs under the action of G if $I_1 = I_2 = \cdots = I_r = 0$ defines the orbit of the trivial ODE under G. That is an equation is trivializable under G iff all fundamental invariants vanish for this equation.

Second order ODEs

- Equation $y'' = F(x, y, y')$ up to point transformations.
- Fundamental invariants:

$$I_4 = F_{1111} = \frac{\partial^4 F}{(\partial y')^4};$$

$$I_4^* = \frac{1}{6}F_{11xx} - \frac{1}{6}F_1 F_{11x} - \frac{2}{3}F_{01x} + \frac{2}{3}F_1 F_{01} + F_{00} - \frac{1}{2}F_0 F_{11}.$$

- Tresse (1896) via differential invariants of pseudogroups, Élie Cartan (1924) via (generalized) projective connections.
- Parabolic geometry of type $(A_2, \{\alpha_1, \alpha_2\})$.

Third order ODEs

- Equation $y''' = F(x, y, y', y'')$ up to contact transformations.
- Fundamental invariants:

$$I_4 = F_{2222} = \frac{\partial^4 F}{(\partial y'')^4};$$

$$W_3 = -F_0 - \frac{1}{3}F_1F_2 - \frac{2}{27}F_2^3 + \frac{1}{2}F_{1x} + \frac{1}{3}F_2F_{2x} - \frac{1}{6}F_{2xx}.$$

- K.W. Wünschmann (1905) for computation of W, S.-S.Chern (1939) via Cartan equivalence method, H. Sato, A. Y. Yoshikawa (1998) via Tanaka theory and normal Cartan connections.
- Parabolic geometry of type $(C_2, \{\alpha_1, \alpha_2\})$.

Systems of ODE's of higher order

- Equation $\mathcal{E} \subset J^k(\mathbb{R}, \mathbb{R})$ resolved with respect to $y^{(k)}$ is written in jet coordinates as:

$$y_k = F(x, y, y_1, \ldots, y_{k-1}).$$

  It projects to $J^{k-1}(\mathbb{R}, \mathbb{R})$ as a local diffeomorphism and naturally inherits intrinsic structures:
- Contact distribution from $J^{k-1}(\mathbb{R}, \mathbb{R})$:

$$C = \langle dy - y_1\, dx, dy_1 - y_2\, dx, \ldots, dy_{k-2} - y_{k-1}\, dx \rangle^\perp$$

- Tangent direction to lifts of solutions:

$$E = \langle D_x = \partial_x + y_1 \partial_y + \cdots + y_{k-1} \partial_{y_{k-2}} + F \partial_{y_{k-1}} \rangle.$$

- If $k \geq 3$, then the "vertical" direction tangent to the fibers of the projection $J^{k-1}(\mathbb{R}, \mathbb{R}) \to J^{k-2}(\mathbb{R}, \mathbb{R})$ (Lie–Backlund theorem): $V = \langle \partial_{y_{k-1}} \rangle$.

Intrinsic geometry

- We have $C = E \oplus V$, and it can be shown that this intrinsic structure uniquely defines ODEs of order ≥ 3 up to the pseudo-group of contact transformations.
- Moreover, contact transformations will preserve all projections $J^{k-1}(\mathbb{R},\mathbb{R}) \to J^{l}(\mathbb{R},\mathbb{R})$, $l \geq 1$.
- In case of point transformations we need to add the completely integrable distribution tangent to the projection $J^{k-1}(\mathbb{R},\mathbb{R}) \to J^{0}(\mathbb{R},\mathbb{R}) = \mathbb{R}^2$.
- In case of fiber-preserving transformations we need to add the completely integrable distribution tangent to the projection $J^{k-1}(\mathbb{R},\mathbb{R}) \to \mathbb{R}$ (space of independent variables).
- Again, all these completely integrable distributions together with the contact distribution on $J^{k-1}(\mathbb{R},\mathbb{R})$ define the intrinsic geometry of a given ODE. Two ODEs are (locally) equivalent under the pseudogroup G if and only if their corresponding intrinsic geometries are locally equivalent.

Symmetries of the trivial equation

- Assume now that:
  - $k \geq 2$ for the pseudogroup of fiber-preserving transformations;
  - $k \geq 3$ for the pseudogroup of point transformations;
  - $k \geq 4$ for the pseudogroup of contact transformations.
- In these cases the symmetry algebra of the trivial equation is a semi-direct product of $\mathfrak{gl}(2)$ and an irreducible representation $V = \mathbb{R}^k$. We call it *the symbol algebra* of the k-th order ODE. Note that for $k \geq 4$ the fiber-preserving, point and contact symmetries are the same!
- Explicitly:

$$\mathfrak{g} = \langle \partial_x, x\partial_x, y\partial_y, x^2\partial_x + (k-1)xy\partial_y \rangle \oplus \langle x^i\partial_y, i = 0, \dots, k-1 \rangle.$$

- This Lie algebra acts transitively on the trivial equation $\mathcal{E}_0$ with the 3-dim stabilizer

$$\mathfrak{g}^0 = \langle x\partial_x, y\partial_y, x^2\partial_x + (k-1)xy\partial_y \rangle.$$

Cartan connection

- **Theorem.** *There exists a natural Cartan connection associated with the pair* $(\mathcal{E}, G)$*. It is defined as a* $\mathfrak{g}$*-valued absolute parallelism* $\omega\colon T\mathcal{G} \to \mathfrak{g}$ *on a certain bundle* $\pi\colon \mathcal{G} \to \mathcal{E}$*.*
- Notion of *an associated connection* depends on the pseudogroup, as different pseudogroups define different intrinsic geometries.
- For example, we say that the connection ω is associated with the fiber-preserving geometry of ODEs, if for any section $s\colon \mathcal{E} \to \mathcal{G}$ the pull back of the 1-form $s^*\omega$ maps distribution $E = \langle D_x \rangle$ to $\mathfrak{gl}(2) \subset \mathfrak{g}$ and the tangent direction to the fibers of $J^k(\mathbb{R}, \mathbb{R}) \to \mathbb{R}$ to $\mathbb{R}^k + \mathfrak{g}^0$.
- Similarly we define Cartan connections associated with pseudogroups of contact and point transformations.
- Natural Cartan connection means that its curvature $\Omega = d\omega + 1/2[\omega, \omega]$ satisfies special normalization conditions that guarantee its uniqueness.

Lie algebra cohomology defines fundamental invariants

- The symbol algebra $\mathfrak{g}$ can be equipped with a natural grading, if we assign weights to variables x, y as: $\deg x = 1$, $\deg y = k$. For example, its negative part $\mathfrak{g}_-$ is a graded nilpotent Lie algebra generated by two elements of degree -1:

$$\mathfrak{g}_{-1} = \langle \partial_x, x^{k-1}\partial_y \rangle.$$

- We are interested in the cochain complex $C^k(\mathfrak{g}_-, \mathfrak{g})$ that computes Lie algebra cohomology of $\mathfrak{g}_-$ with values in $\mathfrak{g}$. It controls the curvature of Cartan connections associated to $(\mathcal{E}, G)$. Namely, the curvature can be interpreted as a function κ on $\mathcal{G}$ with values in $C^2(\mathfrak{g}_-, \mathfrak{g})$.

Conditions on the curvature (contact transformations)

- The case of contact pseudogroup can be treated via so-called Tanaka theory of structures on filtered manifolds. Namely, we define a "good" scalar product on $\mathfrak{g}$ and
$$\partial^*\colon C^r(\mathfrak{g}_-, \mathfrak{g}) \to C^{r-1}(\mathfrak{g}_-, \mathfrak{g})$$
dual to the standard cohomology differentials:
$$\partial\colon C^{r-1}(\mathfrak{g}_-, \mathfrak{g}) \to C^r(\mathfrak{g}_-, \mathfrak{g})$$
- A Cartan connection associated to the contact geometry of k-th order ODE $\mathcal{E}$ is called *normal*, if $\partial^*\kappa = 0$, where κ is its curvature function.
- Fundamental invariants correspond to the elements of $H^2_+(\mathfrak{g}_-, \mathfrak{g})$.
- (!!) Note that many elements of $H^2_+(\mathfrak{g}_-, \mathfrak{g})$ still produce zero invariants due to flatness of contact distribution. Extra algebraic conditions required to identify elements of the cohomology that leads to the non-trivial fundamental invariants.

Conditions on the curvature (fiber-preserving transformations)

- Consider now the case of the pseudo-group of fiber-preserving transformations.
- Define a subcomplex $\check{C}$ of the cohomology complex $C(\mathfrak{g}_-, \mathfrak{g})$:

$$\check{C}^r = \{\alpha \in C^r(\mathfrak{g}_-, \mathfrak{g}) \mid \alpha(\wedge^r \mathbb{R}^k) \subset \mathbb{R}^k\}.$$

- The construction of the normal Cartan connection associated to the geometry of ODEs under fiber-preserving transformations is identical to the contact case if we replace the complex $C^r(\mathfrak{g}_-, \mathfrak{g})$ with $\check{C}^r$!
- Similar construction works in case of point geometry of ODEs: it boils down to defining yet another subcomplex of $C^r(\mathfrak{g}_-, \mathfrak{g})$. The rest of the construction is unmodified.
- Fundamental invariants correspond to the 2nd cohomology of the corresponding complexes.

Generalized Wilczynski invariants

- Consider a linear ODE:

$$y^{(k)} + p_{k-1}(x)y^{(k-1)} + \cdots + p_0(x)y(x) = 0$$

  up to a smaller pseudogroup $(x, y) \mapsto (\lambda(x), \mu(x)y)$, which preserves the class of linear equations.
- The canonical Laguerre-Forsyth form is defined by conditions: $p_{k-1} = p_{k-2} = 0$.
- Then the following expressions become fundamental invariants for the class of linear equations:

$$\Theta_r = \sum_{j=1}^{r-1} (-1)^{j+1} \frac{(2r-j-1)!(k-r+j-1)!}{(r-j)!(j-1)!} p_{k-r+j-1}^{(j-1)},$$

  for $r = 3, \ldots, k$.
- *Generalized Wilczynski invariants* W_r, $r = 3, \ldots, k$ for a non-linear scalar ODE are defined as Wilczynski invariants Θ_r evaluated at the linearization of the system.

Fundamental invariants of a scalar ODE up to contact transformations (D.-2001)

Define

$$I_r = \frac{\partial^r F}{(\partial y^{(k-1)})^r}$$

Fundamental invariants of a scalar ODE of order $k \geq 4$ under contact transformations are:

- generalized Wilczynski invariants $W_3, \ldots, W_k$;
- invariant I_3 for 4th order ODE / I_2 for k-th order ODE, $k \geq 5$;
- one or two extra invariants:
  $k = 4$: $J_4 = F_{233} + \frac{1}{6}F_{33}^2 + \frac{9}{8}F_3 F_{333} + \frac{3}{4}F_{333x}$;
  $k = 5$: $J_5 = F_{234} - \frac{2}{3}F_{333} - \frac{1}{2}F_{34}^2 \mod I_2, W_3$;
  $k \geq 6$: $J_3 = F_{k-1,k-2} \mod I_2$;
  $k \geq 7$: $J_4 = F_{k-2,k-2} \mod I_2, J_3, W_3$.

Fundamental invariants of a scalar ODE up to point transformations

Fundamental invariants of a scalar ODE of order $k \geq 3$ under point transformations are:

- generalized Wilczynski invariants $W_3, \dots, W_k$;
- invariant I_3 for 3rd order ODE / I_2 for k-th order ODE, $k \geq 4$;
- extra invariants:
  $k = 3$: $J_4 = F_{122} + \frac{1}{6} F_{22}^2 \mod I_3$, $J_4^* = F_{02} - F_{12x} + F_{22xx}$ mod W_4;
  $k \geq 5$: $J_3 = F_{k-1,k-2} \mod I_2$;
  $k \geq 6$: $J_4 = F_{k-2,k-2} \mod I_2, J_3, W_3$;
  $k \geq 7$: one more additional invariant (no explicit formula yet).

Fundamental invariants of a scalar ODE up to fiber-preserving transformations

Fundamental invariants of a scalar ODE of order $k \geq 2$ under fiber-preserving transformations are:

- generalized Wilczynski invariants $W_3, \ldots, W_k$;
- invariant I_3 for 2rd order ODE / I_2 for k-th order ODE, $k \geq 3$;
- extra invariants:
  $k = 2$: $J_3 = F_{01} - F_{11x}$, $J_4 = F_{00} + \frac{1}{2}F_1 F_{01} - \frac{1}{2}F_0 F_{11} - \frac{1}{2}F_{01x}$;
  $k = 3$: $J_4 = F_{02} - F_{12x} \mod I_2$;
  $k \geq 4$: $J_3 = F_{k-1,k-2} \mod I_2$;
  $k \geq 5$: $J_4 = F_{k-2,k-2} \mod I_2, J_3, W_3$;
  $k \geq 6$: one or two more additional invariants (no explicit formula yet).