Non-commutative integrable discrete systems of a geometric origin

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Outline

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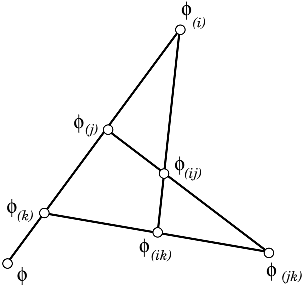
Desargues maps

Maps $\phi : \mathbb{Z}^{\widehat{N}} \to \mathbb{P}^M(\mathbb{D})$, $\widehat{N}, M \geq 2$, such that the points $\phi(\widehat{n})$, $\phi_{(i)}(\widehat{n})$ and $\phi_{(j)}(\widehat{n})$ are collinear, for all $\widehat{n} \in \mathbb{Z}^{\widehat{N}}$, $i \neq j$ *[AD 2010]*

Notation: $\phi_{(i)}(n_1, \dots, n_i, \dots, n_{\widehat{N}}) = \phi(n_1, \dots, n_i + 1, \dots, n_{\widehat{N}})$



Algebraic description in homogeneous coordinates $\boldsymbol{\Phi} : \mathbb{Z}^{\widehat{N}} \to \mathbb{D}^{M+1}$

$$\boldsymbol{\Phi} + \boldsymbol{\Phi}_{(i)} A_{ij} + \boldsymbol{\Phi}_{(j)} A_{ji} = 0, \qquad i \neq j, \qquad A_{ij} : \mathbb{Z}^{\widehat{N}} \to \mathbb{D}^{\times}$$

The non-commutative Hirota system

Gauge transformations: $\boldsymbol{\Phi} = \tilde{\boldsymbol{\Phi}} F$, where $F : \mathbb{Z}^{\widehat{N}} \to \mathbb{D}^{\times}$ - gauge function results in $\tilde{A}_{ij} = F_{(i)} A_{ij} F^{-1}$

One can find homogeneous coordinates such that $A_{ji} = -A_{ij} = U_{ij}^{-1}$

$$\boldsymbol{\Phi}_{(i)} - \boldsymbol{\Phi}_{(j)} = \boldsymbol{\Phi} U_{ij}, \qquad 1 \leq i \neq j \leq \widehat{N},$$

Fact

The gauge functions which do not change the structure of the above linear problem are characterized by the condition $F_{(i)} = F_{(j)}$ for all pairs of indices, i.e. F is a function of $n_1 + n_2 + \cdots + n_{\widehat{N}}$.

$$U_{ij} + U_{ji} = 0, \qquad U_{ij} + U_{jl} + U_{li} = 0,$$
$$U_{li} U_{lj(i)} = U_{lj} U_{li(j)} \qquad \Longrightarrow \qquad U_{ij} = \rho_i^{-1} \rho_{i(j)}$$

[Nimmo 2006]

The Hirota equation

When $\mathbb{D} = \mathbb{F}$ is commutative then the functions U_{ij} can be parametrized in terms of a single potential $\tau : \mathbb{Z}^{\widehat{N}} \to \mathbb{F}$

$$U_{ij} = \frac{\tau \tau_{(ij)}}{\tau_{(i)} \tau_{(j)}}, \qquad 1 \leq i < j \leq \widehat{N}$$

The nonlinear system reads *[Hirota 1981], [Miwa 1982]*

$$\tau_{(i)}\tau_{(jl)} - \tau_{(j)}\tau_{(il)} + \tau_{(l)}\tau_{(ij)} = 0, \qquad 1 \leq i < j < l \leq \widehat{N}$$

The non-commutative KP hierarchy

Let $\widehat{N} = N + 1$, we distinguish the last variable $k = n_{N+1}$, denote also

$$n = (n_1, \ldots, n_N), \quad \boldsymbol{\Phi}(n,k) = \boldsymbol{\Psi}_k(n), \quad U_{N+1,i}(n,k) = u_{i,k}(n)$$

which allows the rewrite a part (that with the distinguished variable) of the linear problem in the form

$$\boldsymbol{\Psi}_{k+1} - \boldsymbol{\Psi}_{k(i)} = \boldsymbol{\Psi}_k u_{i,k}, \qquad i = 1, \ldots, N.$$

[Kajiwara, Noumi, Yamada 2002]

The compatibility of the above linear system reads

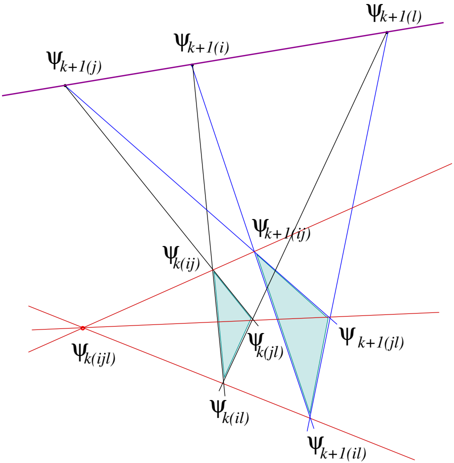
$$u_{j,k} u_{i,k(j)} = u_{i,k} u_{j,k(i)}, \qquad i \neq j,$$

$$u_{i,k(j)} + u_{j,k+1} = u_{j,k(i)} + u_{i,k+1}.$$

The first part allows to define potentials $r_k(n) = \rho_{N+1}(n,k)$ such that $u_{i,k} = r_k^{-1} r_{k(i)}$, while the other equations give the system

$$(r_{k(j)}^{-1} - r_{k(i)}^{-1}) r_{k(ij)} = r_{k+1}^{-1}(r_{k+1(i)} - r_{k+1(j)}), \qquad i \neq j$$

Four dimensional consistency of Desargues maps



[AD 2010]

Outline

Periodic Desargues maps: $\phi_{k+P}(n) = \phi_k(n)$

$$\boldsymbol{\Psi}_{k+P}(n) = \boldsymbol{\Psi}_k(n)\mu_k(n), \quad \mu_{k+1}(n) = \mu_{k(i)}(n), \quad r_{k+P} = r_k\mu_k$$

Matrix linear problem

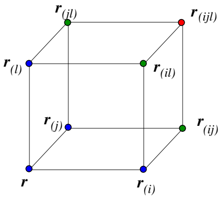
$$(\boldsymbol{\Psi}_1, \dots, \boldsymbol{\Psi}_P)_{(i)} = (\boldsymbol{\Psi}_1, \dots, \boldsymbol{\Psi}_P) \begin{pmatrix} -u_{i,1} & 0 & \cdots & 0 & \mu_1 \\ 1 & -u_{i,2} & 0 & \dots & 0 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & & & -u_{i,P-1} & 0 \\ 0 & 0 & \dots & 1 & -u_{i,P} \end{pmatrix}$$

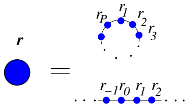
where μ_1 is a function of the variable $n_\sigma = n_1 + \cdots + n_N$.
The corresponding (lattice non-isospectral non-commutative modified Gel'fand–Dikii) system of non-linear equations

$$\begin{aligned} (r_{k(j)}^{-1} - r_{k(i)}^{-1}) r_{k(ij)} &= r_{k+1}^{-1}(r_{k+1(i)} - r_{k+1(j)}), \quad k = 1, \dots, P-1, \\ (r_{P(j)}^{-1} - r_{P(i)}^{-1}) r_{P(ij)} &= \mu_1^{-1} r_1^{-1}(r_{1(i)} - r_{1(j)})\mu_{1(\sigma)} \qquad i \neq j. \end{aligned}$$

Three dimensional consistency of the GD systems

$\boldsymbol{r} = (r_k)$ where $k \in \mathbb{Z}/(P\mathbb{Z})$ – periodic case, or $k \in \mathbb{Z}$ in the full KP case





Multidimensional consistency of a discrete system — possibility of extending the number of independent variables of the system by adding its copies in different directions

Fact

The lattice non-isospectral non-commutative modified Gel'fand–Dikii system is three-dimensionally consistent.

Commutative reductions of the GD systems

Lemma

Given solution r_k, $k = 1, \dots, P$ of the non-isospectral GD system, if the division ring $\mathbb{D}$ is commutative then the function $R = r_1 r_2 \dots r_P$, splits into the product of functions of single variables n_i, $i = 1, \dots, N$, and a function of the sum of independent variables.

Proof.

Multiply all the P equations of of the GD system to get

$$\hat{R}_{(ij)} \hat{R} = \hat{R}_{(i)} \hat{R}_{(j)},$$

where $\hat{R} = RM^{-1}$ and $M_{(\sigma)} = \mu_1 M$. This gives the the factorization $R = \Gamma_1(n_1)\Gamma_2(n_2) \dots \Gamma_N(n_N) M(n_\sigma)$. □

Commutative reduction for $P = 2$

$$r_1 r_2 = G_1^2(n_1)\dots G_N^2(n_N)M(n_\sigma), \qquad \Gamma_i = G_i^2, \quad G_i(n_i{+}1) = F_i(n_i)G_i(n_i)$$

$$r_1 = xG, \qquad r_2 = \frac{GM}{x}, \qquad G = G_1 G_2 \dots G_N$$

non-isospectral non-autonomous lattice modified KdV equations

$$x_{(ij)} = \mu_1 x \, \frac{x_{(i)} F_j - x_{(j)} F_i}{x_{(j)} F_j - x_{(i)} F_i}, \qquad i \neq j$$

Commutative reduction for $P = 3$

$$r_1 r_2 r_3 = G_1^3(n_1)\dots G_N^3(n_N)M(n_\sigma), \qquad G_i(n_i+1) = F_i(n_i)G_i(n_i)$$

$$r_1 = xG, \qquad r_2 = \frac{yG}{x}, \qquad r_3 = \frac{GM}{y}, \qquad G = G_1G_2\dots G_N$$

$$x_{(ij)} = \frac{x}{y}\,\frac{y_{(j)}x_{(i)}F_j - y_{(i)}x_{(j)}F_i}{x_{(j)}F_j - x_{(i)}F_i}, \qquad y_{(ij)} = \mu_1 x\,\frac{y_{(i)}F_j - y_{(j)}F_i}{x_{(j)}F_j - x_{(i)}F_i}, \qquad i \neq j$$

It can be rewritten in scalar form as non-isospectral non-autonomous lattice modified Boussinesq equations

$$\left(\frac{\mu_1}{y_{(ij)}}\left(y_{(i)}F_j - y_{(j)}F_i\right)\right)_{(ij)} - \mu_1 y\left(\frac{F_j}{y_{(j)}} - \frac{F_i}{y_{(i)}}\right) =$$

$$= \left(\frac{y_{(ij)}}{\mu_1 y}\,\frac{y_{(j)}F_j^2 - y_{(i)}F_i^2}{y_{(i)}F_j - y_{(j)}F_i}\right)_{(j)} - \left(\frac{y_{(ij)}}{\mu_1 y}\,\frac{y_{(i)}F_i^2 - y_{(j)}F_j^2}{y_{(j)}F_i - y_{(i)}F_j}\right)_{(i)}, \qquad i \neq j$$

[Nijhoff, Papageorgiou, Capel, Quispel, 1992], [Nijhoff 1999]

Outline

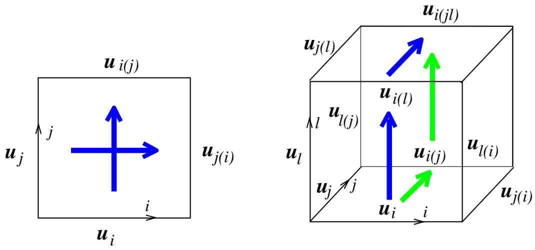
Multidimensional consistency of the KP map

Theorem

The non-commutative KP map (edge system $u_{i,k} = r_k^{-1} r_{k(i)}$)

$$u_{i,k(j)} = (u_{i,k} - u_{j,k})^{-1} u_{i,k}(u_{i,k+1} - u_{j,k+1}), \qquad 1 \le i \ne j \le N,$$

is multidimensionaly consistent



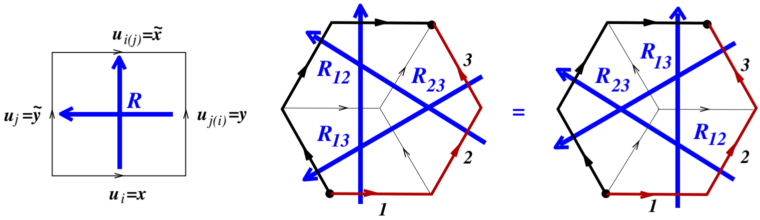
$\boldsymbol{u}_i = (u_{i,k})$, $k \in \mathbb{Z}$ or $k \in \mathbb{Z}/(P\mathbb{Z})$, $u_{i,k+P} = \mu_k^{-1} u_{i,k} \mu_{k(i)}$

From KP map to Yang-Baxter map

A map $R\colon \mathcal{X}\times\mathcal{X}$ is called Yang–Baxter map if

$$R_{12}\circ R_{13}\circ R_{23} = R_{23}\circ R_{13}\circ R_{12}, \qquad \text{in} \quad \mathcal{X}\times\mathcal{X}\times\mathcal{X}$$

If moreover $\pi\circ R\circ\pi\circ R = \mathrm{Id}_{\mathcal{X}\times\mathcal{X}}$, where π is the transposition, then R is called reversible YB map



Non-commutative rational Yang–Baxter maps

Theorem

Given two assemblies of non-commutating variables $\boldsymbol{x} = (x_1, \dots, x_P)$, $\boldsymbol{y} = (y_1, \dots, y_P)$ define polynomials

$$\mathcal{P}_k = \sum_{a=0}^{P-1} \left(\prod_{i=0}^{a-1} y_{k+i} \prod_{i=a+1}^{P-1} x_{k+i} \right), \qquad k = 1, \dots, P,$$

where subscripts in the formula are taken modulo P. If the products $\alpha = x_1 x_2 \dots x_P$ and $\beta = y_1 y_2 \dots y_P$ are central then the map

$$R(\boldsymbol{x}, \boldsymbol{y}) = (\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}), \qquad \tilde{x}_k = \mathcal{P}_k x_k \mathcal{P}_{k+1}^{-1}, \qquad \tilde{y}_k = \mathcal{P}_k^{-1} y_k \mathcal{P}_{k+1},$$

is reversible Yang–Baxter map

commutative case *[Etingov 2003]*

Non-commutative F_{III} map

Fact

The products α and β are conserved (for arbitrary P)

The simplest case: $P = 2$ we put $x = x_1$, $y = y_1$ to get a parameter dependent reversible Yang–Baxter map $R(\alpha, \beta) : (x, y) \mapsto (\tilde{x}, \tilde{y})$

$$\tilde{x} = \left(\alpha x^{-1} + y\right) x \left(x + \beta y^{-1}\right)^{-1},$$
$$\tilde{y} = \left(\alpha x^{-1} + y\right)^{-1} y \left(x + \beta y^{-1}\right),$$

which in the commutative case is equivalent to the F_{III} map in the list of *[Adler, Bobenko, Suris 2004]*

Non-commutative Gel'fand–Dikii systems with centrality assumptions

Proposition

In the P-periodic reduction $u_{i,k+P} = \mu_k^{-1} u_{i,k} \mu_{k(i)}$ of the non-commutative KP system assume centrality of the monodromy factors μ_k and of the products $\mathcal{U}_i = u_{i,1} u_{i,2} \dots u_{i,P} \mu_k^{-1}$. Then $\mathcal{U}_i$ is a function of n_i only.

In particular, for $P = 2$ we obtain in full analogy to the commutative case the non-autonomous, non-isospectral lattice modified KdV equation for non-commutative variable x

$$\left(x_{(j)}^{-1} F_i - x_{(i)}^{-1} F_j \right) x_{(ij)} = \mu_1 \left(x_{(i)}^{-1} F_i - x_{(j)}^{-1} F_j \right) x, \qquad F_i = (\mathcal{U}_i)^{1/P}$$

iso-spectral case *[Bobenko, Suris 2002]*

Conclusion

- Periodic reductions of Desargues maps lead to non-commutative, non-isospectral, and "non-autonomous" analogues of the modified Gel'fand–Dikii hierarchy
- Both (vertex and edge) systems are multidimensionally consistent
- The companion map of the edge system can be found under "centrality" assumption and gives a non-commutative rational invertible Yang-Baxter map
- The non-commutative Gel'fand–Dikii systems under "centrality" assumption give non-isospectral, and non-autonomous (with central non-isospectral and non-autonomous factors) non-commutative equations

References

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