Exact and asymptotic solutions of a system of nonlinear shallow water equations in basins with gentle shores

S.Yu. Dobrokhotov, V.A.Kalinichenko, D.S. Minenkov, V.E. Nazaikinskii

Ishlinsky Institute for Problems in Mechanics RAS, Moscow
and
Moscow institute of Physics and Technology

Supported by Russian Science Foundation

Independent University
Moscow
29 March 2023

PISTON PROBBLEMS for TSUNAMI WAVES
2-D SHALLOW WATER EQUATION

$$\frac{\partial \boldsymbol{\eta}}{\partial t} + div((\eta + D(x))\mathbf{u}) = 0, \quad \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}, \nabla)\mathbf{u} + g\nabla\boldsymbol{\eta} = 0.$$

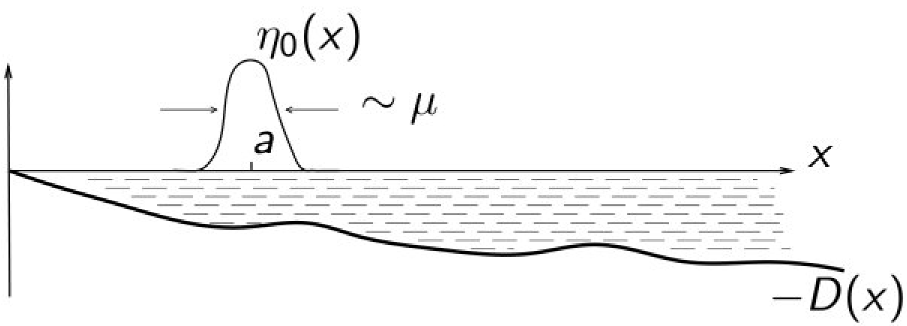
LOCALISED INITIAL DATA

$$\mathbf{u}|_{t=0} = 0, \quad \boldsymbol{\eta}|_{t=0} = \eta^0(\frac{x - x^0}{\mu})$$

$$\mu = \frac{\text{characteristic size of the source}}{\text{characteristic size of the basin}} \quad \text{is small}$$

the free boundary problem $x \in \Omega_t \in \mathbb{R}^2$

$$\eta(\mathbf{x}, t) + D(\mathbf{x})|_{x \in \partial\Omega_t} = 0$$



1) PISTON PROBBLEMS for TSUNAMI WAVES

2-D SHALLOW WATER EQUATION

$$\frac{\partial \boldsymbol{\eta}}{\partial t} + div((\eta + D(x))\mathbf{u}) = 0, \quad \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}, \nabla)\mathbf{u} + g\nabla\boldsymbol{\eta} = 0.$$

LOCALISED INITIAL DATA

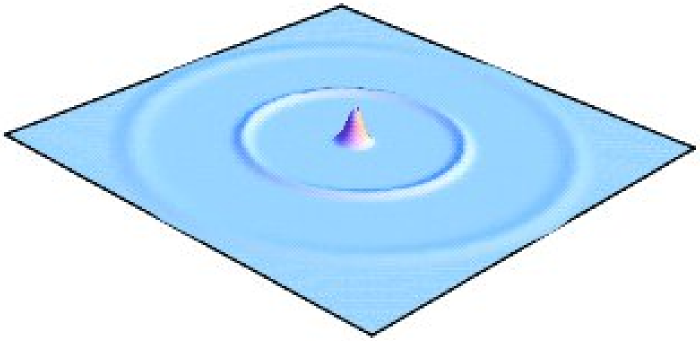
$$\mathbf{u}|_{t=0} = 0, \quad \boldsymbol{\eta}|_{t=0} = \eta^0(\frac{x - x^0}{\mu})$$

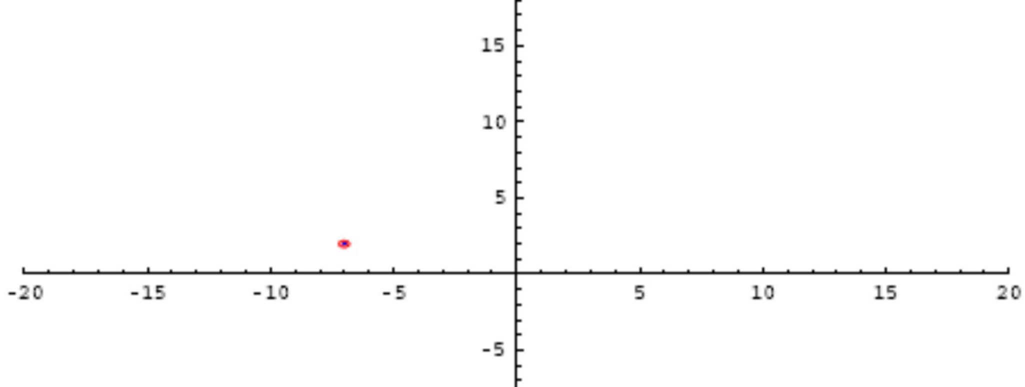
LINEARIZATION IN the OPEN OCEAN

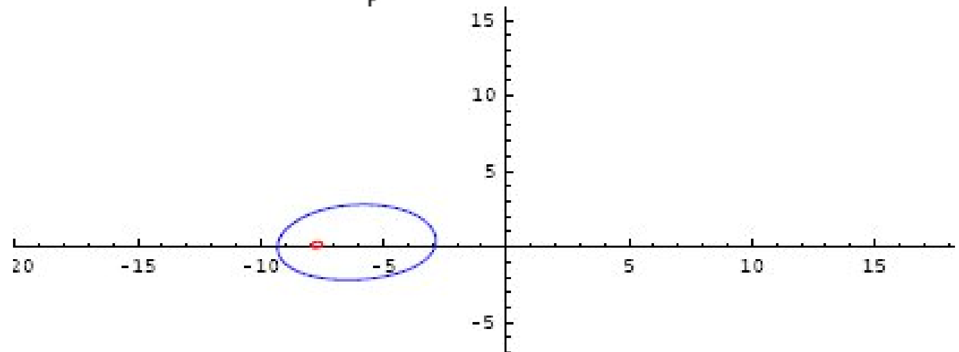
$$\frac{\partial \boldsymbol{\eta}}{\partial t} + div((\eta + D(x))\mathbf{u}) = 0, \quad \frac{\partial \mathbf{u}}{\partial t} + g\nabla\boldsymbol{\eta} = 0 \quad \Longrightarrow \quad \frac{\partial^2 \boldsymbol{\eta}}{\partial t^2} = \nabla(c^2(x)\nabla\boldsymbol{\eta}).$$

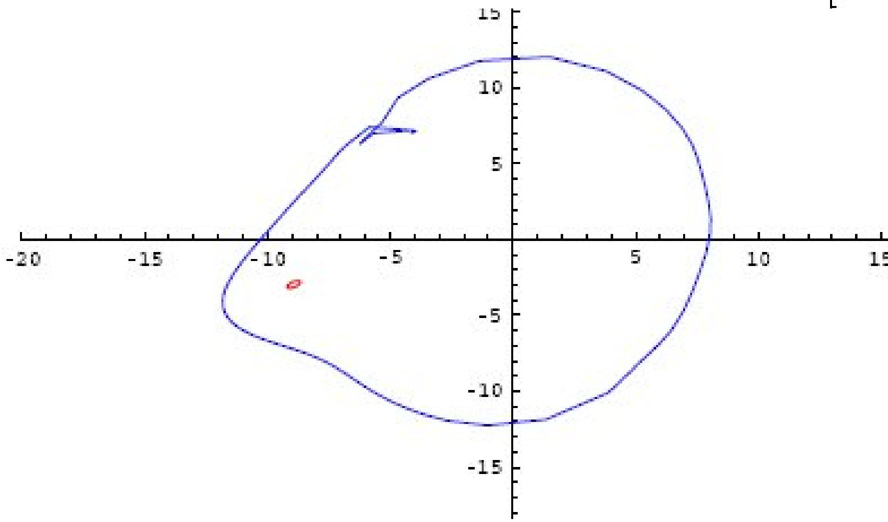
$c^2(x) = gD(x)$ is a slow varying function

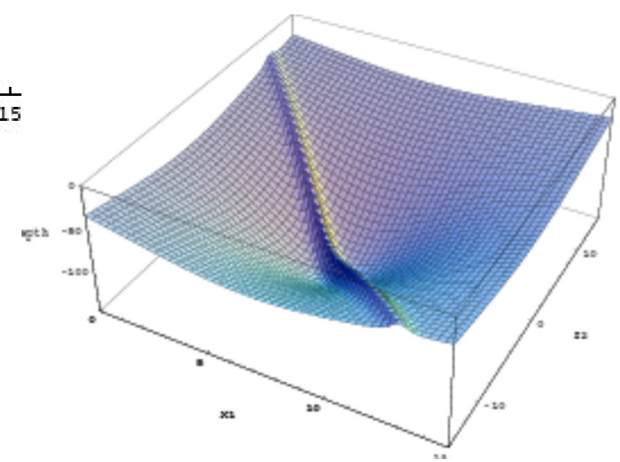
$$\mu = \frac{\text{characteristic size of the source}}{\text{characteristic size of the basin}} \quad \text{is small}$$




15
10
5
-5
-20
-15
-10
-5
5
10
15
20


15
10
5
-5
20
-15
-10
-5
5
10
15


15
10
5
-5
-10
-15
-20
-15
-10
-5
5
10
15

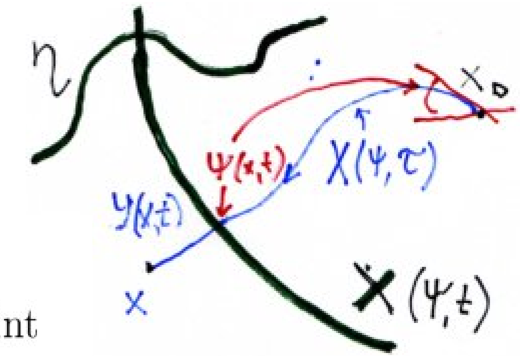


The asymptotics near regular points of the front edge and

Heisenberg uncertainty principle: $\dim\Lambda_t = 2$, $\dim\pi_x(\Lambda_t) = 1$!!!!!

The leading edge front = nonstandard caustic (the new formulas are working)

The wave field outside the focal points



$$\eta \approx \sqrt{\frac{\mu}{|X_\psi(\psi,t)|}}\sqrt[4]{\frac{D(x_0)}{D(\mathbf{x})}}\mathrm{Re}\Big[\mathrm{e}^{-\mathrm{i}\pi \mathrm{m}(\psi,\mathrm{t})/2}\mathrm{F}\Big(\frac{\mathrm{y(x,t)}}{\mu}\sqrt{\frac{\mathrm{D(x_0)}}{D(\mathbf{x})}},\psi\Big)|_{\psi=\psi(\mathrm{x,t})}\Big]$$

Here

$y(x,t)$ is the alternative distance between the point x and the closest point $X(\psi(x,t),t)$ on the front,

$\psi(x,t)$ is the correspondence angle (coordinate) on the front,

$m((\psi,t))$ is the Morse (Maslov) index of this point.

$$F(s,\psi) = \frac{e^{-i\pi/4}}{\sqrt{2\pi}}\int_0^\infty \widetilde{\eta}^0(\rho\mathbf{n}(\psi))\sqrt{\rho}e^{is\rho}\,d\rho. \qquad \widetilde{\eta}^0(k) = \frac{1}{2\pi}\int_{\mathbb{R}^2}\eta^0(z)e^{i\langle k,z\rangle}\,dz\ , \quad \mathbf{n}(\psi) = \begin{pmatrix}\cos\psi\\ \sin\psi\end{pmatrix}$$

Asymmetric Sretenskii, Cherkesov, Dotsenko, Sergievskii, Le Mehaute, Wang, Chia-Chi Lu source

$$\eta^0(z) = \frac{A}{(1+(z_1/B_1)^2+(z_2/B_2)^2)^{3/2}}, \qquad F(s,\psi) = \frac{Ae^{-i\frac{\pi}{4}}}{2\sqrt{2}\left(\sqrt{B_1^2\cos^2\psi + B_2^2\sin^2\psi} - is\right)^{3/2}}.$$

Caustics: the leading front edge

with strong focal (turning) points $X_\psi = 0$ + shore $D(x) = 0$

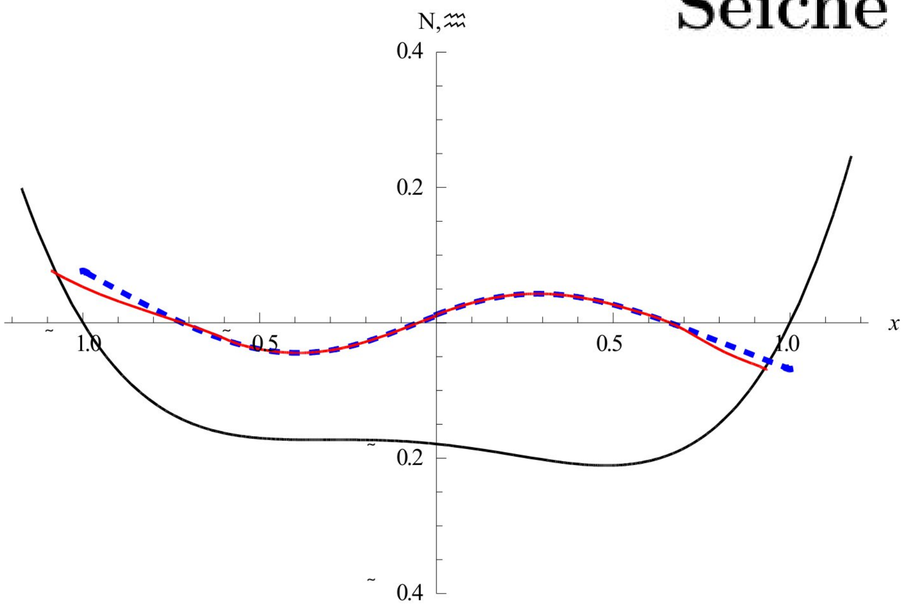
2) 1-D Shallow water: two shores

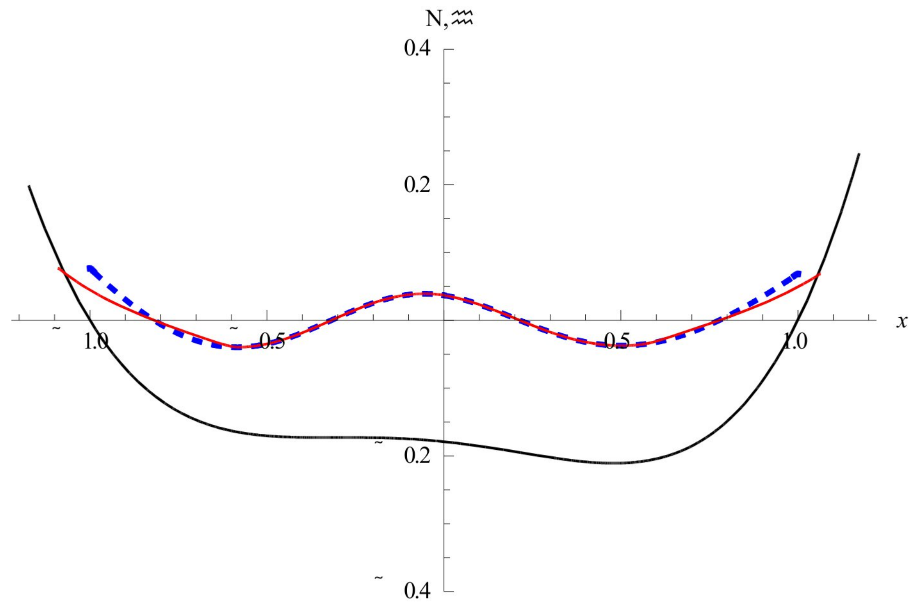
$$\eta_t + \left((D(x)+\eta)u\right)_x = 0, \quad u_t + g\eta_x + uu_x = 0$$

Depth: $D(a) = D(b) = 0 \quad D'(a) \neq 0, \quad D'(b) \neq 0$

Boundary condition at two variable boundaries x(t): $\eta(x_a(t),t) + x_a(t) = 0$

Seiche





3) Bessel functions in nonlinear problems of coastal waves and billiards with semi-rigid walls

The shallow water equations

$$\eta_t + \langle \nabla, D(x)\mathbf{u} \rangle + \varepsilon \langle \nabla, \eta \mathbf{u} \rangle = 0, \quad \mathbf{u}_t + g\nabla \eta + \varepsilon \langle \mathbf{u}, \nabla \rangle \mathbf{u} = 0, \quad (\varepsilon \eta(x,t) + D(x))|_{\Gamma(t)} = 0$$

Bottom $\quad D(x_1, x_2) = D(\rho) = a(R^2 - \rho^2)$

The reduction to the wave equation and Laplace-Beltrami type equation

$$\frac{\partial^2 N}{\partial t^2} = \langle \nabla, gD(x)\nabla N \rangle \qquad N = Re(e^{i\omega t}\psi(x))$$

$$\hat{\mathcal{H}}\psi \equiv -\mu^2 < \nabla, gD(x_1, x_2)\nabla \psi >= \omega^2 \psi, \quad (x_1, x_2) \in \Omega.$$

Nonstandard Liouville tori

$$\Lambda^2_{\omega,c} = \left\{H = a(R^2-\rho^2)\left(p_\rho^2 + \frac{p_\phi^2}{\rho^2}\right) = \omega^2, p_{\phi^2} = c^2\right\}$$

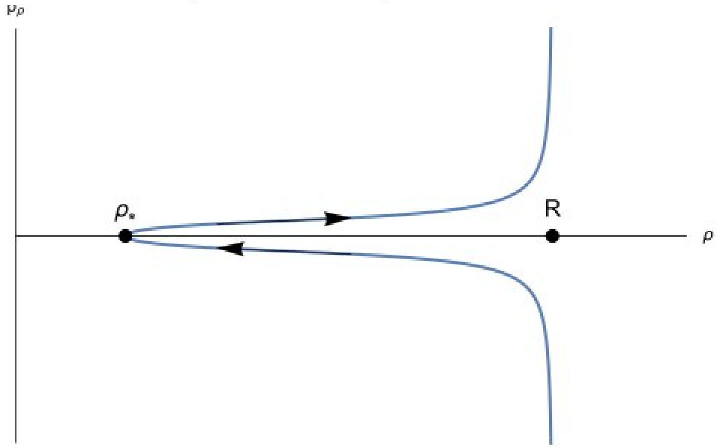


Рис. 1: Проекция лагранжева многообразия Λ на плоскость (ρ, p_ρ)

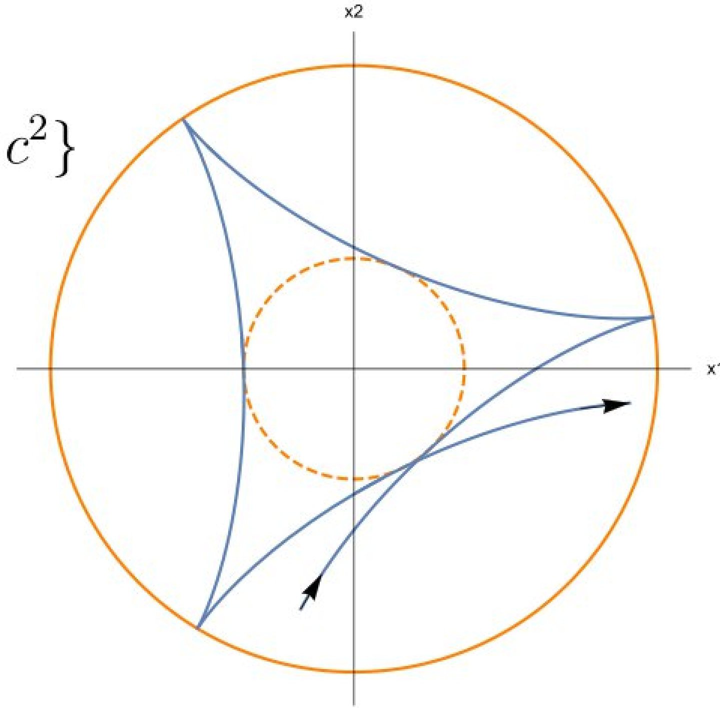
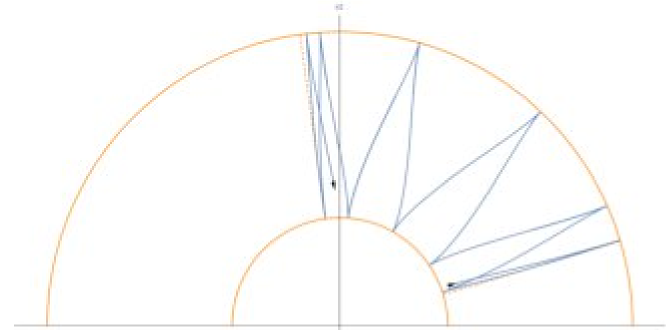


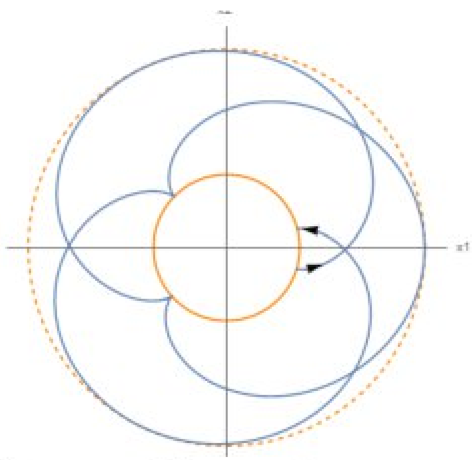
Рис. 2: Проекция траекторий гамильтоновой системы на плоскость (x_1, x_2). Сплошная окружность – граница берега (полужесткая стенка), пунктирная окружность – простая каустика (мягкая стенка)

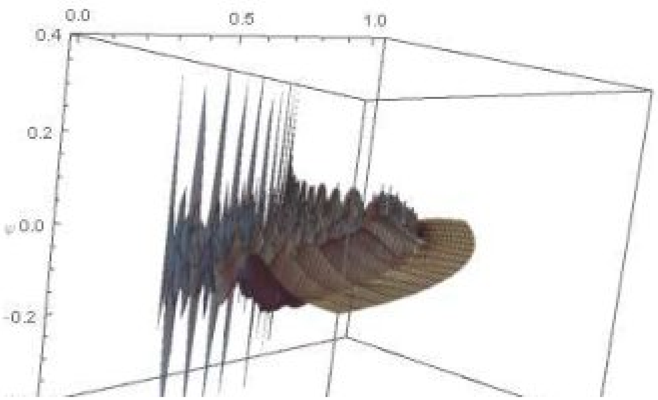
Billiard with semi-rigid walls

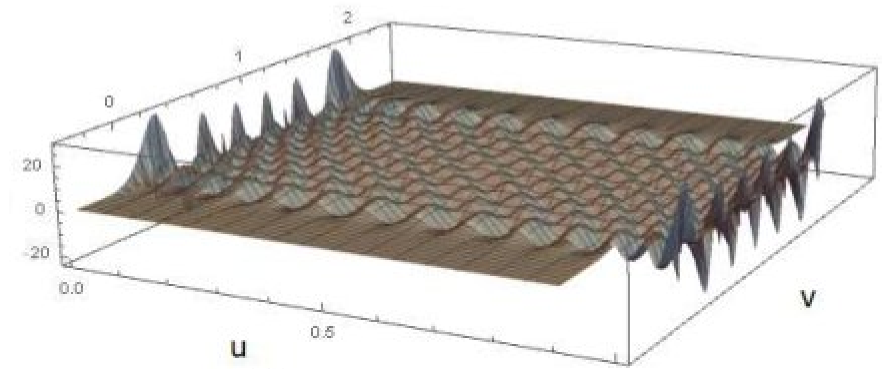
2-D seiche

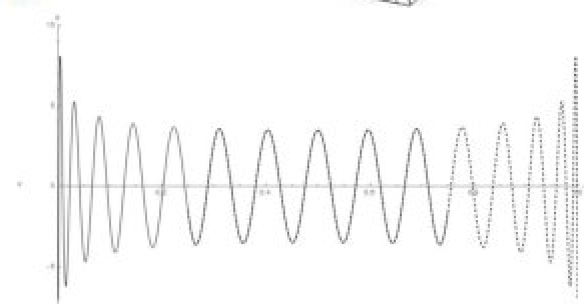


2-D Coastal waves









A. Yu. Anikin, S. Yu. Dobrokhotov, V. E. Nazaikinskii, A. V. Tsvetkova, NONSTANDARD LIOUVILLE TORI AND CAUSTICS IN ASYMPTOTICS IN THE FORM OF AIRY AND BESSEL FUNCTIONS FOR 2D STANDING COASTAL WAVES, St. Petersburg Math. J. Vol. 33 (2022), No. 2, 185-205

The first step:

Linearization of free-boundary problem for the water waves $\Longrightarrow$

Wave equation with localized source

$\Omega \subset \mathbf{R}^2$ a domain;

$\partial\Omega$ smooth

wave propagation in Ω
from a source localized near $x_0 \in \Omega$:

$$\eta_{tt} - \langle \nabla, c^2(x)\nabla \rangle \eta = 0$$

$$\eta|_{t=0} = V(\mu^{-1}(x - x_0)), \quad \eta_t|_{t=0} = 0$$

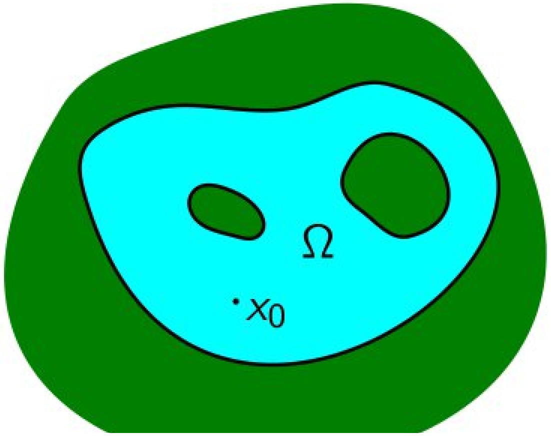
$V(y) \in C^\infty$ decays at ∞; $\mu \to 0$

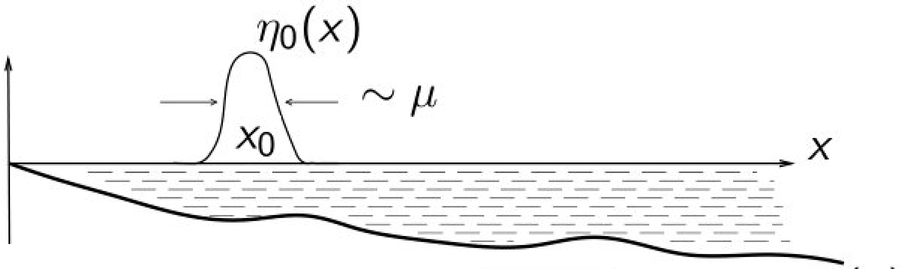
$c^2(x) \in C^\infty$; $c^2(x)|_{\partial\Omega} = 0$;

$\nabla c^2(x)|_{\partial\Omega}$ vanishes nowhere

$$\mu = \frac{\text{characteristic size of the source}}{\text{characteristic size of the basin}}$$ is small

Task: find asymptotic solution as $\mu \to 0$





$c(x) = \sqrt{gD(x)}$

Oleinik, Radkevich (1969) no "classical" boundary conditions needed:
$\|\eta_t\|^2_{L^2} + \|c^2(x)\nabla\eta\|^2_{L^2}$ is bounded

General

E. N. Pelinovskii, *Hydrodynamics of Tsunami Waves* (Nizhnii Novgorod, 1996)

Yu. I. Shokin, L. B. Chubarov, A. G. Marchuk, and K. V. Simonov, *Numerical Experiment in Tsunami Problem* (Nauka, Siberian Division, Novosibirsk, 1989).

V.P. Maslov, *Perturbation Theory and Asymptotic Methods* (Moscow University, Moscow, 1965; Dunod, Paris, 1972).

V.P. Maslov and M.V. Fedoryuk, *Semiclassical Approximation for the Equations of Quantum Mechanics* (Nauka, Moscow, 1976; Reidel, Dordrecht, 1981).

Global asymptotics without run-up

S. Yu. Dobrokhotov, S. Ya. Sekerzh-Zenkovich, B. Tirozzi, and T. Ya. Tudorovskiy, "Description of a Tsunami Wave Propagation on the Ground of the Maslov Canonical Operator," Dokl. Akad. Nauk **409** (2), 171–175 (2006) [Doklady Mathematics **74** (1) 592–596 (2006)].

S. Dobrokhotov, A. Shafarevich, and B. Tirozzi, "Localized Wave and Vortical Solutions to Linear Hyperbolic Systems and Their Application to the Linear Shallow Water Equations," Russ. J. Math. Phys. **15** (2), 192–221 (2008).

S. Yu. Dobrokhotov, R. Nekrasov, and B. Tirozzi, "Asymptotic Solutions of the Linear Shallow-Water Equations with Localized Initial Data," J. Engrg. Math. **69** (2–3), 225–242 (2011).

S. Yu. Dobrokhotov, S. Ya. Sekerzh-Zenkovich, B. Tirozzi, and B. Volkov, "Explicit Asymptotics for Tsunami Waves in Framework of the Piston Model," Russ. J. Earth Sciences **8** (ES403), 1–12 (2006).

M. V. Berry, "Focused Tsunami Waves," Proc. R. Soc. A **463** (2087), 3055–3071 (2007).

Run-up

G.F. Carrier and H.P. Greenspan, Water waves of finite amplitude on a sloping beach, *J. Fluid Mech.* **4**(1), 97–109 (1958).

В. А. Фок, "О каноническом преобразовании в классической и квантовой механике" *Вестн. Ленинградск. ун-та*, 1959, № 16, 67–70.

E. Pelinovsky and R. Mazova, "Exact Analytical Solutions of Nonlinear Problems of Tsunami Wave Run-Up on Slopes with Different Profiles," Natur. Hazards **6**, 227–249 (1992).

T. Vukašinac and P. Zhevandrov, "Geometric Asymptotics for a Degenerate Hyperbolic Equation," Russ. J. Math. Phys. **9** (3), 371–381 (2002).

S.Yu. Dobrokhotov and B. Tirozzi, "Localized Solutions of the One-Dimensional Nonlinear Shallow Water Equations with Velocity $c = \sqrt{x}$," Uspekhi Mat. Nauk **65** (1) (391), 185–186 (2010).

S.Yu. Dobrokhotov, V.E. Nazaikinskii, and B. Tirozzi, "Asymptotic Solutions of the Two-Dimensional Model Wave Equation with Degenerating Velocity and with Localized Initial Data," Algebra i Analiz **22** (6), 67–90 (2010).

V.E. Nazaikinskii, "Phase Space Geometry for a Wave Equation Degenerating on the Boundary of the Domain," Mat. Zametki, **92** (1), 153–156 (2012).

S. Yu. Dobrokhotov, V. E. Nazaikinskii, B. Tirozzi, Two-dimensional wave equation with degeneration on the curvilinear boundary of the domain and asymptotic solutions with localized initial data, Russian Journal of Mathematical Physics, Vol. 20, No. 4, 2013, pp. 389-401

S.Yu.Dobrokhotov, V.E.Nazaikinskii, Characteristics with Singularities and the Boundary Values of the Asymptotic Solution of the Cauchy Problem for a Degenerate Wave Equation, Mathematical Notes, 2016, Vol. 100, No. 5

V. A. Chugunov, S.A. Fomin, W. Noland, B.R. Sagdiev, Tsunami run up on a sloping beach, https://doi.org/10.1002/cmm4.1081.

S. Yu. Dobrokhotov, D. S. Minenkov, and V. E. Nazaikinskii, On Asymptotic Solutions of the Cauchy Problem for a Nonlinear System of Shallow Water Equations in a Basin with Gently Sloping Banks, Russian Journal of Mathematical Physics, Vol. 29, No. 1, 2022, pp. 28-36

S. Yu. Dobrokhotov, V. E. Nazaikinskii, A.V. Tsvetkova, Nonlinear effects and splash of coastal waves in the framework of shallow water theory, Proc. Steklov Inst. Math., (2023)

S. Yu. Dobrokhotov, V. A. Kalinichenko, D. S. Minenkov, V. E. Nazaikinskii Asymptotics of Long Standing Waves in One-Dimensional Basins with Shallow Coasts: Theory and Experiment, Appl.Math.Mech./Fluid Dynamics (2023)

Difficulties of the numerical analysis: free boundary in nonlinear case, there is no standard boundary conditions in the linear case

Linear asymptotic analysis: non standard caustics appear and interact, they are: the leading front edge (very strong caustic) and the shoreline

Linear case: shoreline $\partial\Omega = \{D(x) = 0\}$, the domain Ω is bounded by $\partial\Omega$.

Assumption: $\partial\Omega$ is smooth and the normal derivative to the shore is not degenerate$\frac{\partial D}{\partial \mathbf{n}}|_{\Gamma_0} = \nabla D|_{\Gamma_0} \neq 0$.

The main defect in the linear model:
it does not describe the splash (run up)

Nonlinear case: the free boundary problem

$$\eta(\mathbf{x}, t) + D(\mathbf{x}) = 0.$$

$\Longrightarrow$ **Carrier-Greenspan transform or its asymptotic modification near the beach**

1-D SHALLOW WATER EQUATION OVER NONUNIFORM BOTTOM WITH THE DEPTH $D(x)$

$$\eta_t + \frac{\partial}{\partial x}[v(\eta + D(x))] = 0, \quad v_t + vv_x + g\eta_x = 0$$

$\eta(x,t) \rightarrow$ free elevation

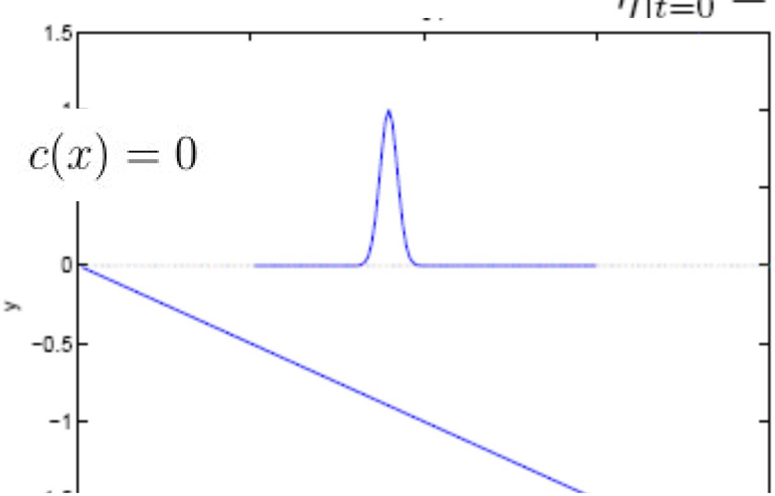
$v(x,t) \rightarrow$ velocity

$D(x) = \gamma^2 x + O(x^2) \rightarrow$ depth

$$c^2(x) = gD(x)$$

CAUCHY PROBLEM:

$$\eta|_{t=0} = e(x)V\left(\frac{x-a}{\mu}\right), \quad v|_{t=0} = 0$$



$$\eta_t + \frac{\partial}{\partial x}[v(\eta + \gamma x)] = 0, \; v_t + vv_x + g\eta_x = 0$$

CARRIER-GREENSPAN TRANSFORM

THE LINEAR WAVE EQUATION for $N(\tau, y), U(\tau, y)$:

$$N_\tau + \frac{\partial}{\partial y}(\gamma^2 y U) = 0, \quad U_\tau + g N_y = 0, \qquad g = 1,\ \gamma = 1$$

CONSIDER the SYSTEM

$$x = y - N(\tau, y) + \frac{1}{2}U^2(\tau, y), \quad t = \tau + U(\tau, y)$$

Let it defines one-to-one map from $\{y \geq 0, \tau \in \mathbb{R}\}$ to the value area of the right hand side

THEN

$$\eta(t, x) = N(\tau, y) - \frac{1}{2}U^2(\tau, y), \quad v(t, x) = U(\tau, y)$$

are the solution to the **ORIGINAL NONLINEAR SYSTEM** in a **PARAMETRIC FORM**

$$\eta_t + \frac{\partial}{\partial x}[v(\eta + \gamma x)] = 0,\ v_t + v v_x + g\eta_x = 0$$

Remark.

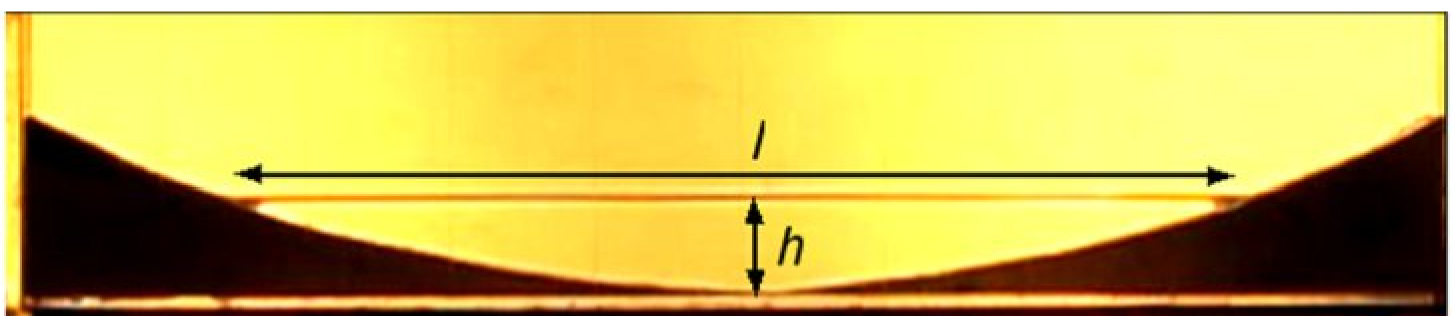
The 1-D *linear* shallow water equation is a Tricomi type equation. But the methods of studying such equations here do not give anything reasonable. ***It is necessary to investigate a nonlinear problem.***

$$\text{Standing waves } N = \cos(\omega_n t)\tilde{N}_n$$

Parabolic bottom: exact solutions to linear problem

$$D = D_0((y/\beta)^2 - 1), \quad \omega_n^2 = \frac{gD_0}{\beta^2} n(n+1), \quad N = AL_n(\frac{y}{\beta}),$$

here D_0 is the maximum depth, 2β is the basin size, $\tilde{N}_n$
$L_n(z)$ is the n-th *Legendre* polynomial .



Shallow water: slopping bottom, Carrier—Greenspan

$$\eta_t + \left((D(x)+\eta)u\right)_x = 0, \quad u_t + g\eta_x + uu_x = 0$$

Depth $D(x) = \gamma(x-a)$:

)

Withount loss of generality: g=1, a=0,, $\gamma = 1$

Carrier—Greenspan transform.

$$t = \tau + U, \quad x = y - N + U^2/2, \quad \eta = N - U^2/2, \quad u = U \Longleftrightarrow$$

$$J \equiv \frac{\partial(\tau,y)}{\partial(t,x)} = 1 + \eta_x - u_t - \eta_x u_t + \eta_t u_x, \quad J^{-1} \equiv \frac{\partial(t,x)}{\partial(\tau,y)} = 1 - N_y + U_\tau - N_y U_\tau + N_\tau U_y + UU_y$$

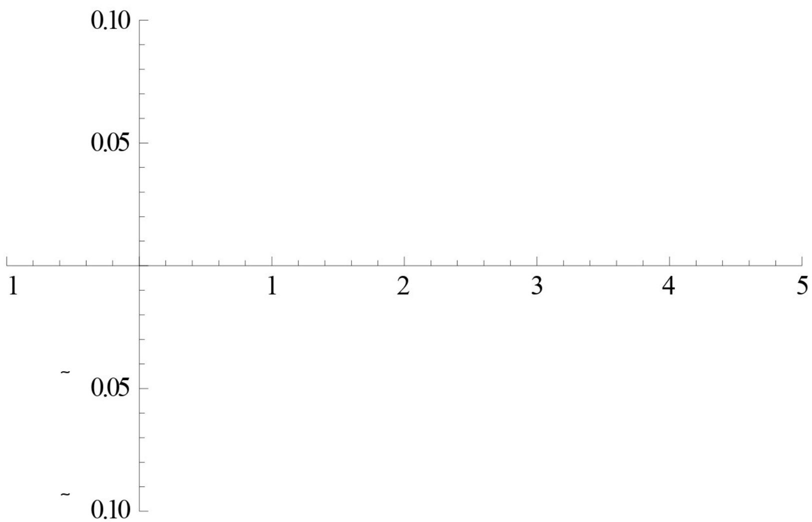
Theorem (C—G).

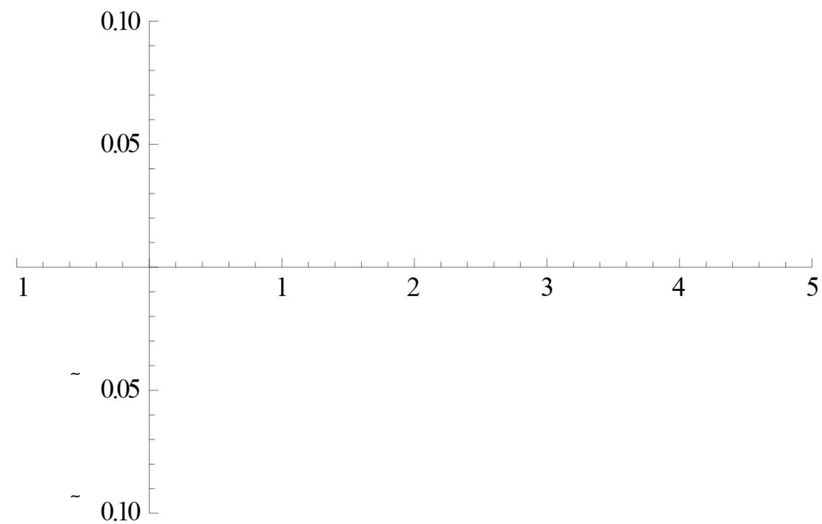
If $J > 0$, $J < \infty$
then shallow water
is equivalent to linearized SW:

$$\begin{pmatrix} \dfrac{\partial v}{\partial t} + \dfrac{\partial[\eta + v^2/2]}{\partial x} \\ \dfrac{\partial \eta}{\partial t} + \dfrac{\partial[(\eta + x)v]}{\partial x} \end{pmatrix} = \frac{\partial(\tau,y)}{\partial(t,x)} \begin{pmatrix} \dfrac{\partial U}{\partial \tau} + \dfrac{\partial N}{\partial y} \\ \dfrac{\partial N}{\partial \tau} + \dfrac{\partial[yU]}{\partial y} \end{pmatrix} = 0$$

)

Linear and nonlinear interaction with the focal point (shore): jump of the Maslov index, the Hilbert transform, the profile metamorphosis and the creation of the "N-wave" (smoothed Dirac-δ function $\to$ smoothed $1/x$- Sokhotskiy function)





Exact solitary-type and smooth shock-type solutions

$$u_{tt} - c^2 \Delta u = 0\,, \qquad c \in \mathbb{R}\,, \quad c > 0$$

$$u(x,t) = \operatorname{Re} \frac{A(1 + ict/\mu)}{((1 + ict/\mu)^2 + |x|^2/\mu^2)^{3/2}}$$

$$|x| = \sqrt{x_1^2 + x_2^2}\,, \qquad A, \mu \in \mathbb{R}\,, \quad \mu > 0$$

L.Sretenskii, S.F. Dotsenko, B.Yu. Sergievskii and L.V. Cherkasov,

S. Wang, B. Le Mehaute and Chia-Chi Lu, .

$$u|_{t=0} = \frac{A}{(1 + |x|^2/\mu^2)^{3/2}}\,, \qquad u_t|_{t=0} = 0$$

$$t \gg \mu/c$$

$$u(x,t) = \frac{\sqrt{\mu} A}{2\sqrt{2ct}} \operatorname{Re} \frac{i}{(i + y/\mu)^{3/2}} + O(\mu^{3/2}), \qquad y = |x| - ct$$

S. Yu. Dobrokhotov, B. Tirozzi, Localized solutions of one-dimensional nonlinear shallow-water equations with velocity $c = \sqrt{x}$, Russian Math. Surveys, 65:1 (2010), 177-179//

Exact 1-D solution

$$N(z,\tau) = \operatorname{Re} \frac{A(\tau + ib)}{(z - (\tau + ib)^2/4)^{3/2}},$$

The smoothed shock waves $\to \int_{-\infty}^{t} \dots$

$$u(x,t) = \frac{A}{c}\,\mathrm{Re}\frac{i}{\sqrt{(1+ict/\mu)^2 + |x|^2/\mu^2}}$$

$$u|_{t=0} = 0\,, \qquad u_t|_{t=0} = \frac{A}{\mu(1+|x|^2/\mu^2)^{3/2}}$$

$t \gg \mu/c$

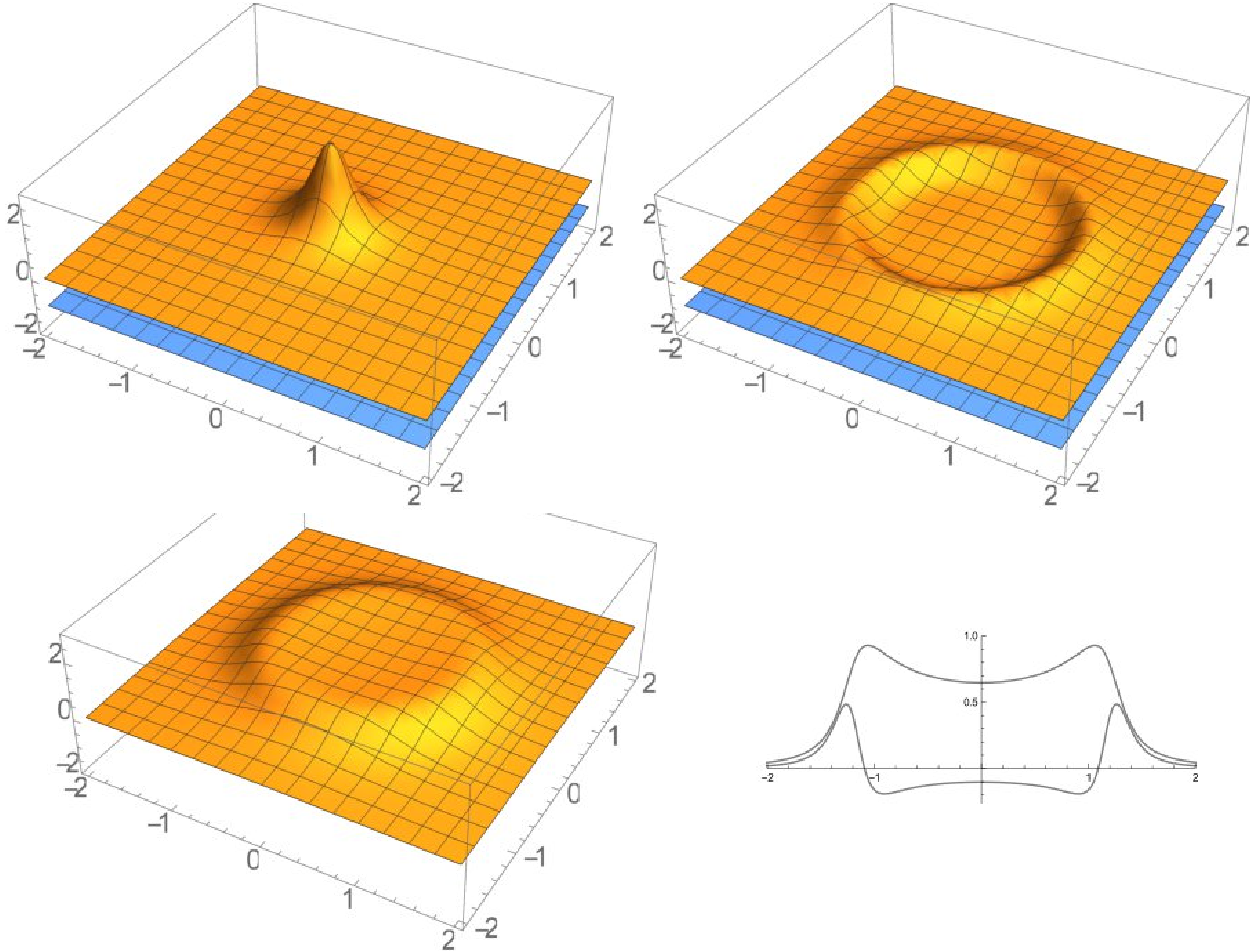
$$u(x,t) = \frac{\sqrt{\mu}A}{c\sqrt{2ct}}\,\mathrm{Re}\frac{i}{(i+y/\mu)^{1/2}} + O\left(\mu^{3/2}\right)$$

$$y = \sqrt{x_1^2 + x_2^2} - ct\,.$$

A. V. Aksenov, S. Yu. Dobrokhotov, K. P. Druzhkov, Exact Step-Like Solutions of One-Dimensional Shallow-Water Equations over a Sloping Bottom, Math. Notes, 104:6 (2018), 915-921

Exact 1-D solution

$$N = 4\,\mathrm{Re}\,\frac{A}{\sqrt{y - (\tau + ib)^2/4}}$$



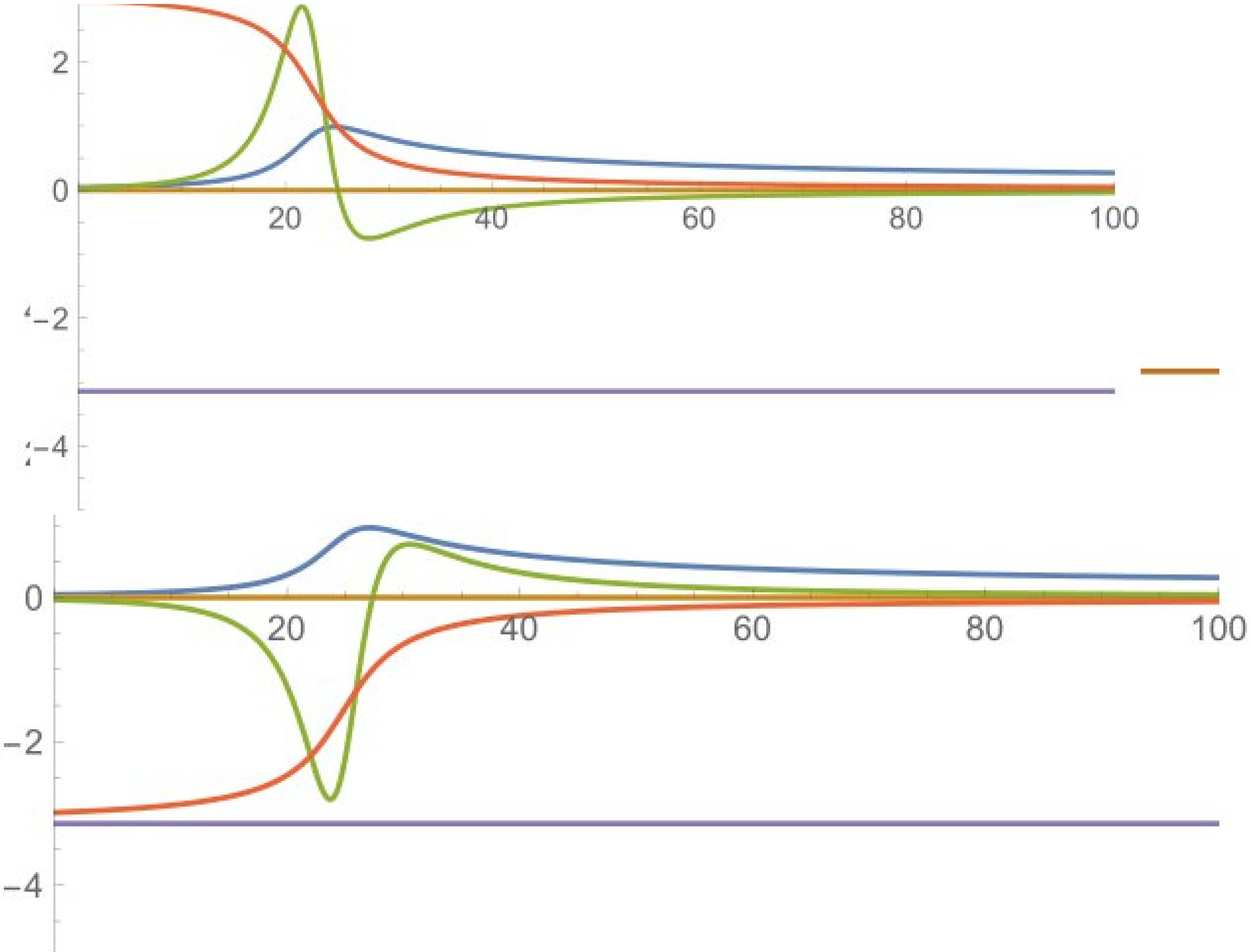
2
0
−2
−2
−1
0
1
2
2
1
0
−1
−2
1.0
0.5
−2
−1
1
2

The profile metamorphosis: Reflected wave = Hilbert transform (influent wave)

$$\Uparrow$$

$y = 0$ is the focal point, the Maslov index jumps

The solutions with $\mathrm{Im}A = 0$ **belong to the spectrum of the Hilbert transform**



Nonlinear shallow water equations with sloppy bottom

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{\partial \eta}{\partial x} = 0, \quad \frac{\partial \eta}{\partial t} + \frac{\partial[(\eta + x)v]}{\partial x} = 0.$$

Transformation:

$$z = x - t^2/2, \quad u = v - t, \quad h = \eta + x.$$

Standard shallow water equations with even bottom

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial z} + \frac{\partial h}{\partial z} = 0, \quad \frac{\partial h}{\partial t} + \frac{\partial[hu]}{\partial z} = 0.$$

Linear system

$$\frac{\partial U}{\partial \tau} + \frac{\partial N}{\partial y} = 0, \quad \frac{\partial N}{\partial \tau} + \frac{\partial [yU]}{\partial y} = 0.$$

Transformation

S. Yu. Dobrokhotov, S. B. Medvedev, D. S. Minenkov, On Replacements Reducing One-Dimensional Systems of Shallow-Water Equations to the Wave Equation with Sound Speed $c^2 = x$, Math. Notes, 93:5 (2013), 704-714

$$t = \tau + U, \quad z = y - N - \tau U - \tau^2/2, \quad h = y + U^2, \quad u = -\tau.$$

Standard nonlinear shallow water equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial z} + \frac{\partial h}{\partial z} = 0, \quad \frac{\partial h}{\partial t} + \frac{\partial [hu]}{\partial z} = 0.$$

The Jacobian:

$$\frac{\partial(t,z)}{\partial(\tau,y)} = 1 - 2\,\frac{\partial N}{\partial y} + \left(\frac{\partial N}{\partial y}\right)^2 - y\left(\frac{\partial U}{\partial y}\right)^2.$$

Yu. A. Chirkunov, S. Yu. Dobrokhotov, S. B. Medvedev, D. S. Minenkov, Exact solutions of one-dimensional nonlinear shallow water equations over even and sloping bottoms, Theoret. and Math. Phys., 178:3 (2014), 278-298

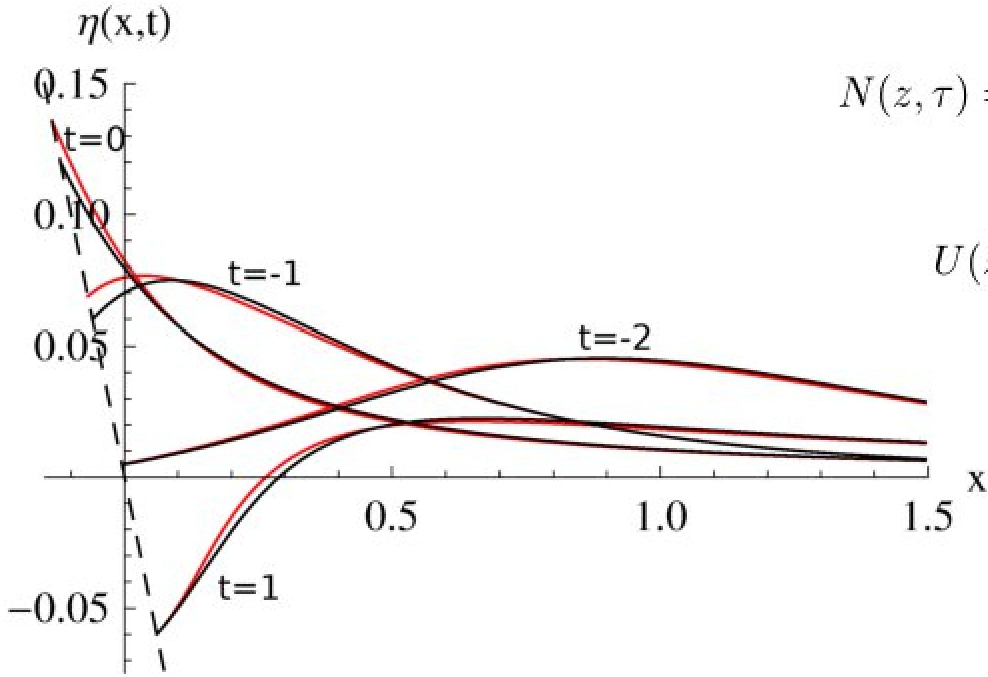
Reduced Carrier-Greenspan transformation (simplification):

$$D(y) = D(x) + \eta(x,t)\rho(x),\ \tau = t,\ N(y,\tau) = \eta(x,t),\ U(y,\tau) = u(x,t),$$

here $\rho(x)$ is a cut-off function, $\rho = 1$ near x^0 : $\quad D(x) = 0$ and $\rho = 0$ outside of the neighborhood of the point x^0.

The main property: **the boundary becomes fixed**

Example: exact solitary solution



$$N(z,\tau) = \mathrm{Re}\,\frac{A(\tau + ib)}{(z - (\tau + ib)^2/4)^{3/2}},$$

$$U(z,\tau) = 2\,\mathrm{Re}\,\frac{A}{(z - (\tau + ib)^2/4)^{3/2}},$$

Reduced Carrier-Greenspan transformation (simplification):

$$D(y) = D(x) + \eta(x,t)\rho(x),\ \tau = t,\ N(y,\tau) = \eta(x,t),\ U(y,\tau) = u(x,t),$$

here $\rho(x)$ is a cut-off function, $\rho = 1$ near x^0 : $D(x) = 0$ and $\rho = 0$ outside of the neighborhood of the point x^0.

The first formula defines the passage from independent coordinates x, t to coordinates independent coordinates y, t

The main property: *the boundary becomes fixed*

The main conjecture: *For almost all non-breaking long waves, an approximate (asymptotic) solution of the problem can be obtained as follows: first, a linear problem is solved, and then, in the vicinity of the coastline, the solution is determined in parametric form using the simplified Carrier-Greenspan transformation.*

Important remarks: 1) the proposed change of variable works in 1-D and 2-D cases;

2) the proposed conjecture means that a) amplitudes of considered waves are small enough and b) the change $D(y) = D(x) + \eta(x,t)\rho(x)$ defines the transition to the variables y, t which transfers the original nonlinear system with a sufficiently large nonlinearity (in the vicinity of the shore) to a nonlinear system with a sufficiently small nonlinearity. *The linear part of reduced system is no more but the linearized system of shallow water equations with a fixed boundary. Our hypothesis means that in a reduced system, nonlinear terms play the role of corrections and can be discarded in the main approximation.*

Our global aim is a rigorous proof of this conjecture.

Algorithm for constructing an approximation solutions

$$\eta_t + \langle \nabla, (D(x)+\eta)\mathbf{u}\rangle = 0, \quad \mathbf{u}_t + \langle \mathbf{u}, \nabla\rangle \mathbf{u} + g\nabla\eta = 0, \quad t\in[0,T],$$

$$\eta\Big|_{t=0} = \eta^{(0)}(x), \quad \mathbf{u}\Big|_{t=0} = \mathbf{u}^{(0)}(x),$$

$$(x,t)\in\overline{\Omega}_t\times[O,T] \qquad \eta(\mathbf{x},t)+D(\mathbf{x})|_{x\in\partial\Omega_t} = 0 \qquad \nabla D(x)\neq 0$$

Step 1. Construct a solution $N(y,t)$, $\mathbf{U}(y,t)$, $(y,t)\in\overline{\Omega}_0\times[O,T]$ (exact or approximate) of the "naive" linearization of the original problem

$$N_t(y,t) + \langle \nabla_y, D(y)\mathbf{U}(y,t)\rangle = 0, \ \mathbf{U}_t(y,t) + g\nabla_y N(y,t) = 0, \quad (y,t)\in\overline{\Omega}_0\times[0,T],$$

$$N\Big|_{t=0} = \eta^{(0)}(y), \quad \mathbf{U}\Big|_{t=0} = \mathbf{u}^{(0)}(y), \quad y\in\overline{\Omega}_0.$$

Step 2. Specify an approximation solution $\eta(x,t), \mathbf{u}(x,t)$ of the original problem by parametric formulas

$$x = y - N(y,t)\frac{\rho(y,t)\nabla_y D(y)}{(\nabla_y D(y))^2}, \quad \eta = N(y,t), \quad \mathbf{u} = \mathbf{U}(y,t)$$

the boundary $\partial\Omega_t$: $y\in\partial\Omega_0$

The main ideas of proof

An artificial small parameter $\varepsilon : \eta \to \varepsilon\tilde{\eta}, \quad \mathbf{u} \to \varepsilon\tilde{\mathbf{u}}$

$$\eta_t + \langle \nabla, D(x)\mathbf{u} \rangle + \varepsilon\langle \nabla, \eta\mathbf{u} \rangle = 0, \quad \mathbf{u}_t + g\nabla\eta + \varepsilon\langle \mathbf{u}, \nabla \rangle \mathbf{u} = 0, \quad t \in [0, T],$$

$$\eta \,\big|_{t=0} = \eta^{(0)}(x, \varepsilon), \quad \mathbf{u} \,\big|_{t=0} = \mathbf{u}^{(0)}(x, \varepsilon),$$

$$D(x) + \varepsilon\eta(x, t, \varepsilon) > 0 \quad \text{for } x \in \Omega(t, \varepsilon), \quad D(x) + \varepsilon\eta(x, t, \varepsilon) = 0 \quad \text{for } x \in \partial\Omega(t, \varepsilon).$$

$$\psi = \begin{pmatrix} \eta \\ \mathbf{u} \end{pmatrix}$$

$$\mathscr{L}\psi + \varepsilon b(\psi, \nabla\psi) = 0, \quad t \in [0, T], \qquad \psi \,\big|_{t=0} = \psi^{(0)},$$

$$\mathscr{L} = \begin{pmatrix} \partial_t & \nabla \circ D(x) \\ g\nabla & \partial_t \end{pmatrix}, \qquad b(\psi, \nabla\psi) = \begin{pmatrix} \langle \nabla, \eta\mathbf{u} \rangle \\ \langle \mathbf{u}, \nabla \rangle \mathbf{u} \end{pmatrix}.$$

Definition 1. *By a solution of original problem we mean an admissible pair* (Ω, ψ) *such that (for the numbers* $\varepsilon \geq 0$ *for which it is defined) it satisfies the initial conditions* $(\Omega, \psi)|_{t=0} = (\Omega^{(0)}, \psi^{(0)})$ *and the function* $\psi(x, t, \varepsilon)$ *satisfies the original equations.*

Definition 2. *We say that pairs* (Ω_1, ψ_1) *and* (Ω_2, ψ_2) *of this kind coincide with accuracy up to* $O(\varepsilon^n)$ *and write* $(\Omega_1, \psi_1) \equiv (\Omega_2, \psi_2) \bmod O(\varepsilon^n)$ *if there is a family of diffeomorphisms* $f(\,\cdot\,, \varepsilon)$ *that differ from the identity diffeomorphism*[2] *by* $O(\varepsilon^n)$ *and such that* $f(\Omega_1) = \Omega_2$ *and* $\psi_1 - f^*(\psi_2) = O(\varepsilon^n)$*. (We use a similar terminology also for the case in which the objects under consideration depend on the parameter* $t \in [0, T]$*.)*

Definition 3. *By an asymptotic solution of the original problem up to* $O(\varepsilon^n)$ *we mean an admissible pair* (Ω, ψ) *such that (for those* $\varepsilon \geq 0$ *for which this pair is defined) it satisfies the initial conditions* $(\Omega, \psi)|_{t=0} \equiv (\Omega^{(0)}, \psi^{(0)}) \bmod O(\varepsilon^n)$ *and the function* $\psi(x, t, \varepsilon)$ *gives the discrepancy* $O(\varepsilon^n)$ *when substituted into the original equations.*

The original equation in new variables and the problem with the fix boundary

$$\mathcal{L}_y \Psi + \varepsilon B(\Psi, \nabla_y \Psi, \varepsilon) = 0, \quad (x,t) \in \overline{\Omega}_0 \times [0,T], \qquad \Psi\big|_{t=0} = \Psi^{(0)}, \quad x \in \overline{\Omega}_0,$$

The change of variables

$$\mathcal{L}_y = \begin{pmatrix} \partial_t & \nabla_y \circ D(y) \\ g\nabla_y & \partial_t \end{pmatrix}$$

$$\nabla_x = (\mathcal{J}_0 + \varepsilon \mathcal{J}_1)^{-1} \nabla_y, \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} - \varepsilon \mathcal{J}_2 (\mathcal{J}_0 + \varepsilon \mathcal{J}_1)^{-1} \nabla_y .$$

$$B(\Psi, \nabla_y \Psi, \varepsilon)$$

$$= \begin{pmatrix} \mathcal{J}_2(\mathcal{J}_0 + \varepsilon \mathcal{J}_1)^{-1} N_y + \dfrac{\langle \nabla, D(y)\mathbf{U} \rangle - \big\langle (\mathcal{J}_0 + \varepsilon \mathcal{J}_1)^{-1} \nabla, \big(D(G(y, \varepsilon N)) + \varepsilon N\big)\mathbf{U} \big\rangle}{\varepsilon} \\ \mathcal{J}_2(\mathcal{J}_0 + \varepsilon \mathcal{J}_1)^{-1} \nabla \mathbf{U} - \big\langle \mathbf{U}, (\mathcal{J}_0 + \varepsilon \mathcal{J}_1)^{-1} \nabla \big\rangle \mathbf{U} - \dfrac{(\mathcal{J}_0 + \varepsilon \mathcal{J}_1)^{-1} - I}{\varepsilon} N_y \end{pmatrix}$$

$$\mathcal{L}\psi(x,t,\varepsilon) + \varepsilon b(\psi(x,t,\varepsilon), \nabla \psi(x,t,\varepsilon)) = \mathcal{L}_y \Psi(y,t,\varepsilon) + \varepsilon B(\Psi(y,t,\varepsilon), \nabla_y \Psi(y,t,\varepsilon), \varepsilon)$$

The standard perturbation theory

$$\Psi^{(0)} \sim \sum_{j=0}^{\infty} \Psi_j^{(0)} \varepsilon^j, \quad \Psi \sim \sum_{j=0}^{\infty} \Psi_j \varepsilon^j,$$

$$B(\Psi, \nabla\Psi, \varepsilon) \sim \sum_{j=0}^{\infty} \varepsilon^j B_j(\Psi_0, \dots, \Psi_j, \nabla_y \Psi_0, \dots, \nabla_y \Psi_j),$$

$$\mathcal{L}_y \Psi_0 = 0, \qquad \Psi_0|_{t=0} = \Psi_0^{(0)},$$

$$\mathcal{L}_y \Psi_j = -B_{j-1}(\Psi_0, \dots, \Psi_{j-1}, \nabla_y \Psi_0, \dots, \nabla_y \Psi_{j-1}), \quad \Psi_0|_{t=0} = \Psi_j^{(0)}, \quad j = 1, 2, \dots$$

Proposition 3. *Let v be a smooth vector function on the cylinder $\overline{\Omega}_0 \times [0, T]$, and let u_0 be a smooth function on $\overline{\Omega}_0$. Then there exists a smooth solution of the Cauchy problem $\mathcal{L}_y u = v$, $u|_{t=0} = u_0$ in a cylinder $\overline{\Omega}_0 \times [0, T]$, and this solution is unique.*

The linear inhomogeneous equations

$$\eta_t + \langle \nabla, D(x)\mathbf{u} \rangle = f_1(x,t), \quad \mathbf{u}_t + g\nabla\eta = f_2(x,t),$$
$$\eta \Big|_{t=0} = \eta^{(0)}(x), \quad \mathbf{u} \Big|_{t=0} = \mathbf{u}^{(0)}(x)$$

or

$$\eta_{tt} - \langle \nabla, gD(x)\nabla\eta \rangle = f_{1t} - \langle \nabla, D(x) f_2 \rangle, \quad \eta \Big|_{t=0} = \eta^{(0)}, \quad \eta_t \Big|_{t=0} = f_1 - \langle \nabla, D\mathbf{u}^{(0)} \rangle$$

The wave behavior near the shore.
The shore is a nonstandard caustic:
no standard boundary conditions

Linear approximation

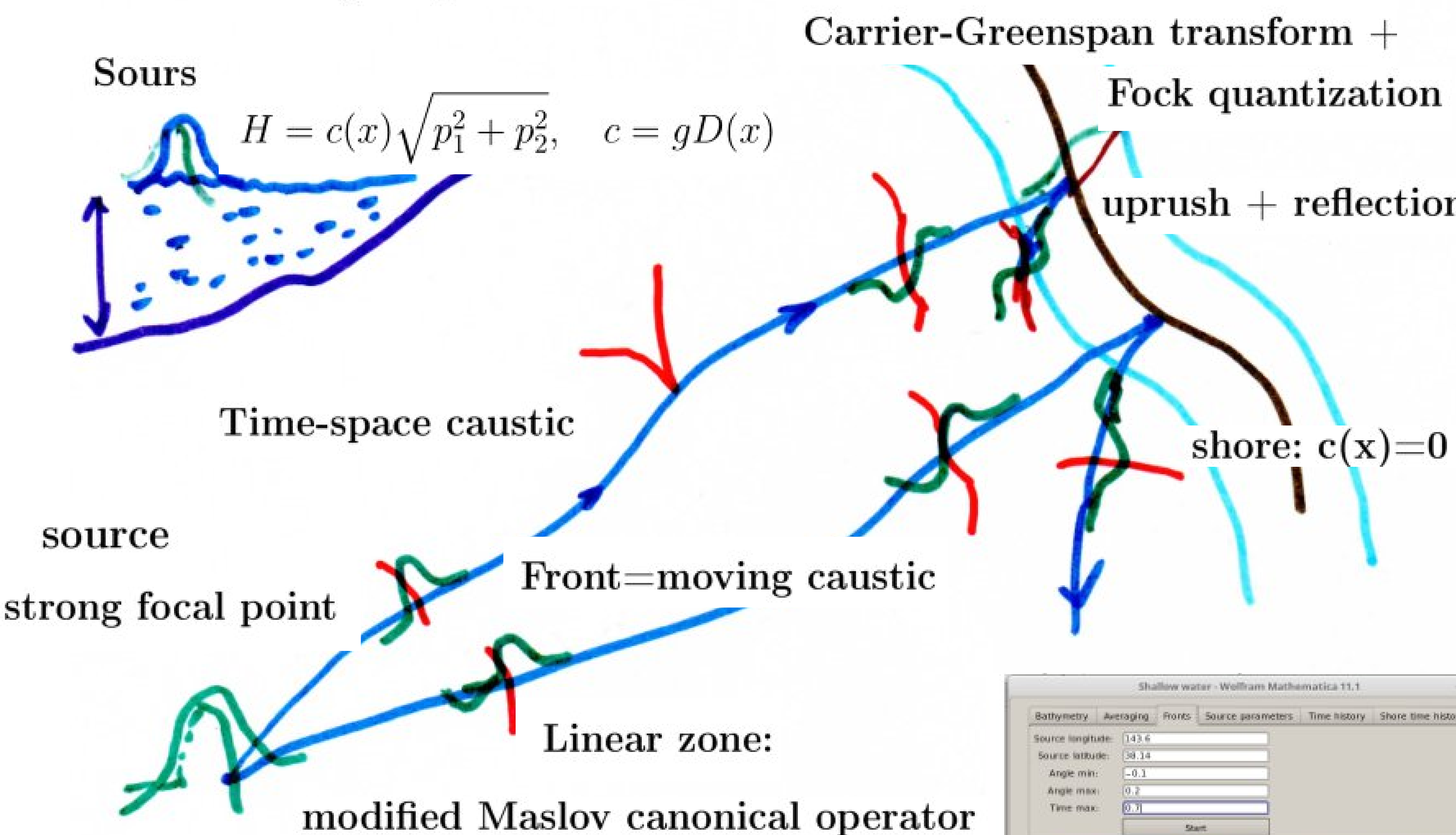
The Fock quantization of canonical transforms and the modified Maslov canonical operator for wave's constructions near beach

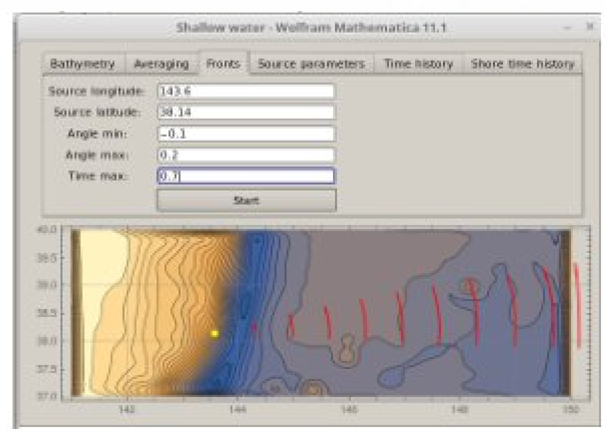
+ **UNIFORMIZATION:** passage to 3-D problem

The Nazaikinskii's part

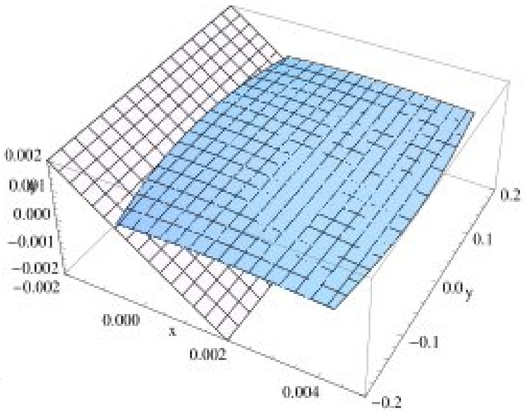
About applications

1) Generalization of semiclassical approximation and ray expansions

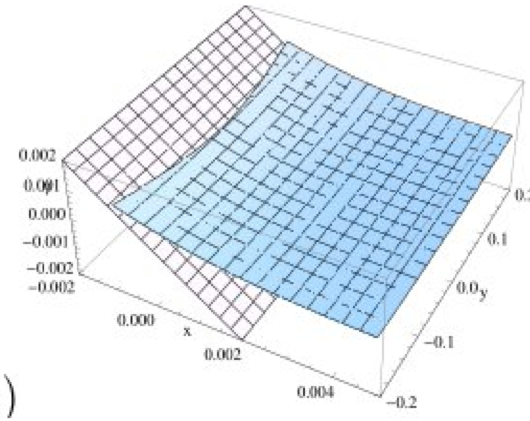




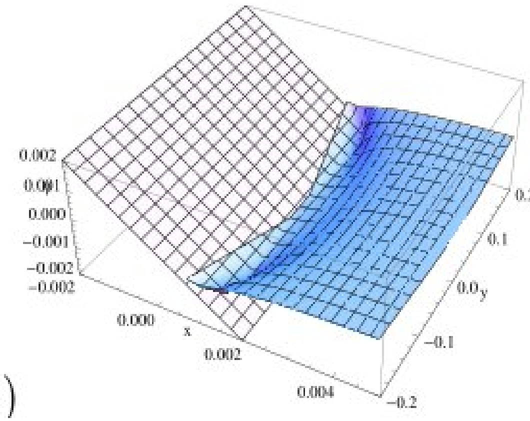
CAUSTICS ARE EVERYWHERE!


0.002
0.001
0.000
−0.001
−0.002
−0.002
0.000
x
0.002
0.004
0.2
0.1
0.0
y
−0.1
−0.2

(a)


0.002
0.001
0.000
−0.001
−0.002
−0.002
0.000
x
0.002
0.004
0.2
0.1
0.0
y
−0.1
−0.2

(b)


0.002
0.001
0.000
−0.001
−0.002
−0.002
0.000
x
0.002
0.004
0.2
0.1
0.0
y
−0.1
−0.2

(c)

2) Bessel functions in nonlinear problems of coastal waves and billiards with semi-rigid walls

The shallow water equations

$$\eta_t + \langle \nabla, D(x)\mathbf{u}\rangle + \varepsilon\langle \nabla, \eta\mathbf{u}\rangle = 0, \quad \mathbf{u}_t + g\nabla\eta + \varepsilon\langle \mathbf{u}, \nabla\rangle\mathbf{u} = 0, \quad (\varepsilon\eta(x,t) + D(x))|_{\Gamma(t)} = 0$$

Bottom $\quad D(x_1, x_2) = D(\rho) = a(R^2 - \rho^2)$

The reduction to the wave equation and Laplace-Beltrami type equation

$$\frac{\partial^2 N}{\partial t^2} = \langle \nabla, gD(x)\nabla N\rangle \qquad N = Re(e^{i\omega t}\psi(x))$$

$$\hat{\mathcal{H}}\psi \equiv -\mu^2 < \nabla, gD(x_1, x_2)\nabla\psi >= \omega^2\psi, \quad (x_1, x_2) \in \Omega.$$

From linear to nonlinear solutions

$$x = y - N(y,t)\frac{\varrho(y)\nabla D(y)}{\|\nabla D(y)\|^2}, \qquad \eta = N(y,t), \qquad \mathbf{u} = \mathbf{U}(y,t)$$

Nonstandard Liouville tori

$$\Lambda^2_{\omega,c} = \left\{H = a(R^2 - \rho^2)\left(p_\rho^2 + \frac{p_\phi^2}{\rho^2}\right) = \omega^2, p_{\phi^2} = c^2\right\}$$

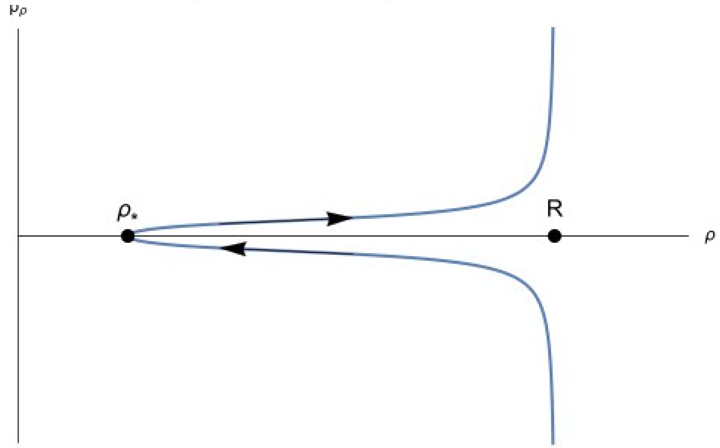


Рис. 1: Проекция лагранжева многообразия Λ на плоскость (ρ, p_ρ)

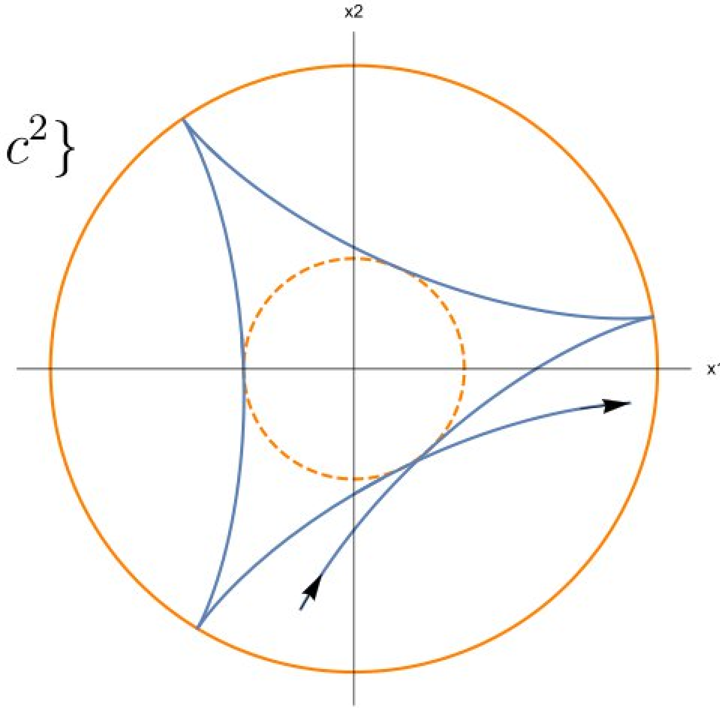


Рис. 2: Проекция траекторий гамильтоновой системы на плоскость (x_1, x_2). Сплошная окружность – граница берега (полужесткая стенка), пунктирная окружность – простая каустика (мягкая стенка)

Linear asymptotics $\rho_* + \delta < \rho \leq R$,

$$\psi(x) \approx \frac{\mu^{-1/2}(-1)^n A e^{im\phi}\sqrt{2\pi}\,|\Psi(\rho)|^{1/2}}{\left(4a(R^2-\rho^2)(\rho^2(ac^2+\omega^2)-ac^2R^2)\right)^{1/4}} \mathrm{J}_0\left(\frac{\Psi(\rho)}{\mu}\right)\Bigg|_{\substack{\rho=\rho(x),\\ \phi=\phi(x)}},$$

$$\Psi(\rho) = \frac{1}{2\sqrt{a}}\Bigg(\sqrt{ac^2}\arccos\left(\frac{2ac^2(R^2-\rho^2)-\rho^2\omega^2}{\rho^2\omega^2}\right)+$$

$$+\sqrt{ac^2+\omega^2}\arccos\left(\frac{-2ac^2(R^2-\rho^2)-R^2\omega^2+2\rho^2\omega^2}{R^2\omega^2}\right)-\pi\sqrt{ac^2}\Bigg), \quad \rho \leq R.$$

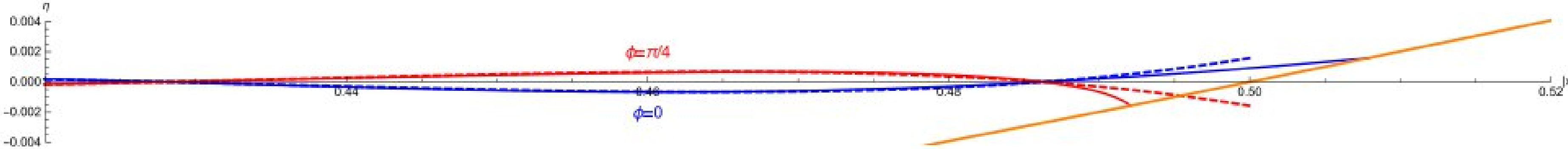
Linear asymptotics $\qquad 0 < \rho < R - \delta$

$$\psi(x) \approx \left. \frac{\mu^{-1/6} A e^{im\phi} 2\sqrt{\pi}\,|\Phi(\rho)|^{1/4}}{\left(4a(R^2-\rho^2)(\rho^2(ac^2+\omega^2)-ac^2R^2)\right)^{1/4}} \mathrm{Ai}\left(-\frac{\Phi(\rho)}{\mu^{2/3}}\right)\right|_{\substack{\rho=\rho(x),\\ \phi=\phi(x)}},$$

где

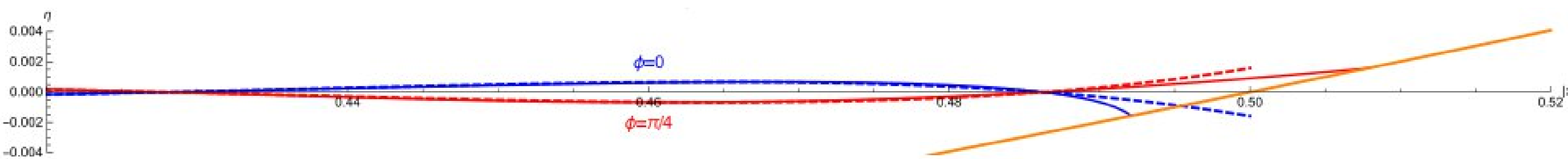
$$\Phi(\rho) = \begin{cases} \Bigg(\frac{3}{2}\Bigg(\frac{1}{2\sqrt{a}}\Bigg(\pi\sqrt{ac^2+\omega^2} - \sqrt{ac^2}\arccos\left(\frac{2ac^2(R^2-\rho^2)-\rho^2\omega^2}{\rho^2\omega^2}\right) - \\ -\sqrt{ac^2+\omega^2}\arccos\left(\frac{-2ac^2(R^2-\rho^2)-R^2\omega^2+2\rho^2\omega^2}{R^2\omega^2}\right)\Bigg)\Bigg)\Bigg)^{2/3}, \quad \rho \geq \rho_*, \\ \Bigg(\frac{3}{2}\Bigg(\frac{1}{2\sqrt{a}}\Bigg(\sqrt{ac^2}\,\mathrm{arctanh}\left(\frac{2\sqrt{ac^2(\rho^2-R^2)(ac^2(-R^2+\rho^2)+\rho^2\omega^2}}{2ac^2(R^2-\rho^2)-\rho^2\omega^2}\right) + \\ \sqrt{ac^2+\omega^2}\,\mathrm{arctanh}\left(\frac{2\sqrt{(ac^2+\omega^2)(\rho^2-R^2)(ac^2(-R^2+\rho^2)+\rho^2\omega^2}}{2ac^2(-R^2+\rho^2)-R^2\omega^2+2\rho^2\omega^2}\right)\Bigg)\Bigg)\Bigg)^{2/3}, \quad 0 < \rho < \rho_*. \end{cases}$$

$t = 0$, масштаб осей $1:1$



$t = 0$, масштаб осей $1:1$

η



$t = 0.7$, масштаб осей $1:1$

3) Some verification for 1-D standing waves:

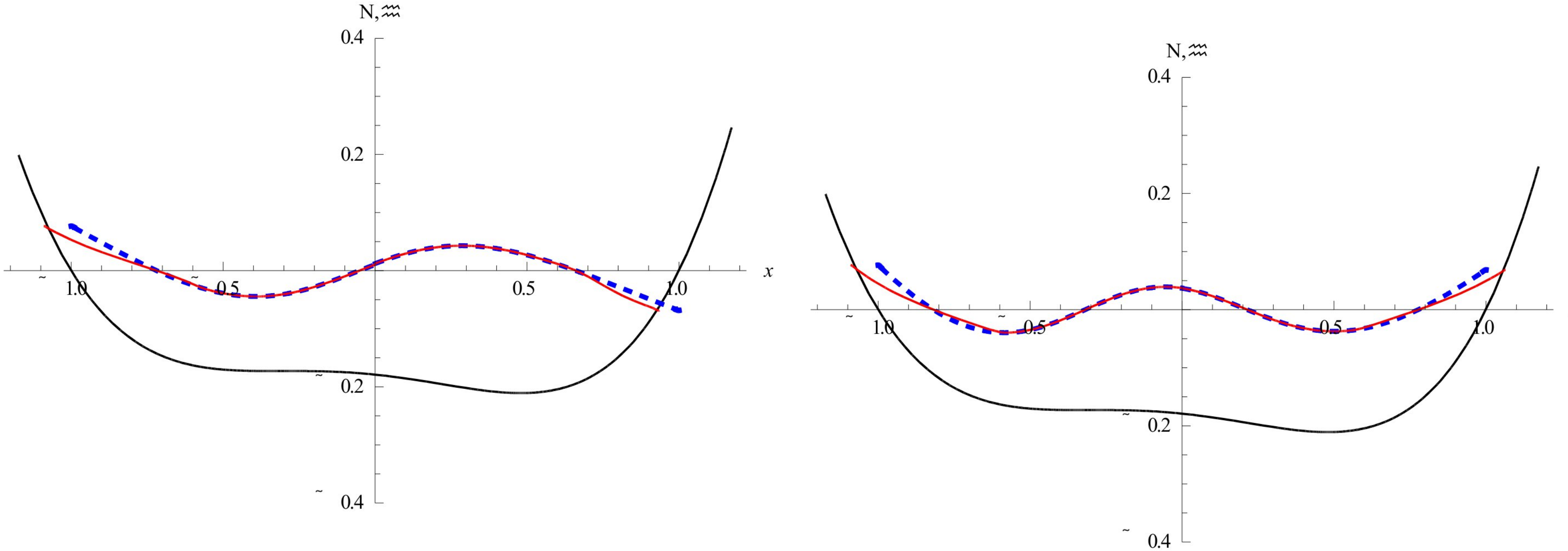
time-periodic nonlinear long waves in extended basins.

Shallow water: two shores

$$\eta_t + \left((D(x)+\eta)u\right)_x = 0, \quad u_t + g\eta_x + uu_x = 0$$

Depth: $D(a) = D(b) = 0 \quad D'(a) \neq 0, \quad D'(b) \neq 0$

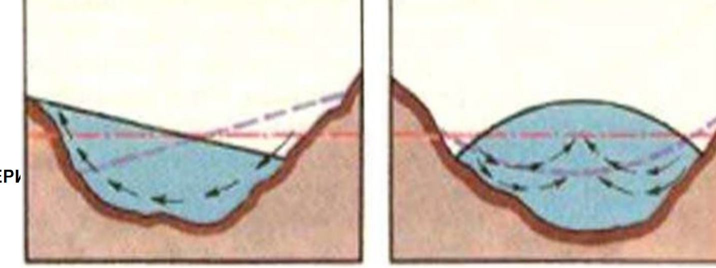
Boundary condition at two variable boundaries x(t): $\eta(x_a(t),t) + x_a(t) = 0$



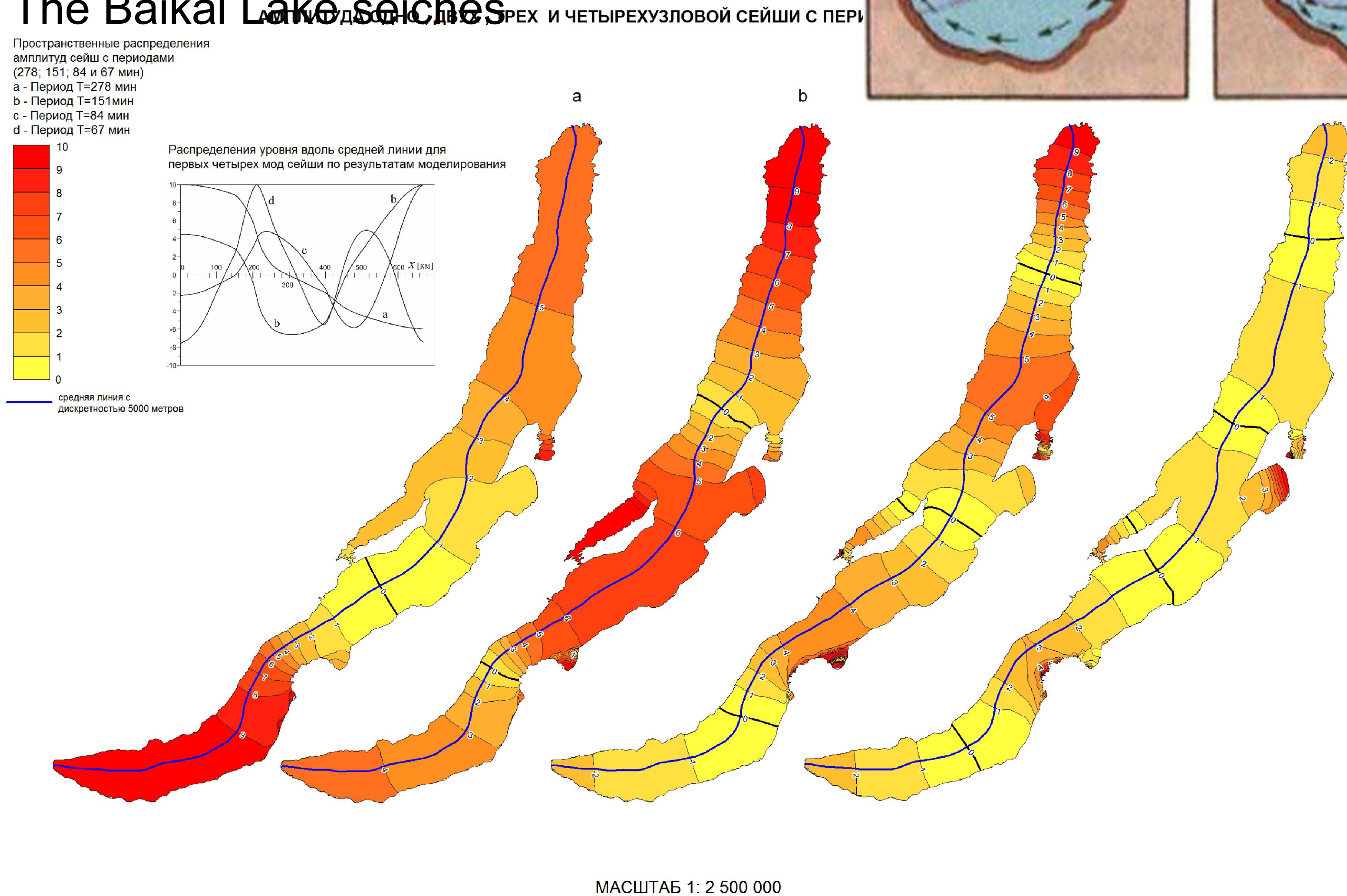
Motivation: Seiches – standing waves

A ***seiche*** is a standing wave in an enclosed or partially enclosed body of water.

One- and two-nodes seiches



The Baikal Lake seiches



G.F. Carrier and H.P. Greenspan, Water waves of finite amplitude on a sloping beach, *J. Fluid Mech.* **4**(1), 97–109 (1958).

В. А. Фок, “О каноническом преобразовании в классической и квантовой механике” *Вестн. Ленинградск. ун-та*, 1959, № 16, 67–70.

E. Pelinovsky and R. Mazova, “Exact Analytical Solutions of Nonlinear Problems of Tsunami Wave Run-Up on Slopes with Different Profiles,” Natur. Hazards **6**, 227–249 (1992).

T. Vukašinac and P. Zhevandrov, “Geometric Asymptotics for a Degenerate Hyperbolic Equation,” Russ. J. Math. Phys. **9** (3), 371–381 (2002).

S.Yu. Dobrokhotov and B. Tirozzi, “Localized Solutions of the One-Dimensional Nonlinear Shallow Water Equations with Velocity $c = \sqrt{x}$,” Uspekhi Mat. Nauk **65** (1) (391), 185–186 (2010).

A. B. Rabinovich, Seiches and Harbor Oscillations, Handbook of Coastal and Ocean Engineering, pp. 193-236 (2009) DOI: 10.1142/9789813204027_0011

S.V. Smirnov, K.M. Kucher, N.G. Granin, I.V. Sturova, 2014, Izvestiya, Atmospheric and Oceanic Physics, Vol. 50, No. 1, pp. 92–102.

V. A. Kalinichenko, S. V. Nesterov and A. N.So, Standing surface waves in a rectangular tank with local wall and bottom irregularities // Fluid Dyn. 2017. Vol. 52. No. 2. P. 230-238.

S. Yu. Dobrokhotov, V. E. Nazaikinskii, Nonstandard Lagrangian Singularities and Asymptotic Eigenfunctions of the Degenerating Operator $-\frac{d}{dx}D(x)\frac{d}{dx}$, Proc. Steklov Inst. Math., 306 (2019), 74-89

Experimental device (no tray)

Диапазон частот	0.1-5 Гц
Диапазон амплитуд	0.03-7.5 см
Угловые смещения	< 8′
Коэффициент нелинейных искажений	2%
Точность измерения периода	3×10^{-3} с
Допустимая масса сосуда с жидкостью	50 кг
Точность измерения пространственных характеристик волновых движений	1 мм

The electromechanical vibration stand provides vertical oscillation of the basin

Shallow water: two shores

$$\eta_t + \left((D(x)+\eta)u\right)_x = 0, \quad u_t + g\eta_x + uu_x = 0$$

Depth: $D(a) = D(b) = 0 \quad D'(a) \neq 0, \quad D'(b) \neq 0$

Boundary condition at two variable boundaries x(t): $\eta(x_a(t),t) + x_a(t) = 0$

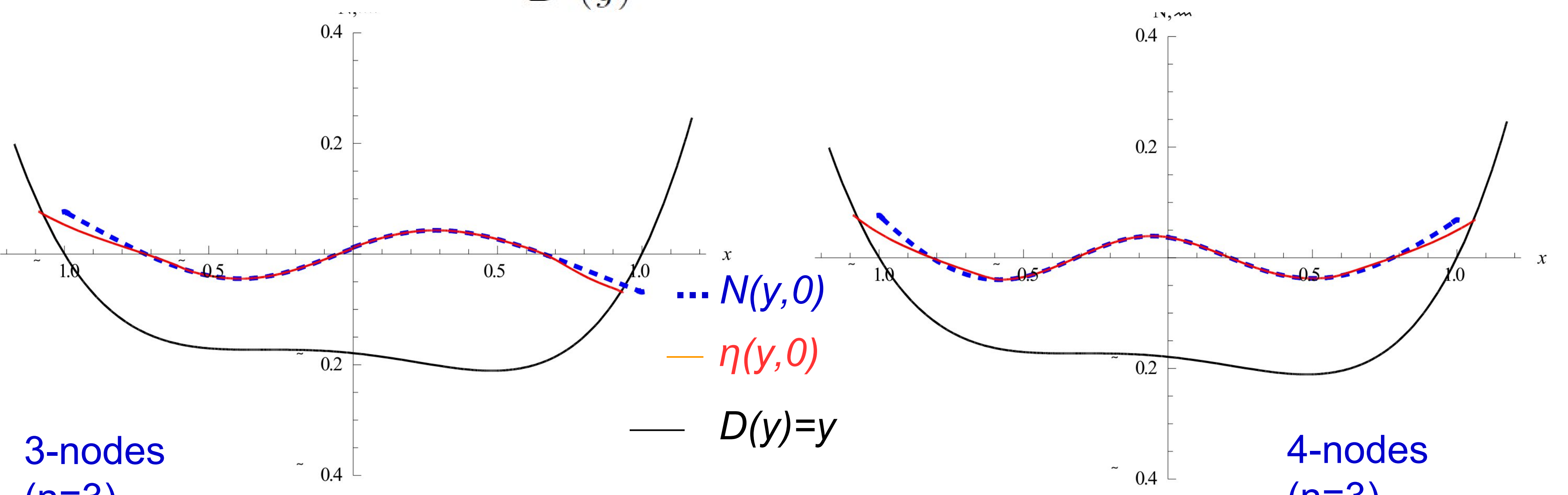
Reduced Carrier—Greenspan transform with cutting function ρ:

$$D(y) = D(x) + \eta(x,t)\rho(x), \quad \tau = t, \quad N(y,\tau) = \eta(x,t), \quad U(y,\tau) = u(x,t)$$

The leading term is defined from linearized shallow water with 2 fixed boundaries:

$$N_\tau + (D(y)U)_y = 0, \quad U_\tau + N_y = 0, \quad y \in [a,b], \quad E^2 = \|N\|^2 + \|\sqrt{y}\,U\|^2 < \infty$$

Finally: $x = y - \varepsilon N_1 \frac{\rho(y)}{D'(y)}, \quad \eta(x,t) = N(y,t), \quad u(x,t) = U(y,t)$



3-nodes (n=3)

4-nodes (n=3)

Parabolic bottom: exact solutions to linear problem

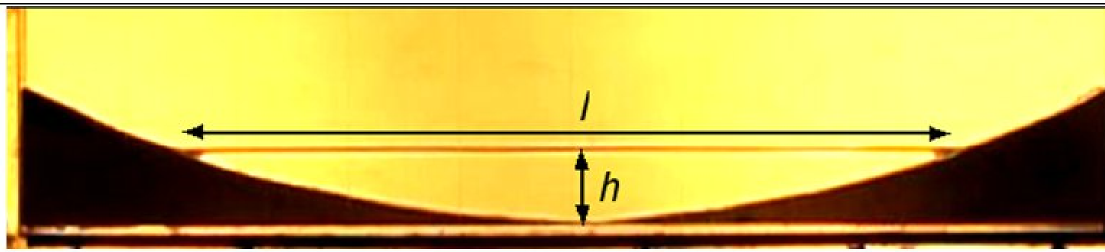
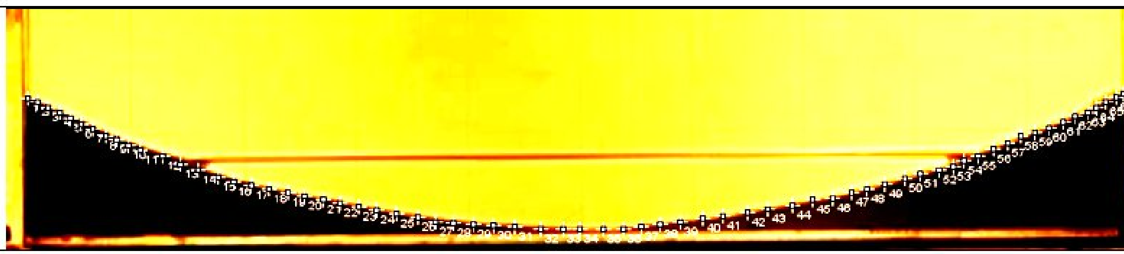
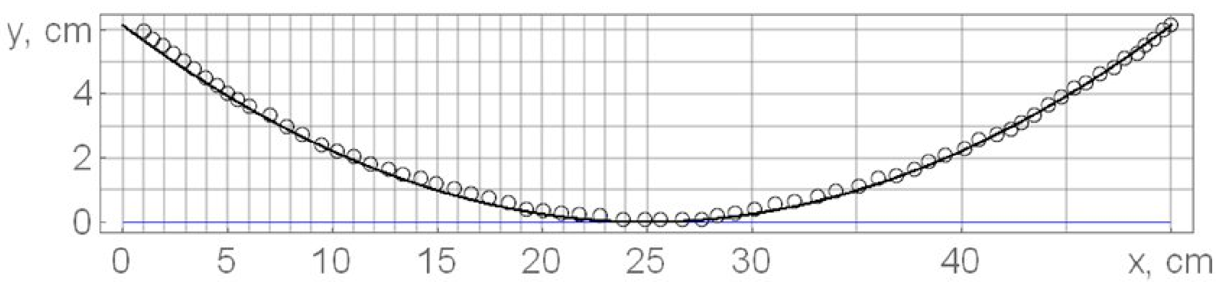
$$D = D_0((y/\beta)^2 - 1), \quad \omega_n^2 = \frac{gD_0}{\beta^2} n(n+1), \;\; N_n^0(y) = L_n(\frac{y}{\beta})$$

here D_0 is the maximum depth, 2β is the basin size,
$L_n(z)$ is the n-th *Legendre* polynomial .

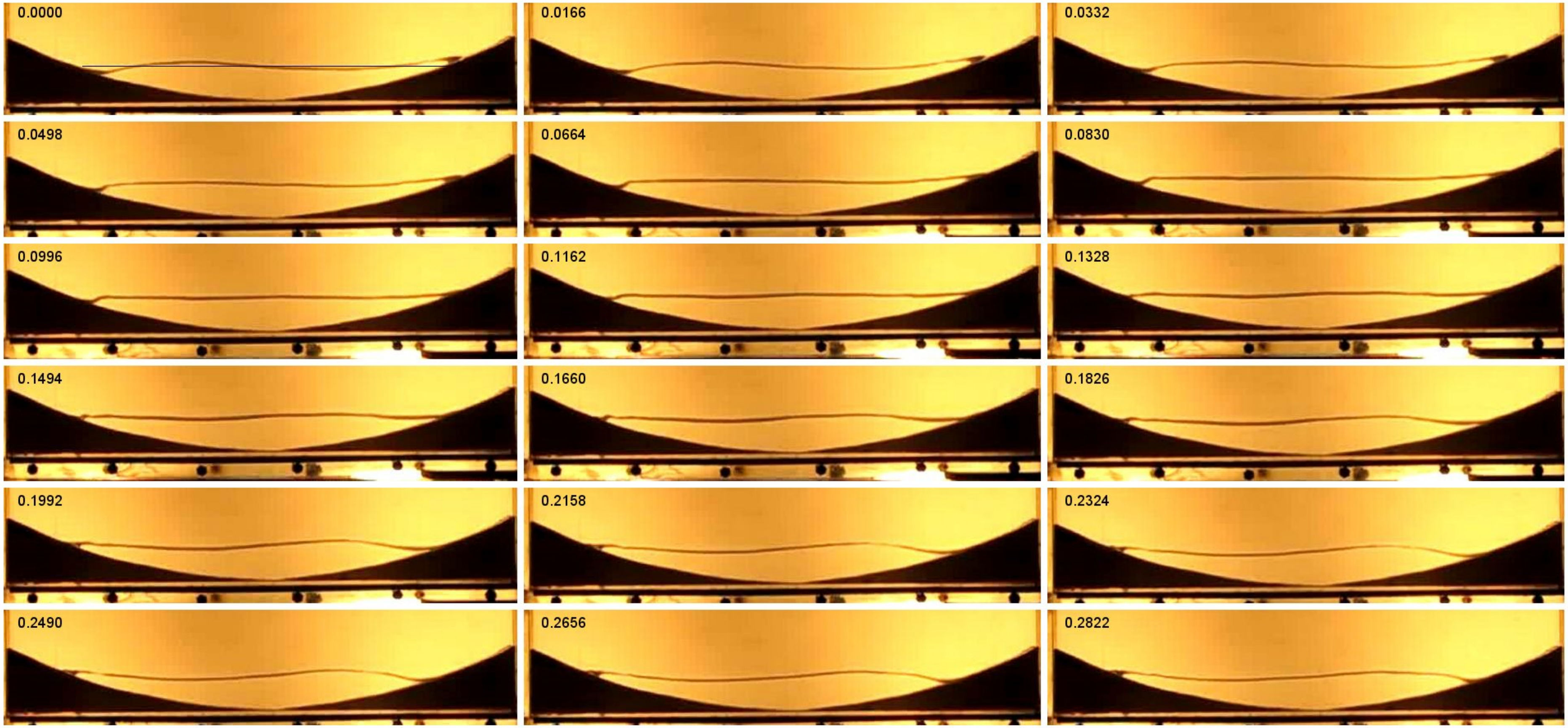
$$N = \mathrm{Re}\big(e^{i\omega_n t} N_n^0(y)\big)$$

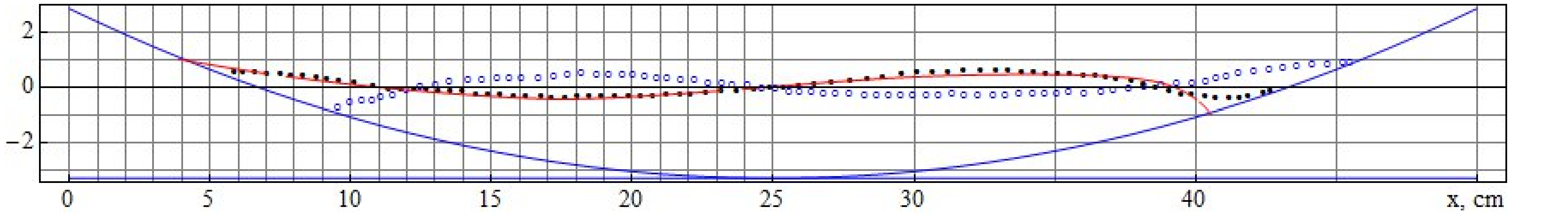
The basin with the parabolic bottom

The sizes: 50x4x50 см

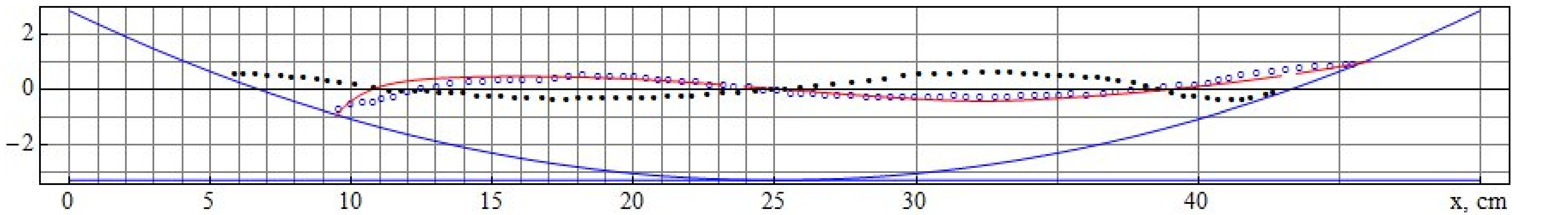
	l = 35.6 cm – длина свободной поверхности жидкости (зеркало) h = 3.3 cm – глубина жидкости
	
	$y = 0.01(x-25)^2$ (cm) Сплошная кривая – расчетный профиль дна Точки – оцифровка фото

Standing wave with small amplitudes

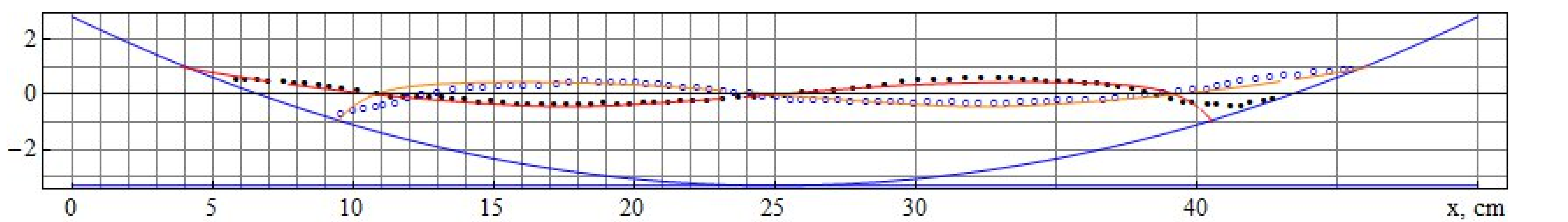




2
0
−2
0
5
10
15
20
25
30
40
x, cm

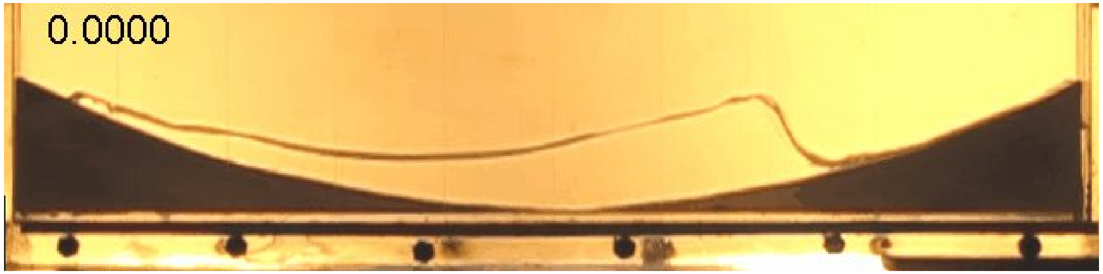


2
0
−2
0
5
10
15
20
25
30
40
x, cm



2
0
−2
0
5
10
15
20
25
30
40
x, cm

Standing waves with large amltudes



The wave behavior near the shore. The shore is a nonstandard caustic: no standard boundary conditions

Linear approximation

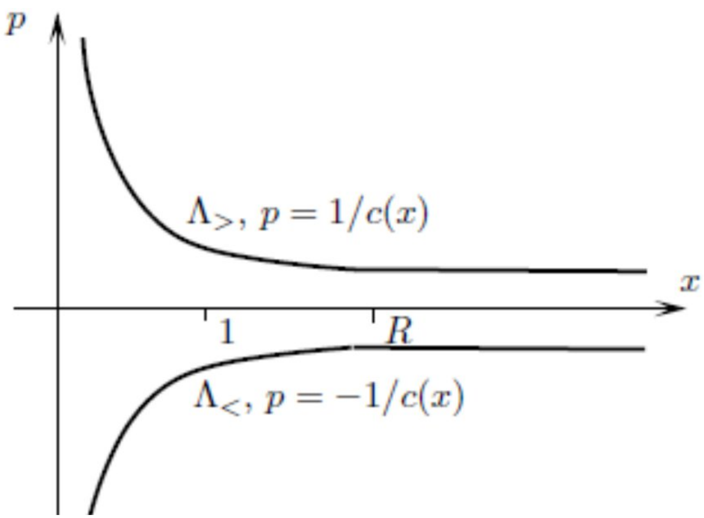
The Fock quantization of canonical transforms and the modified Maslov canonical operator for wave's constructions near beach

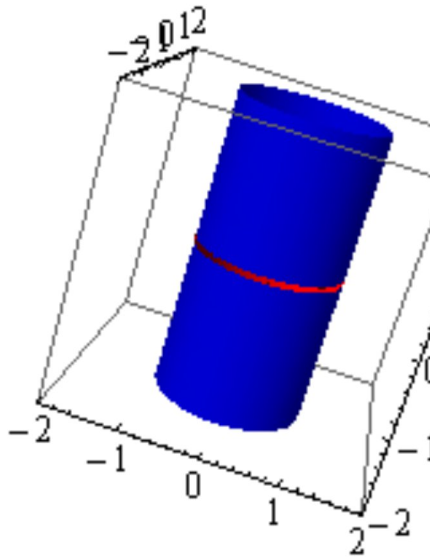
+ UNIFORMIZATION: passage to 3-D problem

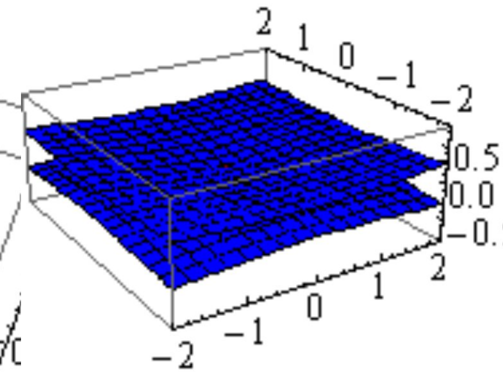
The Nazaikinskii's part

Example

Bessel function







$$\Lambda = \{p^2 x = 1\} \implies \mathbf{J}_0\left(\frac{\sqrt{x}}{h}\right)$$

$$\mathbf{J}_0\left(\frac{\sqrt{x_1^2 + x_2^2}}{h}\right)$$

$$\Lambda^2 = \{p_1^2 + p_2^2 = 1, \quad p_1 x_2 - p_2 x_1 = 0\}$$

$$v(t) = \cos(Qt)v^{(0)} + \frac{\sin(Qt)}{Q}v^{(1)} + \int_0^t \frac{\sin(Q(t-\tau))}{Q}g(\tau)\,d\tau,$$

$$Q = P^{1/2}$$

$$\|v(t)\| \leq \|v^{(0)}\| + T\|v^{(1)}\| + \frac{T^2}{2}\sup_{\tau\in[0,T]}\|g(\tau)\|$$

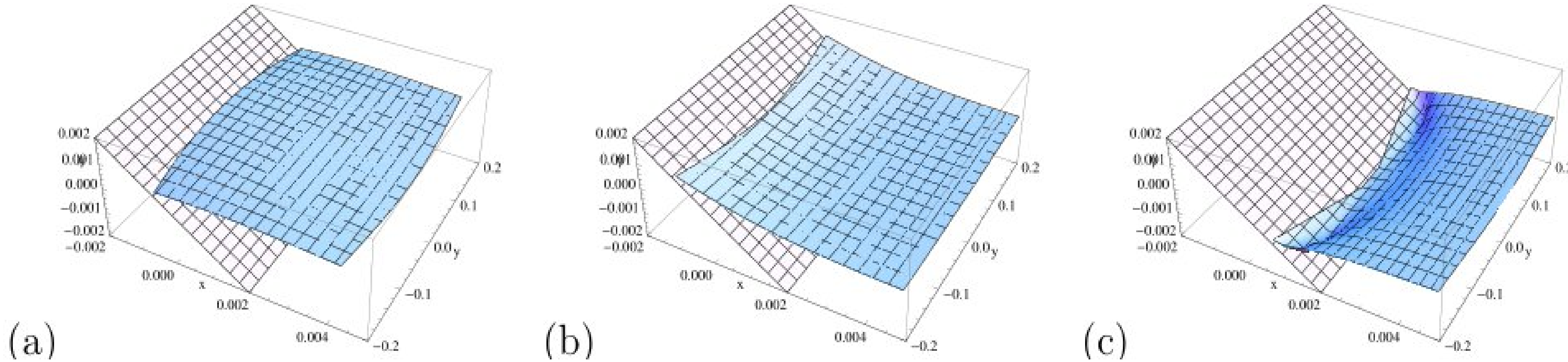
$\|\cdot\|$ is the norm in the space $L^2(M)$

$$\sup_{t\in[0,T]}\|P^k\frac{\partial^j v(t)}{\partial t^j}\| < \infty, \quad k, j = 0, 1, 2, \ldots$$

$$\|w\|_{s+\delta} \leq C_s(\|Pw\|_s + \|w\|_0)$$

$$\|w\|_{k\delta} \leq C_{(k-1)\delta}C_{(k-2)\delta}\cdots C_0(\|P^k w\|_0 + \|P^{k-1}w\|_0 + \cdots + \|w\|_0)$$

$$\sup_{t\in[0,T]}\|\frac{\partial^j v(t)}{\partial t^j}\|_{k\rho} < \infty, \quad j = 0, 1, 2, \ldots ,$$



0.002
0.001
0.000
−0.001
−0.002
−0.002
0.000
x
0.002
0.004
0.2
0.1
0.0
y
−0.1
−0.2
(a)
(b)
(c)

Linearized shallow water: two shores

$$N_\tau + (D(y)U)_y = 0, \quad U_\tau + N_y = 0$$

Depth: $D(a) = D(b) = 0$

$D'(a) \neq 0, \quad D'(b) \neq 0$

No boundary conditions,

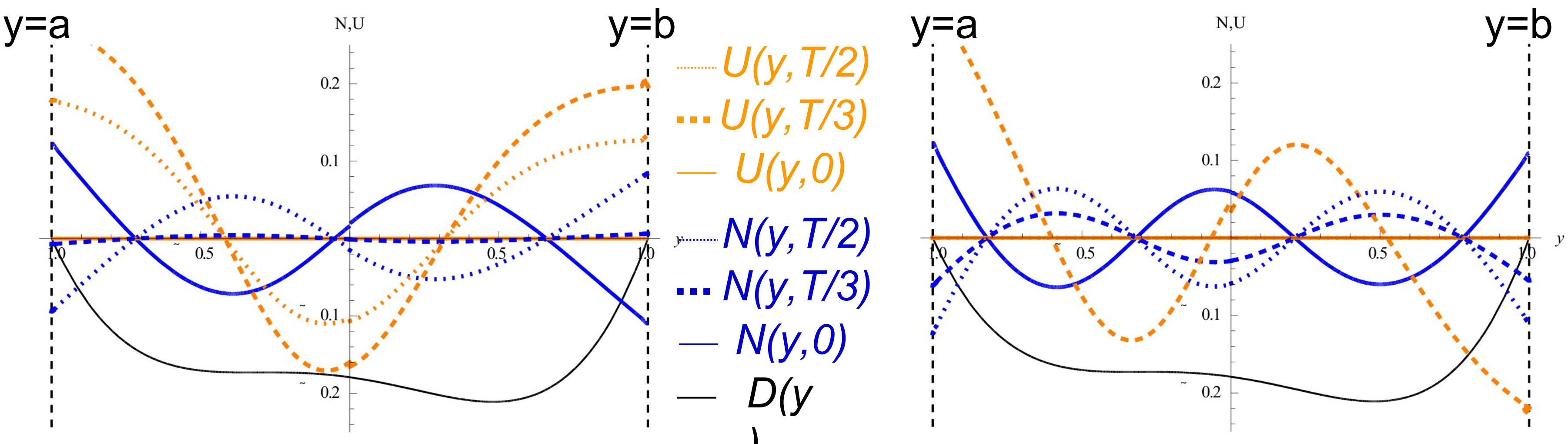
Finite energy condition: $E^2 = \|N\|^2 + \|\sqrt{D(y)}\, U\|^2 < \infty$

Asymptotics. Phase: $S(x,y) = \int_x^y \frac{dy}{\sqrt{D(y)}}, \quad a \le x \le y \le b.$

Quantization: $w_n = \frac{\pi}{S(a,b)}(\frac{1}{2} + n)(1 + O(n^{-1})),\ n \in \mathbb{N}.$

$$N_a(y,\tau) = c\cos(w_n\tau + \tau_0)\mathbf{J}_0(w_n S(a,y))\left(\frac{S(a,y)}{\sqrt{D(y)}}\right)^{1/2} \qquad y \in [a, b-\delta]$$

$$N_b(y,\tau) = c(-1)^n\cos(w_n\tau + \tau_0)\mathbf{J}_0(w_n S(y,b))\left(\frac{S(y,b)}{\sqrt{D(y)}}\right)^{1/2} \qquad y \in [a+\delta, b]$$



3-nodes (n=3)

4-nodes (n=4)

Carrier-Greenspan transformation for standing waves and experimental studies

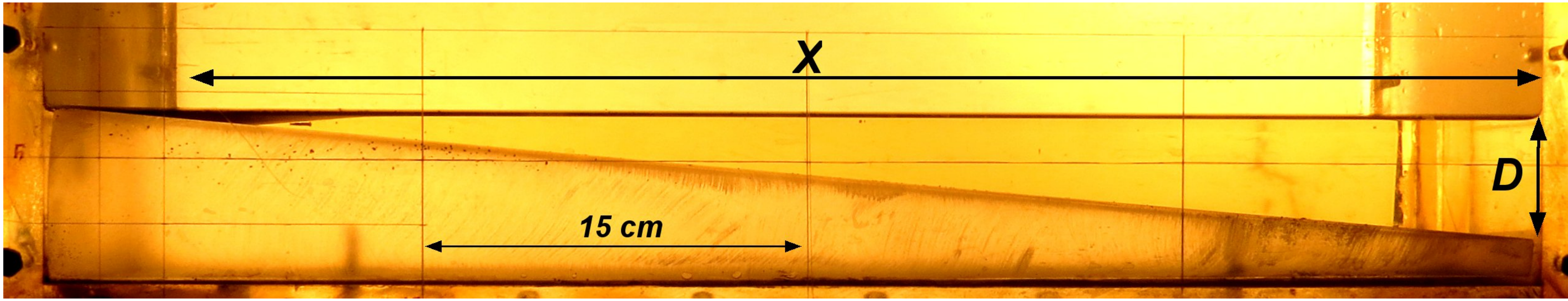
Nonlinear Shallow water equations:

$$\eta_t + \left((D(x)+\eta)u\right)_x = 0, \quad u_t + g\eta_x + uu_x = 0, \quad D = \gamma(x - x^0), \quad u|_{x=b} = 0.$$

Carrier-Greenspan transformation:

The linear equation with nonlinear boundary condition

$$N_\tau + (yU)_y = 0, \quad U_\tau + N_y = 0, \quad U(Y(\tau),\tau) = 0, \quad Y(\tau) = b + N(Y(\tau),\tau),$$
$$\|N\|^2 + \|\sqrt{y}\,U\|^2 < \infty$$



Shallow water: slopping bottom, formal asymptotics

Formal series: $N(y,\tau,\varepsilon) = \varepsilon N_1^{w(\varepsilon)} + \varepsilon^2 N_2^{w(\varepsilon)} + \dots,$

Leading term:

$$N_1^{w(\varepsilon)} \equiv \cos(w_0\tau)\,\mathbf{J}_0(2w(\varepsilon)\sqrt{y}) \qquad w_0 = \mu_n/2$$

$$U_1^{w(\varepsilon)} \equiv \sin(w_0\tau)\,\frac{1}{\sqrt{y}}\,\mathbf{J}_1(2w(\varepsilon)\sqrt{y})$$

Corrected frequency to avoid resonances: $w(\varepsilon) = w_0 + \varepsilon^2 w_2 + \dots$

Boundary condition:

$$U_1(b,\tau) = 0$$

$$U_2(b,\tau) = -U_{1y}(b,\tau)N_1(b,\tau) = \xi_1 \sin(2w_0\tau)$$

First correction: $N_2 = c_2\cos(2w\tau)\,\mathbf{J}_0(4w\sqrt{y}) \qquad c_2 = -\frac{w_0}{2}\mathbf{J}_0(2w_0)\,\mathbf{J}_1'(2w_0)\,(\mathbf{J}_1(4w_0))^{-1}$

Boundary for U_3: $U_3^{w_0}(b,\tau) = \xi_2\sin(3w_0\tau) + \xi_3\sin(w_0\tau) - 2w_2\mathbf{J}_1'(2w_0)\sin(w_0\tau)$

defines phase shift w_2: $w_2 = \xi_3/2\mathbf{J}_1'(2w_0)$

Second correction: $N_3 = c_3\cos(3w_0\tau)\,\mathbf{J}_0(6w_0\sqrt{y}) \qquad c_3 = \frac{\xi_2}{\mathbf{J}_1(6w_0)}$

Etc...

Shallow water: slopping bottom, the leading term

Formal series: $N(y,\tau,\varepsilon) = \varepsilon N_1^{w(\varepsilon)} + \varepsilon^2 N_2^{w(\varepsilon)} + \ldots,$

The Leading term for linearized system:

$$N_1^{w(\varepsilon)} \equiv \cos(w_0\tau)\,\mathbf{J}_0(2w(\varepsilon)\sqrt{y}) \qquad w_0 = \mu_n/2$$

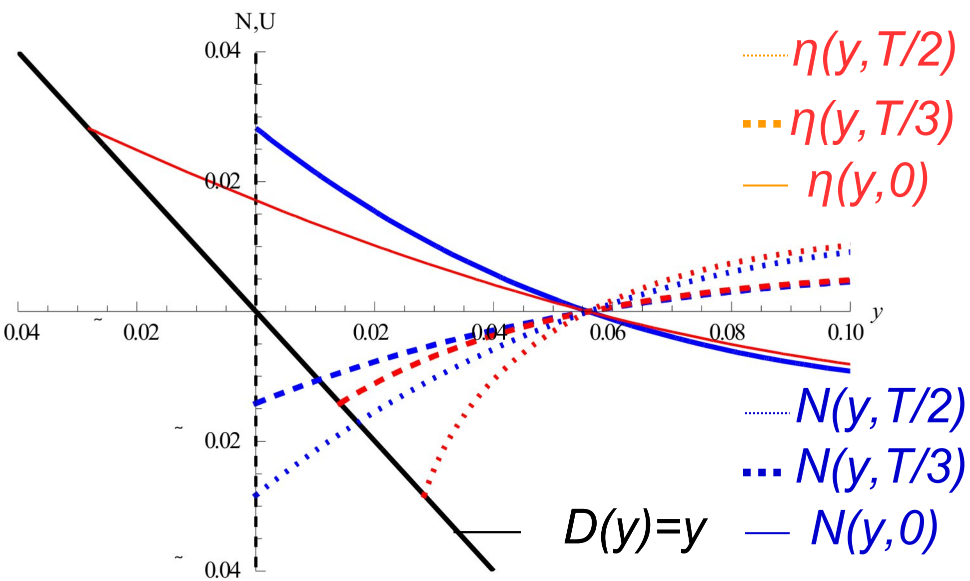
$$U_1^{w(\varepsilon)} \equiv \sin(w_0\tau)\,\frac{1}{\sqrt{y}}\,\mathbf{J}_1(2w(\varepsilon)\sqrt{y})$$

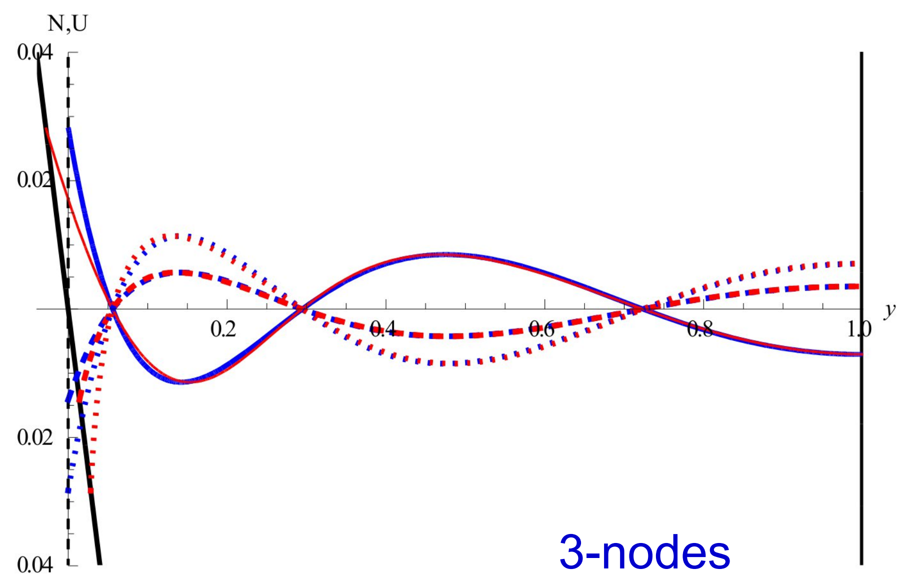
Reduced C—G transform: substitute $\tau(t,y,\varepsilon) = t + O(\varepsilon)$ into *N(y,τ), U(y,τ)*

And get the Leading term for shallow water – parametrically defined via *y ϵ [0,1]*:

$$x = y - \varepsilon N_1^{w_0}(y,t) + O(\varepsilon^2)$$

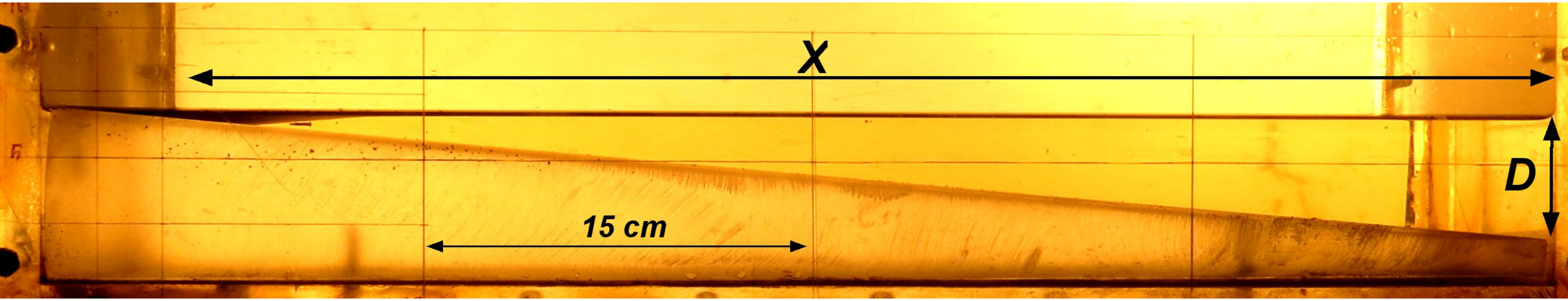
$$\eta(x,t) = \varepsilon N_1^{w_0}(y,t) + O(\varepsilon^2), \quad u(x,t) = \varepsilon U_1^{w_0}(y,t) + O(\varepsilon^2)$$

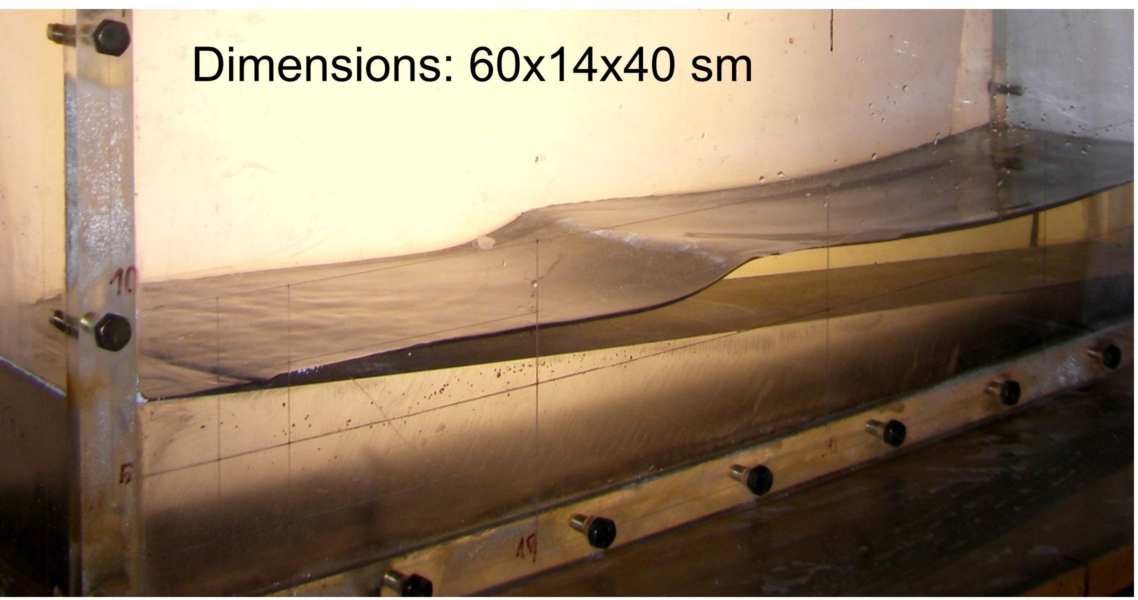




3-nodes

Experimental setup: parametric resonance



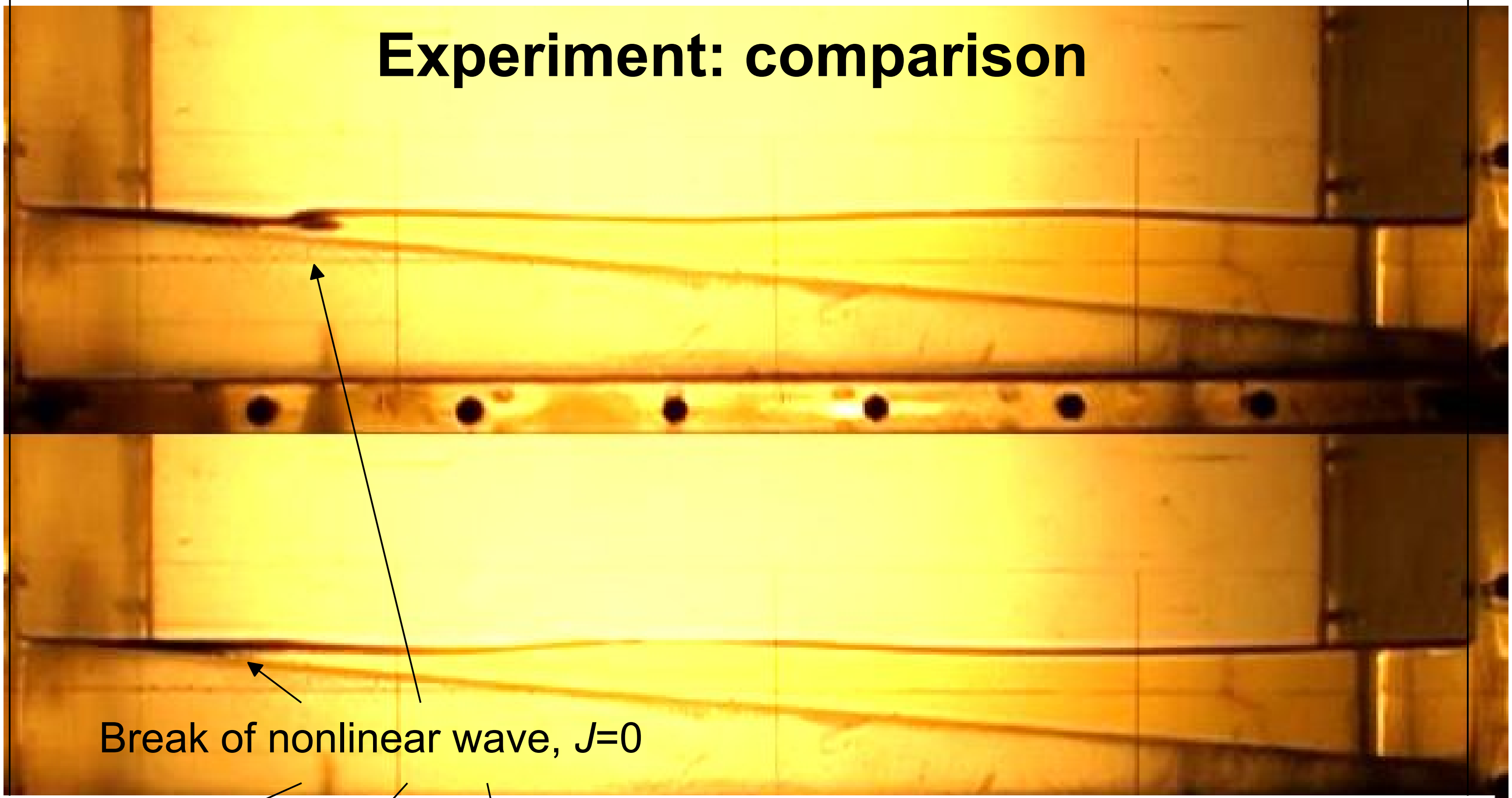


Gravity waves on the free surface
in rectangular vessel
(length = 60sm, width = 14 sm)
with slopping bottom (D:X = 4,5sm : 55 cm)

Surface waves are induced
by vertical oscillations of vessel
with parametric resonance
(Oscillations period = waves period / 2)

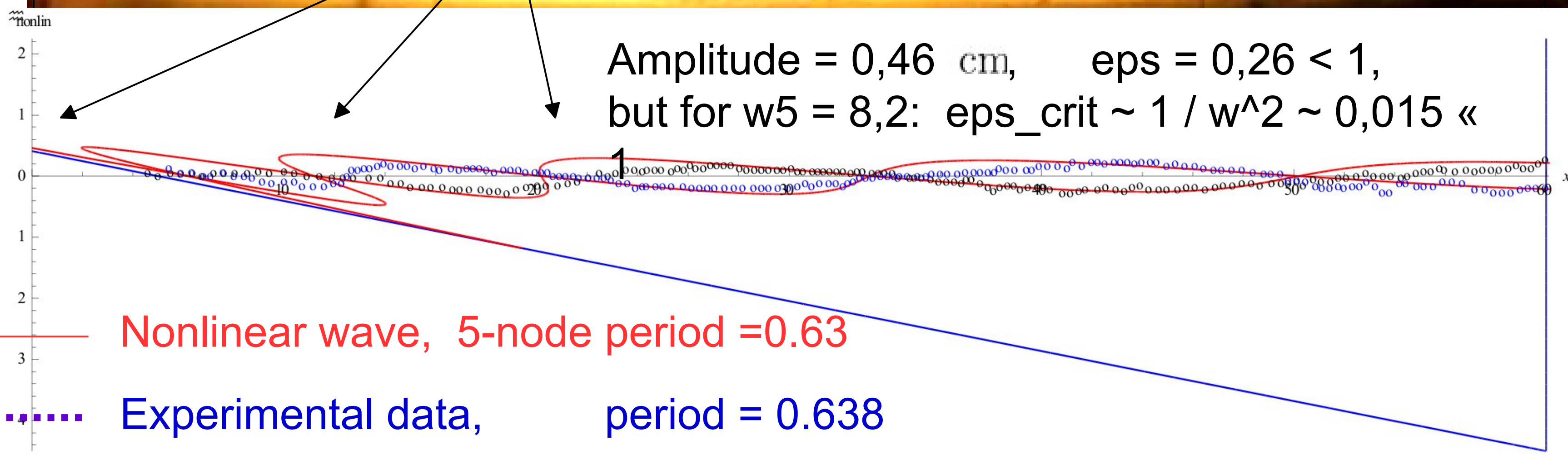
Video 30 and 120 frames per sec,
editing in ImageJ

Experiment: comparison



Break of nonlinear wave, *J*=0

Amplitude = 0,46 cm, eps = 0,26 < 1,
but for w5 = 8,2: eps_crit ~ 1 / w^2 ~ 0,015 « 1



Nonlinear wave, 5-node period =0.63

Experimental data, period = 0.638

Shallow water: two shores

$$\eta_t + \left((D(x)+\eta)u\right)_x = 0, \quad u_t + g\eta_x + uu_x = 0$$

Depth $D(a) = D(b) = 0 \quad D'(a) \neq 0, \quad D'(b) \neq 0$

Boundary condition at two variable boundaries x(t): $\eta(x_a(t), t) + x_a(t) = 0$

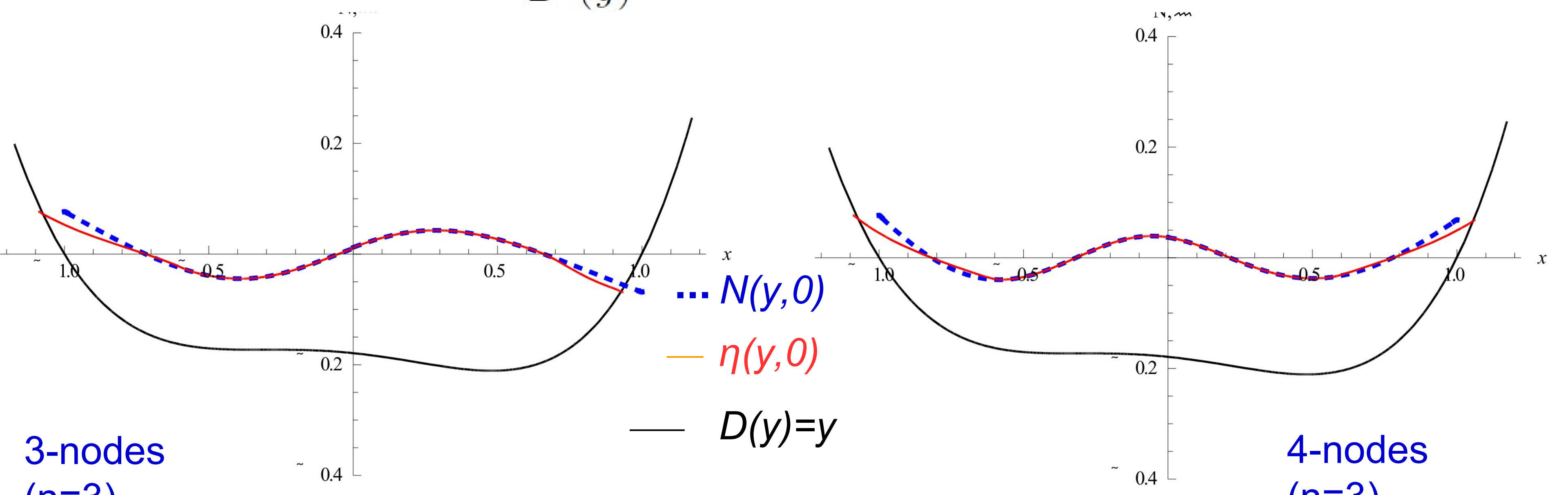
Reduced Carrier—Greenspan transform with cutting function ρ:

$$D(y) = D(x) + \eta(x,t)\rho(x), \quad \tau = t, \quad N(y,\tau) = \eta(x,t), \quad U(y,\tau) = u(x,t)$$

The leading term is defined from linearized shallow water with 2 fixed boundaries:

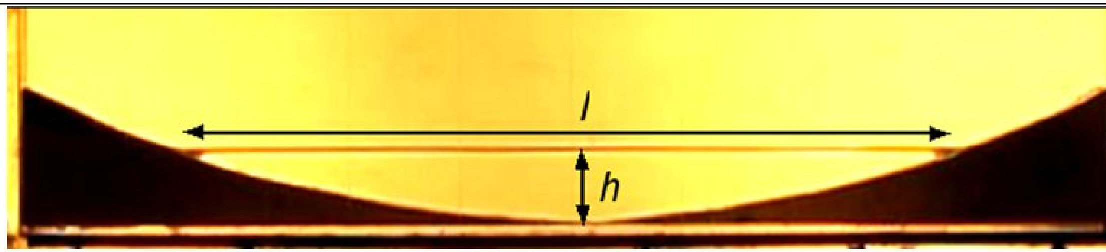
$N_\tau + (D(y)U)_y = 0, \quad U_\tau + N_y = 0$, *y є [a,b],* $E^2 = \|N\|^2 + \|\sqrt{y}\,U\|^2 < \infty$

Finally: $x = y - \varepsilon N_1 \frac{\rho(y)}{D'(y)}, \quad \eta(x,t) = N(y,t), \quad u(x,t) = U(y,t)$

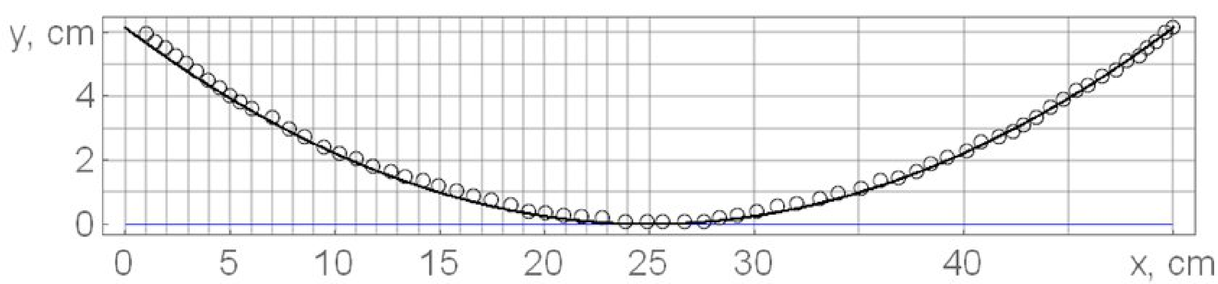


The basin with the parabolic bottom

The sizes: 50x4x50 см



l = 35.6 cm – длина свободной поверхности жидкости (зеркало)
h = 3.3 cm – глубина жидкости

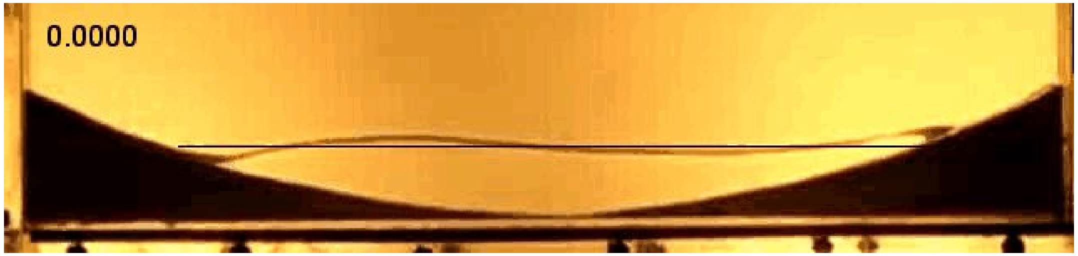


$y = 0.01(x-25)^2$ (cm)

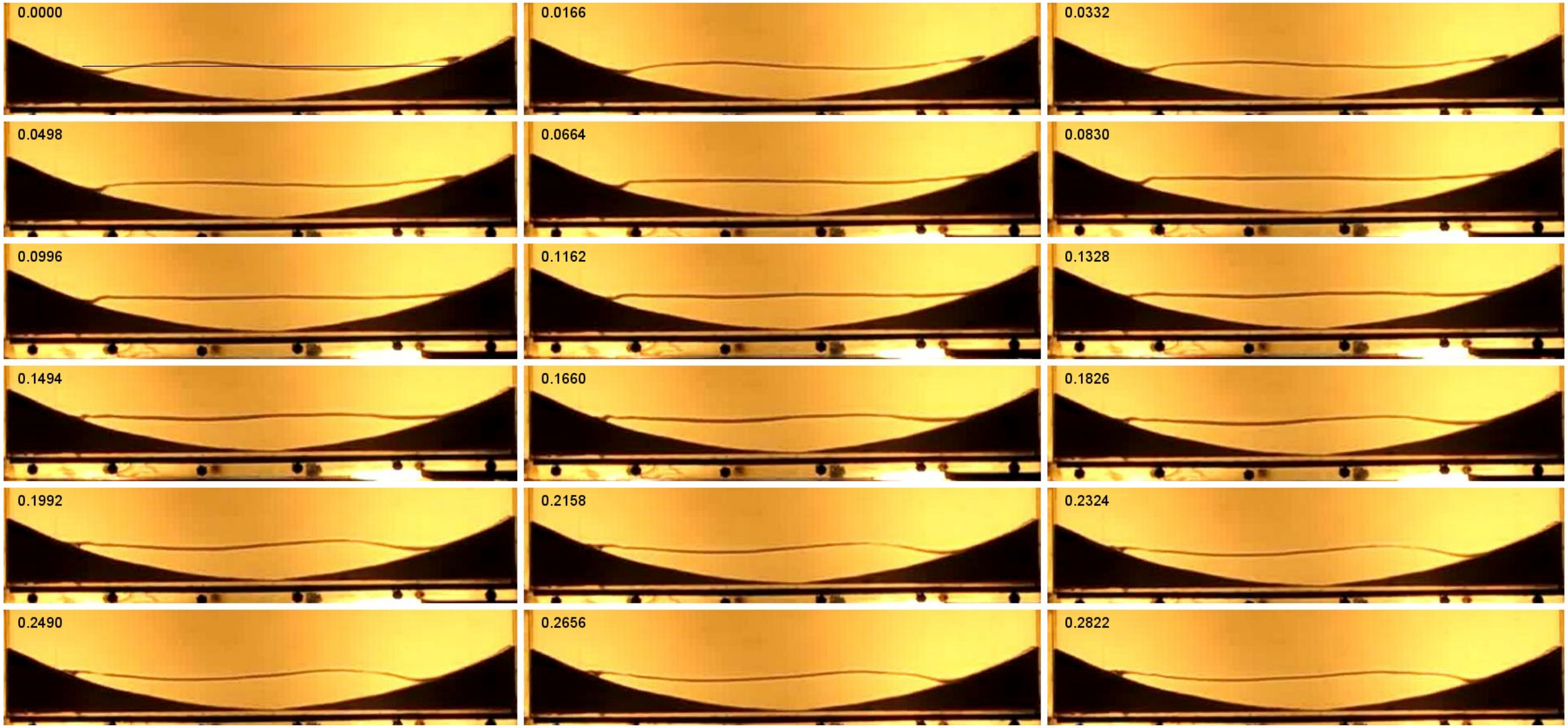
Сплошная кривая – расчетный профиль дна

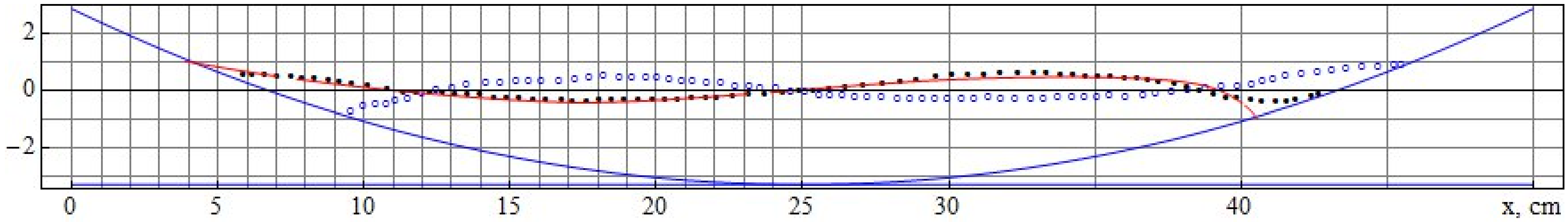
Точки – оцифровка фото

Standing waves with small amplitudes

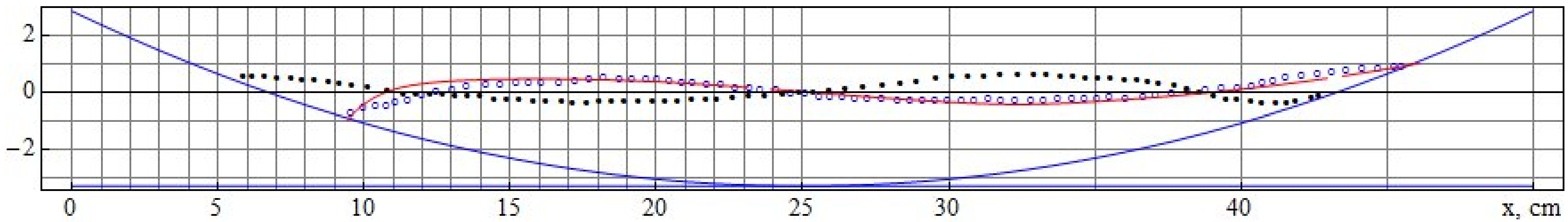


Standing wave with small amplitudes

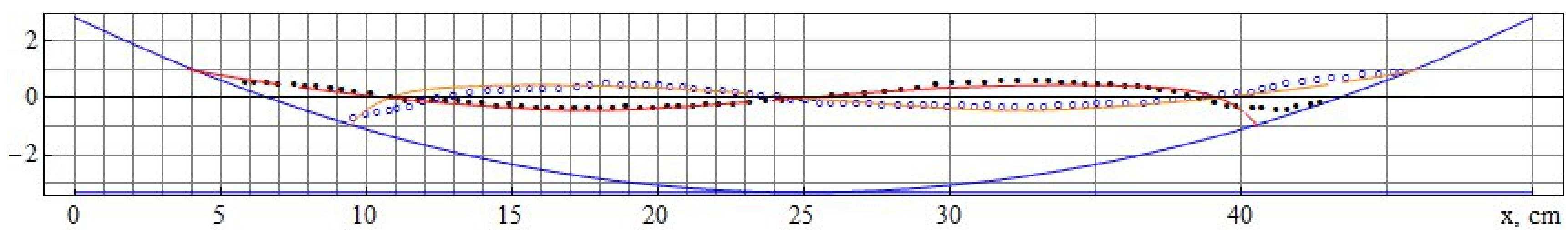




2
0
−2
0
5
10
15
20
25
30
40
x, cm



2
0
−2
0
5
10
15
20
25
30
40
x, cm



2
0
−2
0
5
10
15
20
25
30
40
x, cm

S. Yu. Dobrokhotov, D. S. Minenkov, and V. E. Nazaikinskii, On Asymptotic Solutions of the Cauchy Problem for a Nonlinear System of Shallow Water Equations in a Basin with Gently Sloping Banks, Russian Journal of Mathematical Physics, Vol. 29, No. 1, 2022, pp. 28-36

The question for future: the solutions of the linearized problem depends on additional small parameter μ characterizing the wavelength. The question is at what ratio between and the resulting formulas will work?

Спасибо за внимание!

THANK YOU FOR YOUR ATTENTION!