Monge-Ampère geometry and semigeostrophic equations

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Plan for this talk:

1) semigeostrophic equations

2) Monge-Ampère geometry

3) Main results

4) Examples

5) Outlook and future directions

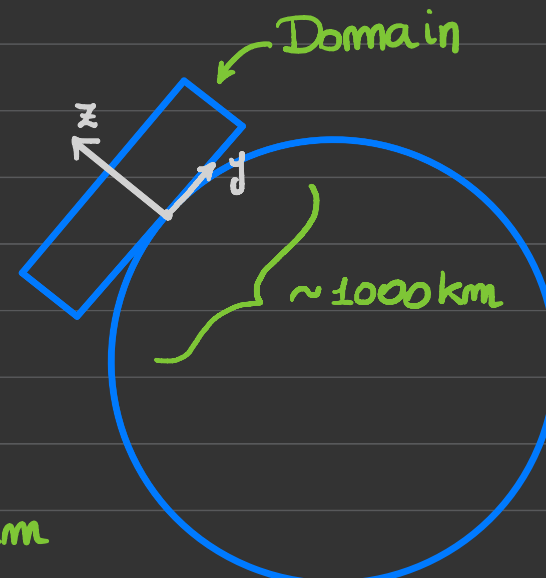
Semigeostrophic equations (Hoskins, 1975)

$$\begin{cases} \dfrac{Du_g}{Dt} - fv + \dfrac{\partial \phi}{\partial x} = 0, \quad \dfrac{Dv_g}{Dt} + fu + \dfrac{\partial \phi}{\partial y} = 0, \quad g\dfrac{\vartheta}{\vartheta_0} = \dfrac{\partial \phi}{\partial z} \\[2ex] \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = 0 \\[2ex] \dfrac{D\vartheta}{Dt} = 0 \end{cases}$$

geopotential $\phi = gh$

potential temperature

$\vartheta = \text{const} \cdot e^{s/c_p}$



$$u_g = -\frac{1}{f}\frac{\partial \phi}{\partial y}, \quad v_g = \frac{1}{f}\frac{\partial \phi}{\partial x}$$ ← geostrophic wind

$$M = fx + v_g, \quad N = fy - u_g$$ ← absolute momentum

Legendre duality (Chynoweth & Sewell 1989)

$$P = \frac{\phi}{f^2} + \frac{x^2+y^2}{2}$$ modified geopotential

$$\underline{X} = (X, Y, Z) := \left(\frac{M}{f}, \frac{N}{f}, \frac{g\theta}{f^2\theta_0} \right)$$ generalized momenta

geostrophic wind relations + hydrostatic balance

$$\Downarrow$$

$$\underline{X} = \nabla P$$

Legendre duality (Chynoweth & Sewell 1989)

$$\left\{ \left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} X \\ Y \end{pmatrix} \right) , \; (z, Z) \right\} \longleftarrow \text{"Dual variables"}$$

ex. $T - P = -zZ$

$\Rightarrow dT - dP = -d(zZ)$

$$\Rightarrow \begin{cases} \dfrac{\partial T}{\partial x} = \dfrac{\partial P}{\partial x} = X \\[2ex] \dfrac{\partial T}{\partial y} = \dfrac{\partial P}{\partial y} = Y \\[2ex] \dfrac{\partial T}{\partial Z} = \cancel{\dfrac{\partial P}{\partial z} dz} - z\, dZ - \cancel{Z\, dz} = -z \end{cases}$$

$$\begin{tikzcd}
 & R(X,Y,Z) \arrow[dr, leftrightarrow, "zZ"] & \\
T(x,y,Z) \arrow[ur, leftrightarrow, "xX+yY"] \arrow[dr, leftrightarrow, "-zZ"'] & & -S(X,Y,z) \\
 & -P(x,y,z) \arrow[ur, leftrightarrow, "-xX-yY"'] &
\end{tikzcd}$$

Vorticity-Streamfunction formulation

- R is the main unknown
- $\underline{X}$ are the independendent variables

2D incompressible Euler eqs.

$$\begin{cases} \Delta \psi = -q \\ \dfrac{Dq}{Dt} = 0 \end{cases}$$

SG equations can be written

kinematics

$$\begin{cases} \det \mathrm{Hess}\, R = \dfrac{1}{q} \\ \dfrac{Dq}{Dt} = 0 \end{cases}$$

dynamics

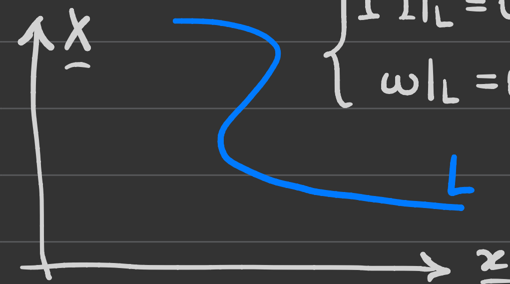
$$\begin{cases} \dfrac{DX}{Dt} = f(y-Y) \\ \dfrac{DY}{Dt} = f(X-x) \\ \dfrac{DZ}{Dt} = 0 \end{cases}$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{u}\frac{\partial}{\partial \underline{x}} = \frac{\partial}{\partial t} + \underline{\dot{X}}\frac{\partial}{\partial \underline{X}} = \frac{\partial}{\partial t} + f\underbrace{(y-Y)}_{\frac{\partial R}{\partial Y}}\frac{\partial}{\partial X} + f\underbrace{(X-x)}_{\frac{\partial R}{\partial X}}\frac{\partial}{\partial Y}$$

Monge-Ampère geometry (Kushner, Lychagin & Rubtsov, 2006)

- symplectic MA equations in 3D

$$\begin{cases} \Omega|_L = 0 \\ \omega|_L = 0 \end{cases}$$



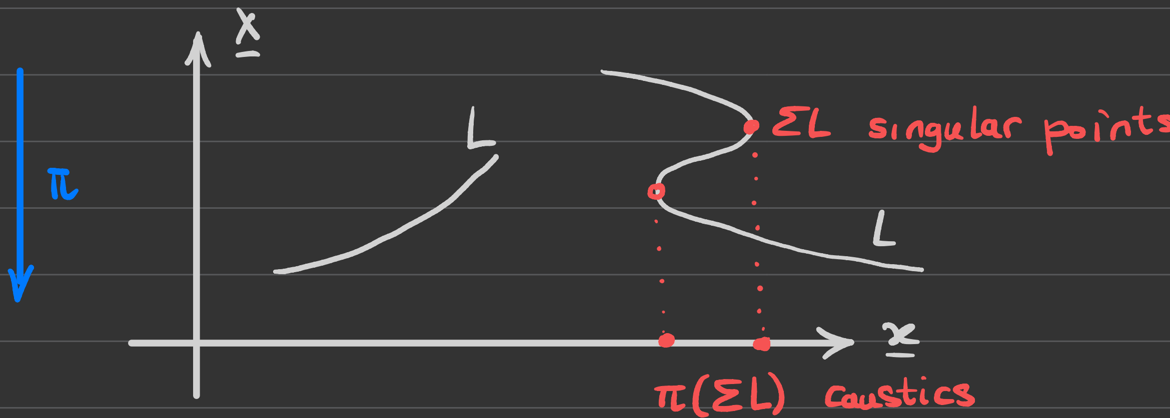
phase space $T^*\mathbb{R}^3$

Monge-Ampère structure

$$\begin{cases} \Omega = dX \wedge dx + dY \wedge dy + dZ \wedge dz \\ \omega = dX \wedge dY \wedge dZ - q\, dx \wedge dy \wedge dz \end{cases}$$

If $\exists\, P(\underline{x})$ s.t. L is the graph of ∇P,

$$\omega|_L = 0 \Rightarrow \boxed{\det \operatorname{Hess} P = q}$$



We can interpret P, R, S, T as generating functions for a solution L.

ex. if L is generated by $T(x,y,z)$, then $\omega|_L = 0$ implies

$$T_{xx}T_{yy} - T_{xy}^2 + 9\,T_{zz} = 0$$

Pseudo-Riemannian geometry

- There is g_ω on T^*R^3 which is compatible with (Ω, ω)

  1) Introduced by Lychogin & Rubtsov (1983)

  2) Banos (2002) $\longrightarrow$ generalized Calabi-Yau structures

- For semigeostrophic equations, g_ω is

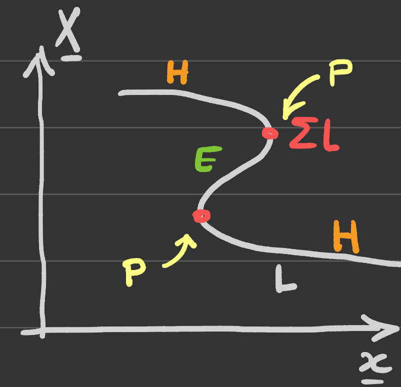
$$\boxed{g_\omega = 2q\, d\underline{x}\, d\underline{X}}$$

What is the meaning of g_ω for SG eqs?

Main results

1) The signature of $h_\omega = g_\omega|_L$ is

- $(1,2)$ on hyperbolic branches of L
- $(3,0)$ on elliptic branches of L



Moreover, h_ω is degenerate on parabolic branches of L

2) Parabolic branches of L are singular

3) Characteristic surfaces $F=0$ satisfy

$$h_\omega^{-1}(dF, dF) = 0 \quad \text{(eikonal equation)}$$

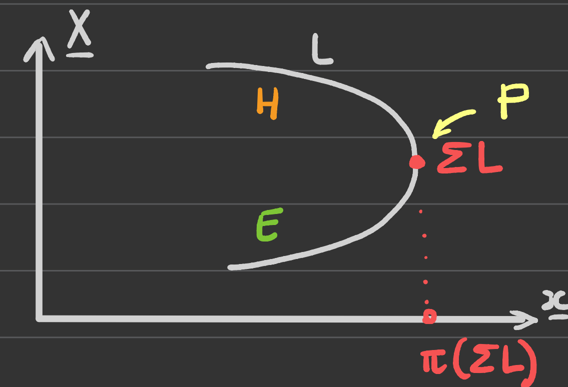
Example 1 : Fold singularity

$$T(x,y,Z) = \underbrace{\frac{y^2}{2}}_{\text{trivial dependence on } y} - \frac{x^2 Z}{2} + \underbrace{\frac{Z^3}{6}}_{\text{Lagrangian fold}} \qquad (q=1)$$

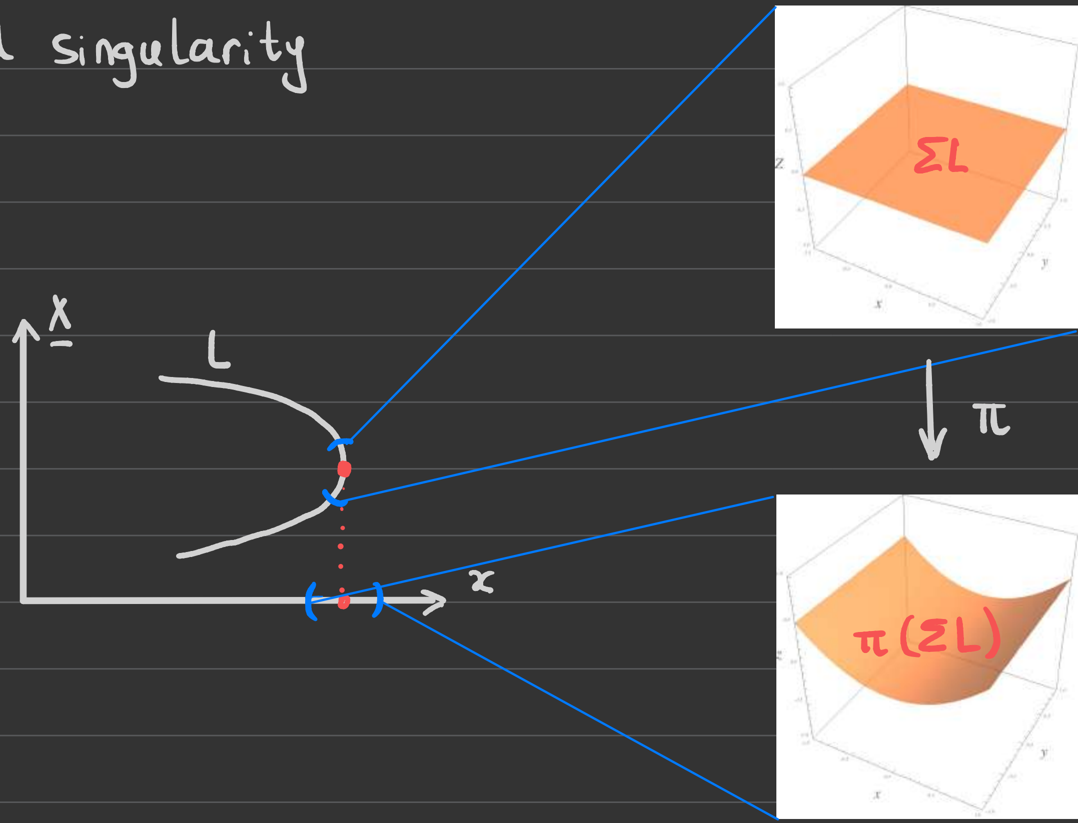
$$L = \left\{ X = \frac{\partial T}{\partial x} = xZ,\ Y = \frac{\partial T}{\partial y} = y,\ z = -\frac{\partial T}{\partial Z} = \frac{x^2}{2} - \frac{Z^2}{2} \right\}$$

$$h_\omega = 2\left(-Z\,dx^2 + dy^2 - Z\,dZ^2\right)$$

$$\Sigma L = \left\{ (x,y,Z) \in L : Z = 0 \right\}$$



Fold singularity



Lagrangian mfd
(x, y, Z)

$\Sigma L = \{ Z = 0 \}$

Physical space
(x, y, z)

$\pi(\Sigma L) = \{ z = \frac{x^2}{2} \}$

Characteristics

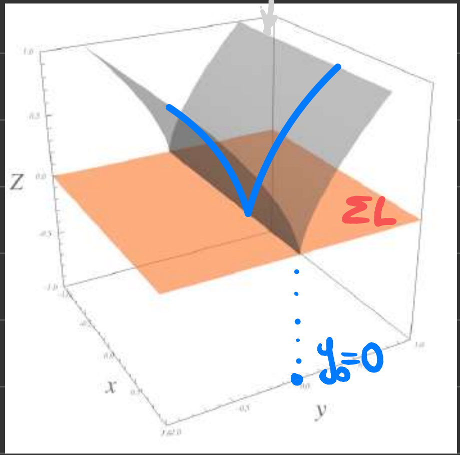
$$F(x,y,Z)=0 \quad \text{s.t.}$$

$$h_\omega^{-1}(dF,dF)=0 \quad \text{(eikonal eq.)}$$

$$\Rightarrow \; -\frac{1}{Z}\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 - \frac{1}{Z}\left(\frac{\partial F}{\partial Z}\right)^2 = 0$$

$$\rightsquigarrow \; F(x,y,Z) = (y-y_0)^2 - \frac{4}{9}Z^3$$

characteristic surface



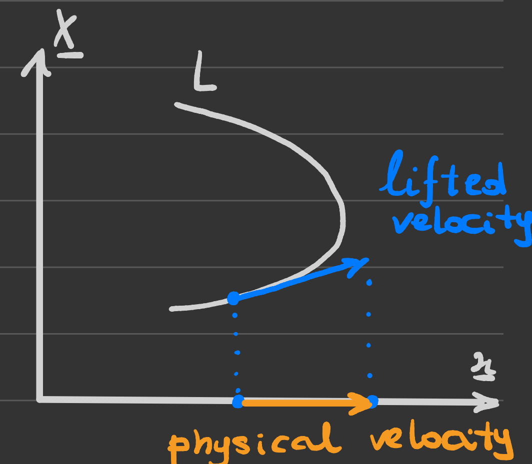
The physical picture

$$
\begin{cases}
\dfrac{DX}{Dt} = f(y-Y), \quad \dfrac{DY}{Dt} = f(X-x), \quad \dfrac{DZ}{Dt} = 0 \\[2ex]
X = \dfrac{\partial T}{\partial x}, \quad Y = \dfrac{\partial T}{\partial y}, \quad z = -\dfrac{\partial T}{\partial Z}
\end{cases}
$$

momentum and energy equations

Lagrangian mfd L

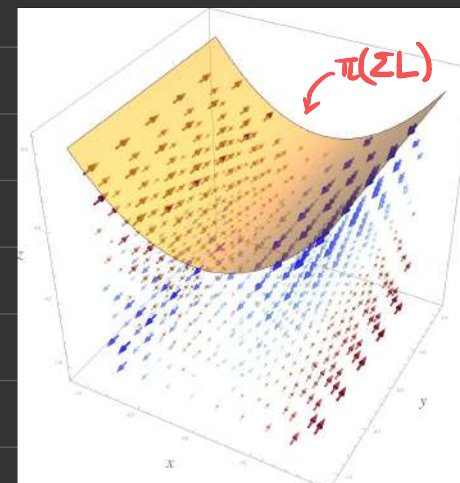
$$
\begin{cases}
\dfrac{\partial X}{\partial t} + \dot{x}\dfrac{\partial X}{\partial x} + \dot{y}\dfrac{\partial X}{\partial y} + \dot{Z}\dfrac{\partial X}{\partial Z} = f(y-Y) \\[2ex]
\dfrac{\partial Y}{\partial t} + \dot{x}\dfrac{\partial Y}{\partial x} + \dot{y}\dfrac{\partial Y}{\partial y} + \dot{Z}\dfrac{\partial Y}{\partial Z} = f(X-x) \\[2ex]
\dot{Z} = 0
\end{cases}
$$



Projection of the velocity field

$$\begin{cases} u = 0 \\ v = v_g = x\left(\sqrt{x^2 - 2z} - 1\right) \\ w = 0 \end{cases}$$

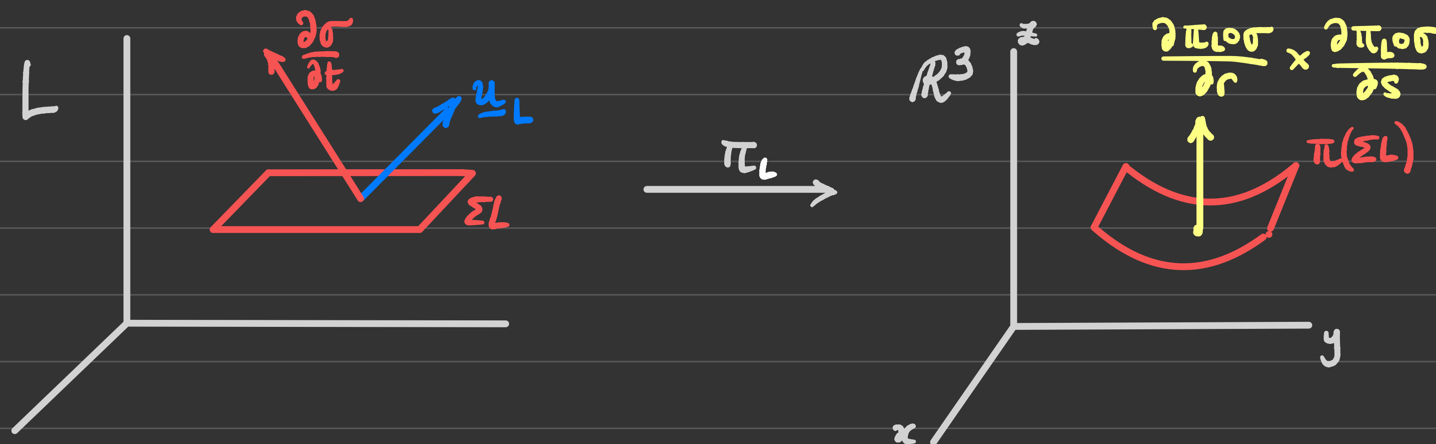
physical velocity field



The velocity field is tangent to the caustics $\pi(\Sigma L)$

Caustics are transported by the velocity field

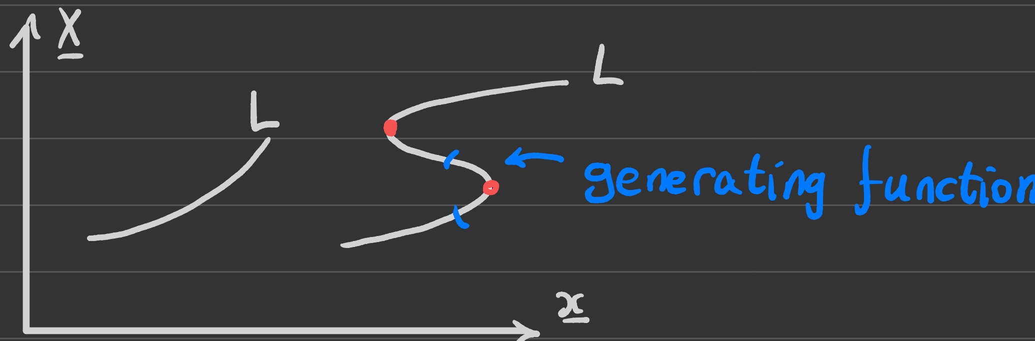
- $\Sigma L \subset L$ parametrized by $\sigma(r,s;t)$ ex. $\sigma(x,y)$



$$d\pi_L\left(\frac{\partial\sigma}{\partial t} - \underline{u}_L\right) \cdot d\pi_L \frac{\partial\sigma}{\partial r} \times d\pi_L \frac{\partial\sigma}{\partial s} = \det\left[d\pi_L\left(\frac{\partial\sigma}{\partial t} - \underline{u}_L, \frac{\partial\sigma}{\partial r}, \frac{\partial\sigma}{\partial s}\right)\right] = 0$$

Consistency of the main results

Def. (Harvey & Lawson, 1982)
The type of a nonlinear PDE at a solution is the type of its linearization about the solution.



Apply the above Def. to the MA eq. satisfied by the generating function.

Example: L generated by $T(x,y,Z)$.

$\Rightarrow$ T satisfies $\quad \dfrac{\partial^2 T}{\partial x^2}\dfrac{\partial^2 T}{\partial y^2} - \dfrac{\partial^2 T}{\partial x \partial y} + 9\dfrac{\partial^2 T}{\partial Z^2} = 0$

Linearization: $T \mapsto T + \delta T$

$$A = \begin{pmatrix} \frac{\partial^2 T}{\partial y^2} & -\frac{\partial^2 T}{\partial x \partial y} & 0 \\ -\frac{\partial^2 T}{\partial x \partial y} & \frac{\partial^2 T}{\partial x^2} & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

LR metric

$$h_\omega = \begin{pmatrix} \frac{\partial^2 T}{\partial x^2} & \frac{\partial^2 T}{\partial x \partial y} & 0 \\ \frac{\partial^2 T}{\partial x \partial y} & \frac{\partial^2 T}{\partial y^2} & 0 \\ 0 & 0 & -\frac{\partial^2 T}{\partial Z^2} \end{pmatrix}$$

$$\Rightarrow \quad \underline{h_\omega = 2\,\mathrm{adj}(A)}$$

Projection singularities

$$L = \left\{ X = \frac{\partial T}{\partial x},\ Y = \frac{\partial T}{\partial y},\ z = -\frac{\partial T}{\partial Z} \right\}$$

$$\Rightarrow \pi|_L(x, y, Z) = \left(x, y, -\frac{\partial T}{\partial Z} \right)$$

$$\Rightarrow d\pi|_L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{\partial^2 T}{\partial x \partial Z} & -\frac{\partial^2 T}{\partial y \partial Z} & -\frac{\partial^2 T}{\partial Z^2} \end{pmatrix}$$

$$\Sigma L = \left\{ \det(d\pi|_L) = -\frac{\partial^2 T}{\partial Z^2} = 0 \right\}$$

Outlook and future directions

- Curvature properties of g_ω and h_ω
- Dynamics and geometric flow
- Connection with optimal transport theory

Extra slides

$$h_\omega^{-1} = \frac{A}{\det(A)}$$ on hyperbolic branches

Classical characteristic surfaces $F = 0$ satisfy

$$\nabla F \cdot A \nabla F = 0$$

$h_\omega^{-1}(dF, dF) = 0$ is solved by Hamilton's method

$$\mathcal{H} = h_\omega^{ij} p_i p_j$$