Methods of tangent and cotangent coverings for Dubrovin-Novikov integrability operators

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Hamiltonian PDEs

An evolutionary system of PDEs

$$F = u^i_t - f^i(t, x, u^j, u^j_x, u^j_{xx}, \ldots) = 0$$

admits a Hamiltonian formulation if

$$u^i_t = A^{ij}\left(\frac{\delta H}{\delta u^j}\right)$$

where A is a Hamiltonian operator, i.e. a differential operator

$$A = a^{ij\sigma} D_\sigma \quad \text{such that} \quad A^* = -A \quad \text{and} \quad [A, A] = 0$$

$D_\sigma = D_x \circ \cdots \circ D_x$ (σ times).

Symmetries

A Hamiltonian equation shows a *correspondence* between conservation laws and symmetries.
Generalized *symmetries* are vector functions $\varphi^i = \varphi^i(u^j, u^j_x, u^j_{xx}, \ldots)$ such that

$$\ell_F(\varphi^i) = D_t\varphi^i - \frac{\partial f^i}{\partial u^j_\sigma} D_\sigma \varphi^j = 0$$

where ℓ_F is the Fréchet derivative.

Conservation laws

A *conservation law* is a one-form $\omega = Adx + Bdt$ which is closed modulo the equation:

$$\bar{d}\omega = \nabla F$$

where $\nabla = a_k^{\tau\sigma} D_{\tau\sigma} F^k$. The vector function

$$\psi_k = \psi_k(u^j, u^j_x, u^j_{xx}, \ldots) = (-1)^{|\tau\sigma|} D_{\tau\sigma} a_k^{\tau\sigma}$$

represents uniquely the conservation law and fulfills the equation

$$\ell_F^*(\varphi^i) = -D_t\psi_i + (-1)^{|\sigma|} D_\sigma \left(\frac{\partial f^j}{\partial u^i_\sigma} \psi_j \right) = 0$$

where ℓ_F^* is the formal adjoint of ℓ_F;

A necessary condition

If an equation admits a Hamiltonian formulation, this implies that A maps conservation laws into symmetries:

$$\ell_F \circ A = (A')^* \circ \ell_F^* \qquad A\colon \text{ almost-Hamiltonian op.}$$

The condition can be extended to all *integrability operators*:

$$\ell_F^* \circ S = S' \circ \ell_F \qquad S\colon \text{ almost-symplectic op.}$$
$$\ell_F \circ R = R' \circ \ell_F \qquad R\colon \text{ recursion operator}$$
$$\ell_F^* \circ C = (C')^* \circ \ell_F^* \qquad C\colon \text{ co-recursion operator}$$

Note that A', S', R', C' are arbitrary.

Almost: it is a necessary condition ...

Cotangent covering

Kersten, Krasil'shchik, Verbovetsky, JGP 2003.
Introducing new variables p_k, p_{kx}, p_{kxx}, ... we can represent operators by linear functions:

$$A(\psi) = a^{ij\sigma} D_\sigma \psi_j \quad \Leftrightarrow \quad A = a^{ij\sigma} p_{j\sigma}$$

Then a Hamiltonian operator fulfills the equations

$$\mathcal{T}^* \colon \begin{cases} \ell_F^*(\mathbf{p}) = -p_{i,t} + (-1)^{|\sigma|} D_\sigma \left(\frac{\partial f^j}{\partial u^i_\sigma} p_j \right) = 0 \\ F = u^i_t - f^i = 0 \end{cases} \quad \text{and} \quad \ell_F(A) = 0.$$

The system $\mathcal{T}^*$ is the *cotangent covering*.

Tangent covering

Introducing new variables q^k, q^k_x, q^k_{xx}, $\ldots$ we can represent operators by linear functions:

$$R(\varphi) = a^{i\sigma}_j D_\sigma \varphi^j \quad \Leftrightarrow \quad A = a^{i\sigma}_j q^j_\sigma$$

Then a recursion operator fulfills the equations

$$\mathcal{T}\colon \begin{cases} \ell_F(\mathbf{q}) = q^i_t - \dfrac{\partial f^i}{\partial u^j_\sigma} q^j_\sigma = 0 \\ F = u^i_t - f^i = 0 \end{cases} \quad \text{and} \quad \ell_F(A) = 0.$$

The system $\mathcal{T}^*$ is the *cotangent covering*.

Invariance

Tangent and cotangent coverings are invariant with respect to point transformations.

The proof is easy for the tangent covering, but it is much more complicated for the cotangent covering, and it needs the absence of differential relations between the components of the linearization.

Example: Hamiltonian operators for KdV

The KdV equation: $u_t = uu_x + u_{xxx}$
The linearization: $\ell_F = D_t - uD_x - u_x - D_{xxx}$
The adjoint linearization: $\ell_F^* = -D_t + uD_x + D_{xxx}$
The cotangent covering for the KdV equation:

$$\begin{cases} p_t = p_{xxx} + up_x \\ u_t = u_{xxx} + uu_x \end{cases}$$

The equation $\ell_F(A) = 0$ has the two solutions:

$$A_1 = p_x \quad \text{or} \quad A_1 = D_x$$

$$A_2 = \frac{1}{3}(3p_{3x} + 2up_x + u_x p) \quad \text{or} \quad A_2 = \frac{1}{3}(3D_{xxx} + 2uD_x + u_x)$$

Example: recursion operator for KdV

The tangent covering of KdV:

$$\mathcal{T}\colon \begin{cases} q_t = u_x q + u q_x + q_{xxx} \\ u_t = u_{xxx} + u u_x \end{cases}$$

Unfortunately, the equation for recursion operators $\ell_F(R) = 0$ has the only trivial solution $R = q$.
However, there is a conservation law on $\mathcal{T}$:
$\omega = q dx + (uq + q_{xx}) dt$. We can introduce a new non-local variable w such that

$$w_x = q, \qquad w_t = uq + q_{xx}.$$

Then we have the non-local recursion operator

$$R = q_{xx} + \frac{2}{3} u q + \frac{1}{3} u_x w \quad \text{or} \quad R = D_{xx} + \frac{2}{3} u + \frac{1}{3} u_x D_x^{-1}$$

Non-local operators

Non-local variables can be introduced on tangent and cotangent covering through the following

Theorem (Kersten, Krasil'shchik, Verbovetsky):

- There is a 1-1 correspondence between cosymmetries of $F = 0$ and conservation laws on $\mathcal{T}$ linear in q^k_σ.
- There is a 1-1 correspondence between symmetries of $F = 0$ and conservation laws on $\mathcal{T}^*$ linear in $p_{k\sigma}$.

Applications to Dubrovin–Novikov operators

The cotangent covering of a hydrodynamic-type system is:

$$\begin{cases} p_{i,t} = (V^k_{i,j} - V^k_{j,i})u^j_x p_k + V^k_i p_{k,x} \\ u^i_t = V^i_j(\mathbf{u})u^j_x \end{cases}$$

A first-order Dubrovin–Novikov Hamiltonian operator:

$$A^i = g^{ij}p_{jx} + \Gamma^{ij}_k u^k_x p_j.$$

Tsarev's compatibility conditions are the coefficients of the linear equation in $p_{k\sigma}$, $\ell_F(A) = 0$:

$$D_t A^i - V^i_{j,k}u^j_x A^k - V^i_j D_x A^j = 0 \quad \Leftrightarrow \quad \begin{cases} g^{ik}V^j_k = g^{jk}V^i_k \\ \nabla_i V^k_j = \nabla_j V^k_i \end{cases}$$

Applications to Ferapontov–Mokhov operators

For every symmetry $\varphi^i\partial/\partial u^i$ there is a conservation law on the cotangent covering of any hydrodynamic-type equation:

$$\omega = \varphi^i p_i dx + V^i_j \varphi^j p_i dt$$

Hydrodynamic-type systems admit time and space translation symmetries: two nonlocal variables $s_x = V^i_j u^j_x p_i$ and $r_x = u^i_x p_i$ which yield a general form of Ferapontov operators:

$$A = g^{ij} p_{j,x} + \Gamma^{ij}_k u^k_x p_j + \alpha V^i_k u^k_x s + \beta(V^i_k u^k_x r + u^i_x s) + \gamma u^i_x r$$

equivalently

$$\begin{aligned} A = {} & g^{ij} D_x + \Gamma^{ij}_k u^k_x + \alpha V^i_k u^k_x D^{-1} V^j_h u^h_x + \\ & \beta(V^i_k u^k_x D_x^{-1} u^j_x + u^i_x D_x^{-1} V^j_h u^h_x) + \gamma u^i_x D_x^{-1} u^j_x \end{aligned}$$

Compatibility conditions

The compatibility conditions between a Ferapontov–Mokhov operator and a hydrodynamic-type system are the coefficients of the linear equation in $p_{k\sigma}$, $\ell_F(A) = 0$:

$$D_t A^i - V^i_{j,k} u^j_x A^k - V^i_j D_x A^j = 0$$

iff

$$\begin{cases} g^{ik} V^j_k = g^{jk} V^i_k \\ \nabla_i V^k_j = \nabla_j V^k_i \\ R^{ij}_{kl} = \alpha(V^i_k V^j_l - V^i_l V^j_k) + \beta(V^i_k \delta^j_l + V^j_k \delta^i_l - V^i_l \delta^j_k - V^j_l \delta^i_k) \\ \qquad\quad + \gamma(\delta^i_k \delta^j_l - \delta^i_l \delta^j_k) \end{cases}$$

Application to third-order DN operators

Dubrovin–Novikov operators can be defined for arbitrary orders. Here we consider the third order ones:

$$
\begin{aligned}
A_3^{ij} =& g^{ij}(\mathbf{u})D_x^3 + b_k^{ij}(\mathbf{u})u_x^k D_x^2 \\
& + [c_k^{ij}(\mathbf{u})u_{xx}^k + c_{km}^{ij}(\mathbf{u})u_x^k u_x^m]D_x \\
& + d_k^{ij}(\mathbf{u})u_{xxx}^k + d_{km}^{ij}(\mathbf{u})u_x^k u_{xx}^m + d_{kmn}^{ij}(\mathbf{u})u_x^k u_x^m u_x^n,
\end{aligned}
$$

Potemin's canonical form in Casimirs:

$$
A_3^{ij} = D_x(g^{ij}D_x + c_k^{ij}u_x^k)D_x
$$

We restrict our consideration to hydrodynamic-type systems in these Casimirs. Then they can be written in conservative form: $V_j^i = (V^i)_{,j}$

Compatibility conditions

Theorem Let A_3 be a Hamiltonian operator. Then $u^i_t = V^i_j u^j_x = (V^i)_{,j} u^j_x$ admits a Hamiltonian formulation with A_3 if and only if

$$\left\{\begin{array}{l} g_{im} V^m_j = g_{jm} V^m_i \\ c_{mkj} V^m_i + c_{mik} V^m_j + c_{mji} V^m_k = 0, \\ V^k_{i,j} = g^{ks} c_{smj} V^m_i + g^{ks} c_{smi} V^m_j \end{array}\right. \tag{1}$$

Theorem. The above system is in involution. Its solution depends on at most $(1/2)n(n+3)$ parameters.

Properties of the systems of conservation laws

Following a construction of Agafonov and Ferapontov (1996-2001) we associate to each system $u^i_t = (V^i)_{,j} u^j_x$ a *congruence of lines* in $\mathbb{P}^{n+1}$ with coordinates $[y^1, \ldots, y^{n+2}]$

$$y^i = u^i y^{n+1} + V^i y^{n+2}$$

Theorem.

- The congruence is *linear*: there are n linear relations between u^i, V^i, $u^i V^j - u^j V^i$.
- The system is linearly degenerate and in the Temple class.
- $V^i = \psi^i_\alpha w^\alpha$, where ψ^i_α is determined by $g_{ij} = \varphi_{\alpha\beta} \psi^\alpha_i \psi^\beta_j$ and w^α are linear functions.

Finding hydrodynamic-type systems

We recall that the metric g_{ij} of a third-order Hamiltonian operator is a Monge metric, and corresponds to a quadratic line complex. There is a projective classification of Monge metrics of Hamiltonian operators for $n = 1, 2, 3, 4$ (Ferapontov, Pavlov, V., JGP 2014, IMRN 2016).

The system of compatibility conditions is

- linear algebraic with respect to g;
- linear algebraic with respect to w^α.

It is very easy to find systems associated with a given operator and operators associated with a given system.

Hamiltonian, momentum and more

The above systems of conservation laws all admit non-local Hamiltonian, momentum and Casimirs. They all are new non-local conserved quantities.
Let us set $\psi^\gamma_k = \psi^\gamma_{km} u^m + \omega^\gamma_k$, and $w^\gamma = \eta^\gamma_m u^m + \xi^\gamma$.
Let us set $u^i = b^i_x$; the system becomes $b^i_t = V^i(\mathbf{b}_x)$.
Theorem.

- Hamiltonian op. $A_3 = -g^{ij}(\mathbf{b}_x) D_x - c^{ij}_k(\mathbf{b}_x) b^k_{xx}$
- Hamiltonian $H = -\int \varphi_{\beta\gamma}[\left(\frac{1}{3}\eta^\gamma_p \psi^\beta_{qm} b^m_x + \frac{1}{2}\omega^\beta_p \eta^\gamma_q\right) b^p b^q + x\left(\frac{1}{2}\psi^\beta_{pq}\xi^\gamma b^p b^q_x + \xi^\gamma \omega^\beta_q b^q\right)]dx$
- n Casimirs $C^\alpha = \int \left(\frac{1}{2}\psi^\alpha_{mk} b^k_x + \omega^\alpha_m\right) b^m dx$
- momentum $P = -\int \left(\frac{1}{3}\varphi_{\beta\gamma}\omega^\beta_q \psi^\gamma_{pm} b^m_x + \frac{1}{2}\varphi_{\beta\gamma}\omega^\beta_p \omega^\gamma_q\right) b^p b^q dx$

Invariance of the hydrodynamic-type system

Theorem. The class of conservative systems of hydrodynamic type possessing third-order Hamiltonian formulation is invariant under reciprocal transformations of the form

$$d\tilde{x} = (a_i u^i + a)dx + (a_i V^i + b)dt$$
$$d\tilde{t} = (b_i u^i + c)dx + (b_i V^i + d)dt$$

Classification results

Theorem. Let $u^i_t = (V^i)_x$ be a hydrodynamic-type system, and suppose that it admits a Hamiltonian formulation via a third-order Dubrovin-Novikov operator whose Casimirs are u^i. Then:

- $n = 2$ The system is linearisable.
- $n = 3$ The system is either linearisable, or equivalent to the system of WDVV equations (to be discussed); from Castelnuovo's classification of linear line congruences.
- $n = 4$ Far more complicated: there exists no classification of linear congruences in $\mathbb{P}^5$.

Example: WDVV equations in 3-comp.

$$\begin{aligned} u^1_t &= u^2_x, \\ u^2_t &= u^3_x, \\ u^3_t &= ((u^2)^2 - u^1 u^3)_x, \end{aligned}$$

with the nonlocal Hamiltonian,

$$H = -\int \left(\frac{1}{2} u^1 \left({\partial_x}^{-1} u^2 \right)^2 + {\partial_x}^{-1} u^2 {\partial_x}^{-1} u^3 \right) dx.$$

Setting $u^1 = f_{xxx}$, $u^2 = f_{xxt}$, $u^3 = f_{xtt}$ we obtain $f_{ttt} = f^2_{xxt} - f_{xxx} f_{xtt}$, which is the simplest case of WDVV equations (Ferapontov, Galvao, Mokhov, Nutku, 1995). It is **bi-Hamiltonian** and up to a non-trivial transformation is the **3-wave equation** (Zakharov, Manakov, ∼1970).

Example: WDVV system in 6-comp.

Dubrovin 1996; Ferapontov-Mokhov 1998; Pavlov-V. 2015. We have a pair of hydrodynamic type systems in conservative form:

$$a^i_y = (v^i(\mathbf{a}))_x, \quad a^i_z = (w^i(\mathbf{a}))_x,$$

where

$$\begin{gathered}
v^1 = a^2, \quad w^1 = a^3, \quad v^2 = a^4, \quad v^3 = w^2 = a^5, \quad w^3 = a^6, \\
v^4 = f_{yyy} = \frac{2a^5 + a^2a^4}{a^1}, \quad v^5 = w^4 = f_{yyz} = \frac{a^3a^4 + a^6}{a^1}, \\
v^6 = w^5 = f_{yzz} = \frac{2a^3a^5 - a^2a^6}{a^1}, \\
w^6 = f_{zzz} = (a^5)^2 - a^4a^6 + \frac{(a^3)^2a^4 + a^3a^6 - 2a^2a^3a^5 + (a^2)^2a^6}{a^1}.
\end{gathered}$$

Monge metric for 6-components WDVV

$$g_{ik}(\mathbf{a}) = \begin{pmatrix} (a^4)^2 & -2a^5 & 2a^4 & -(a^1a^4+a^3) & a^2 & 1 \\ -2a^5 & -2a^3 & a^2 & 0 & a^1 & 0 \\ 2a^4 & a^2 & 2 & -a^1 & 0 & 0 \\ -(a^1a^4+a^3) & 0 & -a^1 & (a^1)^2 & 0 & 0 \\ a^2 & a^1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Remark: the metric can be found in few seconds by computer.

Example: generic value of n

The system of conservation laws:

$$u^1_t = u^2_x, \quad u^2_t = u^3_x, \ldots, \quad u^{n-1}_t = u^n_x, \quad u^n_t = [u^1 u^3 - (u^2)^2]_x.$$

The third-order Hamiltonian operator's Monge metric:

$$g_{ij} = \begin{pmatrix} 2a^2 & -a^1 & 0 & & 1 \\ -a^1 & 0 & & 1 & \\ 0 & & 1 & & \\ & 1 & & & 0 \\ 1 & & & 0 & 0 \end{pmatrix}$$

and the Hamiltonian is

$$H = -\frac{1}{2} a^1 (D^{-1} a^2)^2 + \frac{1}{2} \sum_{m=2}^{N} (D^{-1} a^m)(D^{-1} a^{N+2-m}).$$

Problem: integrability for $n \geqslant 4$?

Open problems

- Integrability for $n \geqslant 4$ of the systems of conservation laws.
- Geometry of WDVV equations. All of them have a third-order H.o.
- Non-local Hamiltonian operators of second and third order: compatibility with hydrodynamic-type systems.
- Extension to symplectic operators, local and non-local.
- Recursion operators?

Symbolic computations

Within the REDUCE CAS (now free software) we use the packages CDIFF and CDE, freely available at `http://gdeq.org`.

CDE (by RV) can compute symmetries and conservation laws, local and nonlocal Hamiltonian operators, Schouten brackets of local multivectors, Fréchet derivatives (or linearization of a system of PDEs), formal adjoints, Lie derivatives of Hamiltonian operators.

Cooperation with AC Norman (Trinity College, Cambridge) to improvements and documentation of REDUCE's kernel.

Forthcoming book, in cooperation with JS Krasil'shchik and AM Verbovetsky: *The symbolic computation of integrability structures for partial differential equations*, to appear in the series Texts and Monographs in Symbolic Computation, Springer, 2017.

Thank you!

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