Vortex-like solitons and Painlevé integrability

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Overview

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$$\left. + \left| D_1 \phi + i D_2 \phi \right|^2 + B - i(\partial_1(\bar{\phi} D_2 \phi) - \partial_2(\bar{\phi} D_1 \phi)) \right] dx^2 \geq \pi N$$

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- Exact solutions on hyperbolic surfaces (constant negative curvature). [Witten77, Manton&Rink2010]

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- Coefficient of the resonance $\Rightarrow$ differential equation for Ω

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- Consider $G(|\phi|)^2 = |\phi|^q$.

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Possible solutions $(m, n) = (3, 6), (4, 4), (6, 3)$.

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- Boundary conditions: given z_0

$$|\phi(z)| \sim_{z\to z_0} |z - z_0|^N$$
$$\sim_{z\to\infty} 1$$

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Remark: $q = 1$ and $4/3$ were found in [Dunajski2012].

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Consider $q = \frac{2}{3}$:

$$h = -3u \sim_{r \to 0} \ln\left[\left(1 - \alpha_N r^{\frac{2}{3}(3-N)}\right)^6 \beta_N \, r^{2N}\right]$$
$$\sim_{r \to \infty} \frac{3\sqrt{3}}{\pi}\left\{1 + 2\cos\left[\frac{\pi}{9}(6-2N)\right]\right\} K_0(r)$$

where α_N, β_N are constants, $N = 1, 2$ and $K_0(r) = \sqrt{\frac{\pi}{2r}}e^{-r}$.

Abelian vortices on conical singularities

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Possible construction of vortices on the universal cover [Contatto,Dorigoni2015].

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