Asymmetric variable separation for the Clebsch model

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F. Magri, T. Skrypnyk, *The Clebsch System*, arXiv:1512.04872.

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Y. Fedorov, F. Magri, T. Skrypnyk *A new approach to separation of variables for the Clebsch integrable system. Part I: Reduction to quadraturs*, arXiv: 2102.03445.

Y. Fedorov, F. Magri, T. Skrypnyk, *A new approach to separation of variables for the Clebsch integrable system. Part II: Inversion of the Abel–Prym map*, arXiv: 2102.03599.

Plan of the talk

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1 Introduction

To begin with we start with several motivational questions and comments.

1. Why separation of variables (SoV) in the Hamilton-Jacobi (HJ) sense is important?

It is important because it is the main method of the integration of the classical equations of motion for integrable hamiltonian systems. It is used also while solving quantum integrable models [11].

2. What is the state of arts in the subject?

There exist three main approaches to the variable separation: a classical one going back to the papers of Stackel [12, 13], Levi-Civitta [14] and Agostinelli [15] and developed later in the papers of Benenti and his school [16, 17] and two modern ones. They are: the “magic recipe” of Sklyanin [11] and the bi-hamiltonian approach of Magri, Falqui and Pedroni [18, 19]. The classical approach is very restricted: it works only for the hamiltonian systems on cotangent bundles T^*N. Two modern approaches cover much wider class of the models, but, unfortunately they are far from being finished.

2. Why the Clebsch model?

The Clebsch system [2] is the only integrable case of Kirckhoff model [1] for which HJ SoV was unknown. It was known on the special submanyfold, where it is equivalent to the Weber model [3].

2 Generalities on Hamilton-Jacobi SoV for the integrable hamiltonian systems

An integrable Hamiltonian system with n degrees of freedom is determined on a $2n$-dimensional symplectic manifold $\mathcal{M}$, which is embedded in the Poisson manyfold $(\mathcal{P}, \{\ ,\ \}_1)$ as a level surface of m Casimir function C_i, by n independent Poisson-commuting first integrals I_j:

$$\{I_i, I_j\}_1 = 0, \quad i, j \in \overline{1, n}.$$

To find HJ separated variables means to find — at least locally — a set of coordinates $x_i, p_j, i, j \in \overline{1, n}$, such that there exist n relations — "equations of separation" — of the following form [11]:

$$\Phi_i(x_i, p_i, I_1, ..., I_n, C_1, ..., C_m) = 0, \quad i \in \overline{1, n}, \tag{1}$$

and the coordinates $x_i, p_j, i, j \in \overline{1, n}$ are (quasi)canonical:

$$\{p_i, x_j, \}_1 = f_i(x_i, p_i)\delta_{ij}, \quad \{x_i, x_j\}_1 = 0, \quad \{x_i, x_j\}_1 = 0, \quad \forall i, j \in \overline{1, n} \tag{2}$$

for some functions f_i, $i \in \overline{1, n}$ on $\mathbb{C}^2$.

It is possible to show that the coordinates of separation x_i satisfy the following equations:

$$\sum_{i=1}^{n} \frac{\partial_{I_k}\Phi_i(x_i, p_i, I_1, ..., I_n, C_1, ..., C_m)}{\partial_{p_i}\Phi_i(x_i, p_i, I_1, ..., I_n, C_1, ..., C_m)} \frac{1}{f_i(x_i, p_i)} \frac{\partial x_i}{\partial t_j} = \delta_{kj}, \quad \forall k, j \in \overline{1, n}, \tag{3}$$

where t_j is a "time" corresponding to the integral I_j, i.e. a parameter along its hamiltonian flow.

From the equations (3) one deduces the Abel-type equations written in the differential form:

$$\sum_{i=1}^{n} \frac{\partial_{I_j}\Phi_i(x_i, p_i, I_1, ..., I_n, C_1, ..., C_m)}{\partial_{p_i}\Phi_i(x_i, p_i, I_1, ..., I_n, C_1, ..., C_m)} \frac{dx_i}{f_i(x_i, p_i)} = dt_j, \quad j \in \overline{1, n}. \tag{4}$$

The equations (4) are the final object of SoV and the starting object for the integration procedure.

3 The “magic recipe”of Sklyanin

In the case when the hamiltonian system under the consideration possess Lax pair formulation there exist the so-called “magic recipe” of Sklyanin [11]. We remarque, that its roots — at least in the classical case — go back to the papers [6] [7], [8]. The “magic recipe” states that in the Lax-integrable case all equations of separation coincide with one spectral curve of the Lax matrix:

$$\Phi_i(x_i, p_i, I_1, ..., I_n, C_1, ..., C_m) = \det(L(x_i) - p_i Id) = 0 \tag{5}$$

and the coordinates of separation coincide with the poles of the “properly normalized” eigenvectors of $L(u)$. Using this normalization one can construct the coordinates and momenta of separations as zeros of certain function $B(u)$ and the values of another function $A(u)$ in these zeros:

$$B(x_i) = 0, \quad p_i = A(x_i). \tag{6}$$

Unfortunately, in the general case the “magic recipe” of Sklyanin dos not answer the question what the “proper” normalization should be. That is why, it is more precise to call the “magic recipe” to be the “magic conjecture”, due to the fact that in the general case there is no concrete “recipe” of how to construct the “proper” normalization of the eigenvectors of $L(u)$, or, equivalently how to construct the functions $B(u)$ and $A(u)$. Another restriction of the method is the condition that all the equations of separation are the same and coincide with a spectral curve of one Lax matrix. As we will show in the present talk, this is too strong the requirement that does not hold true in some cases. That is why it is desirable — even in the Lax integrable case — to have alternative methods of SoV. Two of such the methods we will describe below.

4 The method of the vector fields Z_i

The method of the vector field Z_i in the theory of separation of variables was proposed in the paper of Magri, Falqui and Pedroni [18]. It permits to construct the coordinates of separation x_i for the given bi-hamiltonian system starting from the certain data of the Poisson geometry. In more details, let us assume that there exists a second Poisson bracket $\{\ ,\ \}_2$ compatible with the first one, i.e. any linear combination of the brackets $\{\ ,\ \}_i$: $\{\ ,\ \}_u = u\{\ ,\ \}_1 + \{\ ,\ \}_2$. is a Poisson bracket.

We denote Casimir functions of $\{\ ,\ \}_u$ by $C_k(u)$, $k \in \overline{1,m}$ and assume them to be polynomial [21].

Theorem 4.1 *(Magri, Falqui, Pedroni) Let us consider vector fields Z_k on the Poisson manyfold $\mathcal{P}$ that satisfy the following conditions:*

$$\mathrm{Lie}_{Z_i}\{\ ,\ \}_1 = 0, \qquad i \in \overline{1,m} \tag{7a}$$

$$\mathrm{Lie}_{Z_i}\{\ ,\ \}_2 = \sum_{j=1}^{m} Z_j \wedge [X_j, Z_i], \qquad i \in \overline{1,m} \tag{7b}$$

$$Z_k Z_l(I_i) = 0, \quad i \in \overline{1,n},\ k,l \in \overline{1,m} \tag{7c}$$

$$Z_k(C_l) = \delta_{kl}, \quad k,l \in \overline{1,m}. \tag{7d}$$

Let the roots $u = x_i$, $i \in \overline{1,n}$ of the equation

$$S(u) = \det(Z_i(C_j(u))) = 0, \quad i,j \in \overline{1,m} \tag{8}$$

be functionally independent on generic coadjoint orbits of $\{\ ,\ \}_1$. Then x_i, $i \in \overline{1,n}$ are the coordinates of separation for the considered bi-hamiltonian system.

The theorem above, in principle, provides the coordinates of separation x_i, $i \in \overline{1,n}$. The difficult part of the approach is the necessity to find all vector fields Z_k, $k \in \overline{1,m}$ in order to find the separating polynomial $S(u)$. Indeed, in order to define the separating polynomial in terms of the initial dynamical variables, we need to resolve the system of PDE (7a), (7b), (7c), which is, in general, very complicated task. This task, however, is simplified in certain cases, e.g. when one needs only one vector field Z_k for a certain index k in order to define $S(u)$.

Corollary 4.1 *(Magri, Falqui, Pedroni) Let the Casimir functions C_i, $i \in \overline{1,m-1}$ be the common Casimirs of the both brackets $\{\ ,\ \}_1$ and $\{\ ,\ \}_2$. Then the roots $u = x_i$ of the equation*

$$S(u) = Z_m(C_m(u))) = 0 \tag{9}$$

are the coordinates of separation.

Hence, in the case when all but one Casimirs of the brackets $\{\ ,\ \}_1$ and $\{\ ,\ \}_2$ are the common ones, the problem of the construction of separating polynomial is reduced to the problem of finding of only one vector field Z_m, which satisfies instead of (7c), (7a), (7b) simpler conditions:

$$Z_m(C_i) = 0, \quad i \in \overline{1,m-1}, \quad Z_m(C_m) = 1, \tag{10a}$$

$$Z_m^2(C_m(u)) = 0, \tag{10b}$$

$$\mathrm{Lie}_{Z_m}\{\ ,\ \}_m = 0, \tag{10c}$$

$$\mathrm{Lie}_{Z_m}\{\ ,\ \}_2 = Z_m \wedge [X_m, Z_m], \tag{10d}$$

where X_m is a hamiltonian vector field of the Casimir C_m with respect to the second bracket.

Remark. Observe that although the equations (10) are simpler than the system of equations (7c), (7a), (7b), they still are non-linear PDE which are in general difficult to solve. We will illustrate a possible approach to solution of the problem of finding of the vector field Z_m in the next sections.

5 The method of the differential conditions

This method is due to F. Magri [20]. We will use it in order to find the momenta p_i canonically conjugated to the separated coordinates x_i. The momenta of separation in the general case are not given by the method of the vector fields Z_i. We will expose the method of the differential conditions in the simplest case of the systems with two degrees of freedom and in the convenient for us form, i.e. hereafter we will assume that $n = 2$, $m = 2$. We will hereafter put also $I_1 = H$, $I_2 = K$.

For any function f we define its derivatives along the time flows of the integrals H and K as follows:

$$\dot{f} = \{f, H\}, \quad f' = \{f, K\}. \tag{11}$$

The following Proposition holds true:

Proposition 5.1 *Assume that the separating polynomial has the form*

$$S(u) = u^2 + s_1 u + s_2 = u^2 + Z_2(H)u + Z_2(K), \tag{12}$$

where the vector field Z_2 satisfy the conditions (10) with $m = 2$. Then the functions s_1, s_2 Poisson-commute:

$$\{s_1, s_2\}_1 = 0, \tag{13}$$

and satisfy the following differential equations:

$$s_1' - \dot{s}_2 = 0, \tag{14a}$$

$$s_2' - s_1\dot{s}_2 + s_2\dot{s}_1 = 0. \tag{14b}$$

The above proposition has the following Corollary [22]:

Corollary 5.1 *Let x_1, x_2 be the roots of the polynomial $S(u)$. Then the functions x_1, x_2 Poisson-commute:*

$$\{x_1, x_2\}_1 = 0$$

and satisfy the following differential equations:

$$x_1' + x_2\dot{x}_1 = 0, \quad x_2' + x_1\dot{x}_2 = 0. \tag{15}$$

The above Corollary is used in order to construct the canonically conjugated momenta p_1, p_2 [22]:

Theorem 5.1 *Let the coordinates x_i, $i \in \overline{1,2}$ satisfy the conditions (15) and Poisson-commute. Let the function F_i, $i \in 1,2$ be defined as follows:*

$$F_i = (x_j - x_i)\dot{x}_i, \quad i \in 1,2, \quad j \neq i. \tag{16}$$

Then in the coordinate system consisting of the functions x_1, x_2, H, K, C_1, C_2 we will have

$$F_i = F_i(x_i, x_iH + K, C_1, C_2) \tag{17}$$

and the functions

$$p_i = \int\limits_0^{x_iH+K} \frac{d\lambda}{F_i(x_i, \lambda, C_1, C_2)}, \quad i \in 1,2, \tag{18}$$

are the momenta canonically conjugated to the coordinates x_i, $i \in \overline{1,2}$.

Using this theorem one obtains the explicit form of momenta of separation:

$$p_i = \Psi_i(x_i, x_iH + K, C_1, C_2), \quad \forall i \in 1,2$$

and, consequently, the equations of separation as the final point of SoV:

$$\Phi_i(p_i, x_i, x_iH + K, C_1, C_2) = 0, \quad \forall i \in 1,2.$$

6 The Clebsch model and its bi-hamiltonian structure

The Clebsch model is an integrable model defined on the linear space of the dimension six with the coordinates S_α, T_α, $\alpha \in \overline{1,3}$ that satisfy the standard $e^*(3)$ Poisson brackets:

$$\{S_\alpha, S_\beta\} = \epsilon_{\alpha\beta\gamma} S_\gamma, \quad \{S_\alpha, T_\beta\} = \epsilon_{\alpha\beta\gamma} T_\gamma, \quad \{T_\alpha, T_\beta\} = 0. \tag{19}$$

These brackets possess two Casimir functions:

$$C_2 = \sum_{\alpha=1}^{3} T_\alpha^2, \quad C_1 = \sum_{\alpha=1}^{3} T_\alpha S_\alpha. \tag{20}$$

The quadratic functions:

$$I_1 = H = \sum_{\alpha=1}^{3} S_\alpha^2 + \sum_{\alpha=1}^{3} (j_\beta + j_\gamma) T_\alpha^2, \tag{21}$$

$$I_2 = K = \sum_{\alpha=1}^{3} j_\alpha S_\alpha^2 + \sum_{\alpha=1}^{3} j_\beta j_\gamma T_\alpha^2. \tag{22}$$

are Poisson-commuting integrals of motion. The system is completely integrable: the dimension of the corresponding phase space — a level surface of the Casimir function is four.

The equations of motion of the Clebsch model with respect to the Hamiltonian H read as follows:

$$\frac{dS_\alpha}{dt_1} = (j_\gamma - j_\beta) T_\beta T_\gamma, \qquad \frac{dT_\alpha}{dt_1} = S_\beta T_\gamma - S_\gamma T_\beta. \tag{23}$$

Here t_1 is the time corresponding to the hamiltonian H.

Observe that there is a second Lie-Poisson structure for the Clebsch model, compatible with the first one and having the following explicit form:

$$\{S_\alpha, S_\beta\}_2 = \epsilon_{\alpha\beta\gamma} j_\gamma S_\gamma, \quad \{S_\alpha, T_\beta\}_2 = \epsilon_{\alpha\beta\gamma} j_\beta T_\gamma, \quad \{T_\alpha, T_\beta\}_2 = \epsilon_{\alpha\beta\gamma} S_\gamma. \tag{24}$$

Observe that in the case $j_\alpha \neq 0$, $\alpha \in \overline{1,3}$ the Poisson algebra (24) is isomorphic to $so(4)$. The isomorphism is achieved by the following substitution of variables:

$$S_\alpha \to \sqrt{j_\beta}\sqrt{j_\gamma} S_\alpha, \quad T_\alpha \to \sqrt{j_\alpha} T_\alpha, \qquad \alpha \in \overline{1,3}. \tag{25}$$

In such a way the considered model is isomorphic to the so-called Shottky-Frahm model on $so(4)$.

Observe also that the function C_1 is a common Casimir function for the both brackets. The hamiltonian K is a Casimir function of the structure $\{\ ,\ \}_2$. The integrals are functions H and C_2.

The "Poisson pencil" of the above brackets

$$\{\ ,\ \}_u = u\{\ ,\ \}_1 + \{\ ,\ \}_2 \tag{26}$$

has the following Casimir functions:

$$C_2(u) = C_2 u^2 + Hu + K, \quad C_1(u) = C_1. \tag{27}$$

The function C_1 is a Common Casimir function of the all brackets of the Poisson pencil.

Remark. Hereafter we will assume complete anisotropy i.e. we will assume that $j_\alpha \neq j_\beta$ if $\alpha \neq \beta$.

7 The Weber-Neumann subcase of the Clebsch model

On special coadjoint orbits with $C_1 = 0$ the Clebsch model is equivalent to the Weber-Neumann model [3, 4]. The corresponding coadjoint orbit may be viewed as a cotangent bundle T^*S^2 embedded in six-dimensional linear space with the coordinates $Q_1, Q_2, Q_3, P_1, P_2, P_3$ by the equations:

$$Q_1^2 + Q_2^2 + Q_3^2 = 1, \quad P_1 Q_1 + P_2 Q_2 + P_3 Q_3 = 0, \tag{28}$$

were the coordinates $Q_1, Q_2, Q_3, P_1, P_2, P_3$ satisfy the canonical Poisson brackets:

$$\{P_\alpha, Q_\beta\} = \delta_{\alpha\beta}. \tag{29}$$

In this case there exists the following parametrization of the Lie-algebraic variables:

$$S_\alpha = P_\beta Q_\gamma - P_\gamma Q_\beta, \quad T_\alpha = Q_\alpha, \tag{30}$$

In the result the hamiltonian H of the Clebsch model acquires Weber-Neumann form:

$$H = \sum_{\alpha=1}^{3} P_\alpha^2 + \sum_{\alpha=1}^{3} (j_\beta + j_\gamma) Q_\alpha^2 \tag{31}$$

of the Hamiltonian of motion of the particle on the sphere in the quadratic potential.

8 SoV for the Weber-Neumann subcase of the Clebsch model

Well-known for more than a century is SoV for the subcase $C_1 = 0$ of the Clebsch model [3, 4]. The separating polynomial in this case is written as follows:

$$S(u) = \sum_{\alpha=1}^{3} (u + j_\beta)(u + j_\gamma) T_\alpha^2. \tag{32}$$

Its two roots $u = x_1$, $u = x_2$ satisfy the following equations of separation:

$$(x_i + j_1)(x_i + j_2)(x_i + j_3)p_i^2 + (K + x_i H + x_i^2 C_2) = 0, \quad i \in 1, 2, \tag{33}$$

where the coordinates x_i, p_i, $i \in 1, 2$ are the canonical ones:

$$\{p_i, x_j, \}_1 = \delta_{ij}, \quad \{x_i, x_j\}_1 = 0, \quad \{x_i, x_j\}_1 = 0, \quad \forall i, j \in \overline{1, n}.$$

This SoV is a bi-hamiltonian one: on the surface $C_1 = 0$ the normalized Euler vector field

$$Z_2 = \frac{1}{2C_2} \sum_{\alpha=1}^{3} T_\alpha \frac{\partial}{\partial T_\alpha} \tag{34}$$

satisfies the condition (10). In more details we have that:

$$\mathrm{Lie}_{Z_2}\{\ ,\ \}_1 = 0, \tag{35a}$$

$$\mathrm{Lie}_{Z_2}\{\ ,\ \}_2 = Z_2 \wedge [X_{C_2}, Z_2], \tag{35b}$$

$$Z_2^2(H) = 0, \quad Z_2^2(K) = 0, \quad Z_2^2(C_2) = 0, \quad Z_2^2(C_1) = 0. \tag{35c}$$

$$Z_2(C_2) = 1, \quad Z_2(C_1) = 0. \tag{35d}$$

That is why it defines the separating polynomial by the following formula:

$$S(u) = Z_2(C_2)u^2 + uZ_2(H) + Z_2(K), \tag{36}$$

which exactly coincides with the “made-monic” polynomial (32).

The momenta of separation and equations of separation (33) are obtained using the formula (18).

9 Assymetric SoV for the Clebsch model: the coordinates of separation

In order to obtain the coordinates of separation we use the technique of the vector field $Z = Z_2$. Due to the fact that C_1 is a common Casimir the needed separating polynomial is written as follows:

$$S(u) = Z(C_2)u^2 + Z(H)u + Z(K) \tag{37}$$

where the vector field Z may be normalized as follows: $Z(C_2) = 1$ to make $S(u)$ to be monic. We look for such a vector field Z that the conditions the conditions (35) be satisfied on the whole phase space.

Let us solve the condition (35c) together with the normalization conditions (35d). For this purpose let us consider the general vector field on $e^*(3)$:

$$Z = \sum_{\alpha=1}^{3} A_\alpha \frac{\partial}{\partial S_\alpha} + \sum_{\alpha=1}^{3} B_\alpha \frac{\partial}{\partial T_\alpha}, \tag{38}$$

where A_α, B_β are some functions on the phase space $e^*(3)$.

In order to solve the condition (35c) we assume that vector field Z annuls its components:

$$Z(A_\alpha) = 0, \quad Z(B_\alpha) = 0, \qquad \alpha \in \overline{1,3}. \tag{39}$$

Observe that this is an important technical assumption that permits us to consider instead of set of *differential* equations for the functions A_α, B_β a set of *algebraic* equations for these functions.

Under such the requirement six functions A_α, B_β should satisfy the following six algebraic equations:

$$A_1^2 + A_2^2 + A_3^2 + (j_2 + j_3)B_1^2 + (j_3 + j_1)B_2^2 + (j_1 + j_2)B_3^2 = 0, \tag{40a}$$

$$j_1 A_1^2 + j_2 A_2^2 + j_3 A_3^2 + j_2 j_3 B_1^2 + j_3 j_1 B_2^2 + j_1 j_2 B_3^2 = 0, \tag{40b}$$

$$B_1^2 + B_2^2 + B_3^2 = 0, \tag{40c}$$

$$A_1 B_1 + A_2 B_2 + A_3 B_3 = 0. \tag{40d}$$

$$2(B_1 T_1 + B_2 T_2 + B_3 T_3) = 1, \tag{40e}$$

$$A_1 T_1 + A_2 T_2 + A_3 T_3 + B_1 S_1 + B_2 S_2 + B_3 S_3 = 0, \tag{40f}$$

which are the consequences of the algebraic equations (35c), (35d). The following Proposition holds [22]:

Proposition 9.1 *The equations (40a)-(40f) have the following generic solution:*

$$A_\alpha = \lambda c_\alpha v_\beta v_\gamma, \;\; B_\alpha = \lambda c_\alpha v_\alpha, \quad \textit{where} \tag{41}$$

$$v_\alpha^2 = v + j_\alpha, \tag{42}$$

$$c_\gamma^2 = j_\alpha - j_\beta, \tag{43}$$

v is the function of S_α, T_β satisfying the following irrational equation:

$$c_1 v_2 v_3 T_1 + c_2 v_1 v_3 T_2 + c_3 v_1 v_2 T_3 + c_1 v_1 S_1 + c_2 v_2 S_2 + c_3 v_3 S_3 = 0. \tag{44}$$

and normalization constant λ is given by the following formula $\lambda = \dfrac{1}{2(c_1 T_1 v_1 + c_2 T_2 v_2 + c_3 T_3 v_3)}$.

Remark. Observe, that the equation (44) has eight solutions as the equation of the function v!

For the subsequent we will need to introduce the following auxiliary functions:

$$f_1 = \sum_{\alpha=1}^{3} c_\alpha v_\alpha T_\alpha, \;\; f_2 = \sum_{\alpha=1}^{3} c_\alpha j_\alpha v_\alpha T_\alpha, \;\; f_3 = \sum_{\alpha=1}^{3} c_\alpha v_\beta v_\gamma T_\alpha, \tag{45}$$

$$g_1 = \sum_{\alpha=1}^{3} c_\alpha v_\alpha S_\alpha, \;\; g_2 = \sum_{\alpha=1}^{3} c_\alpha j_\alpha v_\alpha S_\alpha, \;\; g_3 = \sum_{\alpha=1}^{3} c_\alpha v_\beta v_\gamma S_\alpha. \tag{46}$$

The following Theorem holds true [22]:

Theorem 9.1 *Let the vector field Z (38) has the components A_α, B_α as in the Proposition 9.1. Then*

(i) The roots of the polynomial (37) are the Poisson-commuting functions x_1, x_2 having the form:

$$x_1 = v, \qquad x_2 = -v + \frac{f_2 - g_3}{f_1} - (j_1 + j_2 + j_3). \tag{47}$$

(ii) The corresponding canonically conjugated momenta p_1, p_2 are written as follows:

$$p_1 = \frac{(g_1 v + g_2 + v_1 v_2 v_3 f_1)}{2c_1 c_2 c_3 v_1 v_2 v_3}, \quad p_2 = \frac{f_1}{c_1 c_2 c_3}. \tag{48}$$

(iii) The curves of separation $\Phi_i(x_i, p_i, H, K, C_1, C_2)$, have genus three and the following form:

$$\begin{aligned}\Phi_1(x_1, p_1, H, K, C_1, C_2) = 4p_1^2(x_1 + j_1)(x_1 + j_2)(x_1 + j_3) + (x_1^2 C_2 + x_1 H + K) + \\ + 2\sqrt{(x_1 + j_1)(x_1 + j_2)(x_1 + j_3)}\,C_1 = 0,\end{aligned} \tag{49}$$

$$\Phi_2(x_2, p_2, H, K, C_1, C_2) = p_2^4(x_2 + j_1)(x_2 + j_2)(x_2 + j_3) + (x_2^2 C_2 + x_2 H + K)p_2^2 + C_1^2 = 0. \tag{50}$$

Remark. There are three ways to prove this theorem. The first way is to show that defined by Proposition 9.1 vector field Z satisfy also the conditions (35a)- (35b), i.e. define the coordinates of separation indeed. The second way is to show that the coordinates x_i given by (47), Poisson-commute, satisfy the differential conditions (15), then to find explicitly the functions F_i given by (16) and use them in order to find the canonically conjugated momenta and equations of separation. This is done in our papers [22, 25]. Here we propose for your attention the direct proof, which is the simplest one.

Sketch of the Proof. Applying the vector field $Z = Z_2$ with the above components to the polynomial $C_2(u)$ we obtain the explicit form of the polynomial $S(u)$. By the direct check one sees that it factorizes in the product of two linear in u multipliers and has two roots x_1 and x_2 given by (47).

Then we calculate the Poisson brackets among the functions g_α, f_β, v:

$$\{f_1, f_2\}_1 = \frac{c_1c_2c_3f_3^2}{(2v + j_1 + j_2 + j_3)f_1 + g_3 - f_2}, \ \{f_1, f_3\}_1 = -\frac{c_1c_2c_3f_3f_1}{(2v + j_1 + j_2 + j_3)f_1 + g_3 - f_2},$$

$$\{f_1, g_1\}_1 = \frac{c_1c_2c_3f_3f_1}{(2v + j_1 + j_2 + j_3)f_1 + g_3 - f_2}, \ \{f_1, g_2\}_1 = -\frac{c_1c_2c_3vf_3f_1}{(2v + j_1 + j_2 + j_3)f_1 + g_3 - f_2},$$

$$\{f_1, g_3\}_1 = c_1c_2c_3f_1 - \frac{c_1c_2c_3f_3g_1}{(2v + j_1 + j_2 + j_3)f_1 + g_3 - f_2},$$

$$\{f_2, g_1\}_1 = \frac{c_1c_2c_3(f_1^2v_1v_2v_3 + ((v + j_1 + j_2 + j_3)f_1 - f_2)f_3)}{(2v + j_1 + j_2 + j_3)f_1 + g_3 - f_2},$$

$$\{f_2, g_2\}_1 = \frac{c_1c_2c_3(((f_1(v + j_1 + j_2 + j_3) - f_2 + g_3)f_1 - g_1f_3)v_1v_2v_3 - v(v + j_1 + j_2 + j_3)f_3f_1 + vf_3f_2)}{(2v + j_1 + j_2 + j_3)f_1 + g_3 - f_2},$$

$$\{g_3, g_1\}_1 = \frac{c_1c_2c_3(g_1f_2 - g_2f_1)}{(2v + j_1 + j_2 + j_3)f_1 + g_3 - f_2}, \ \{g_3, g_2\}_1 = \frac{c_1c_2c_3(g_1^2v_1v_2v_3 + v(g_1f_2 - g_2f_1))}{(2v + j_1 + j_2 + j_3)f_1 + g_3 - f_2},$$

$$\{f_1, v\}_1 = 0, \ \{g_1, v\}_1 = -\frac{2c_1c_2c_3v_1v_2v_3f_1}{(2v + j_1 + j_2 + j_3)f_1 + g_3 - f_2},$$

$$\{g_2, v\}_1 = -\frac{2c_1c_2c_3v_1v_2v_3((j_1 + j_2 + j_3)f_1 + g_3 - f_2)}{(2v + j_1 + j_2 + j_3)f_1 + g_3 - f_2}, \ \{g_3, v\}_1 = \frac{2c_1c_2c_3v_1v_2v_3g_1}{(2v + j_1 + j_2 + j_3)f_1 + g_3 - f_2},$$

$$\{f_2, v\}_1 = -\frac{2c_1c_2c_3v_1v_2v_3f_3}{(2v + j_1 + j_2 + j_3)f_1 + g_3 - f_2}, \ \{f_3, v\}_1 = -\frac{2c_1c_2c_3v_1v_2v_3f_1}{(2v + j_1 + j_2 + j_3)f_1 + g_3 - f_2}.$$

Using these brackets and the explicit form of x_1 and x_2 we immediately obtain that $\{x_1, x_2\}_1 = 0$. In the same way we find that the variables

$$p_1 = \frac{(g_1 v + g_2 + v_1 v_2 v_3 f_1)}{2c_1 c_2 c_3 v_1 v_2 v_3}, \quad p_2 = \frac{f_1}{c_1 c_2 c_3}, \tag{51}$$

are canonically conjugated to x_1, x_2, i.e:

$$\{p_j, x_i\}_1 = \delta_{ij},\ \{x_i, x_j\}_1 = 0,\ \{p_i, p_j\}_1 = 0,\ \forall i, j \in \overline{1,2}.$$

Finally using the explicit form of the canonical variables p_i, x_i, $i \in 1, 2$ in terms of the functions g_α, f_β, v and expressing in the terms of these functions also the integrals of motion and Casimir functions:

$$C_1 = \frac{1}{c_1^2 c_2^2 c_3^2}(g_1 f_1 (v + j_1 + j_2 + j_3) - (g_2 f_1 + g_1 f_2) - g_3 f_3), \quad C_2 = \frac{1}{c_1^2 c_2^2 c_3^2}(f_1^2 (v + j_1 + j_2 + j_3) - 2 f_2 f_1 - f_3^2),$$

$$H = \frac{1}{c_1^2 c_2^2 c_3^2}\big((j_1^2 + j_2^2 + j_3^2 + j_3 j_1 + j_1 j_2 + j_2 j_3 - v^2) f_1^2 + (-2(j_1 + j_2 + j_3) f_2 + 2 f_3 v_1 v_2 v_3) f_1 + f_2^2 - (v + j_1 + j_2 + j_3) f_3^2$$
$$+ (v + j_1 + j_2 + j_3) g_1^2 - 2 g_1 g_2 - g_3^2\big),$$

$$K = \frac{1}{c_1^2 c_2^2 c_3^2}\big(-((v + j_1 + j_2 + j_2)^2 v + j_1 j_2 j_3) f_1^2 + (2v(v + j_3 + j_1 + j_2) f_2 + 2 v_1 v_2 v_3 (v + j_3 + j_1 + j_2) f_3) f_1 - f_2^2 v$$
$$- 2 f_2 f_3 v_1 v_2 v_3 + (v^2 + v(j_1 + j_2 + j_3) + j_1 j_2 + j_3 j_1 + j_2 j_3)(-f_3^2 + g_1^2) - 2 g_1 g_3 v_1 v_2 v_3 - g_2^2 + g_3^2 v\big).$$

and taking into account that the constraint (44) acquires in $f - g$ coordinates the following form:

$$g_1 + f_3 = 0 \tag{52}$$

we obtain the equations of separation (49)-(50).

Theorem is proven.

10 The reconstruction formulae

Although it is impossible to find *explicitly* the coordinates x_1, x_2 as the functions of S_α, T_α, it is possible to express *explicitly* S_α, T_α using separated variables x_1, x_2, p_1, p_2 and the values of the Casimir functions C_1, C_2. The following Proposition holds true [22]:

Proposition 10.1 *The variables S_α, T_α, $\alpha \in \overline{1,3}$ are expressed via separated coordinates and Casimir functions as follows:*

$$S_\alpha = \frac{1}{c_\beta c_\gamma}\Big(\frac{\sqrt{x_1+j_\beta}\sqrt{x_1+j_\gamma}C_1^2}{2p_2^3(x_1-x_2)^2}+\sqrt{x_1+j_\alpha}\big(\frac{2(x_1+j_\beta)(x_1+j_\gamma)p_1}{(x_1-x_2)^2p_2^2}-\frac{(j_\gamma x_2+j_\beta x_2+j_\beta j_\gamma+2x_1x_2-x_1^2)}{(x_1-x_2)^2p_2}\big)$$
$$\times C_1+\sqrt{x_1+j_\beta}\sqrt{x_1+j_\gamma}\big(\frac{C_2}{2p_2}+\frac{2(x_1+j_3)(x_1+j_2)(x_1+j_1)p_1^2}{(x_1-x_2)^2p_2}-\frac{2(x_2+j_\beta)(x_2+j_\gamma)(x_1+j_1)p_1}{(x_1-x_2)^2}+$$
$$+\frac{(2x_2^3+(3j_\alpha+j_\beta+j_\gamma-x_1)x_2^2+(-2x_1j_\alpha+2j_\alpha(j_\beta+j_\gamma))x_2-((j_\beta+j_\gamma)j_\alpha-j_\beta j_\gamma)x_1+j_1j_2j_3)p_2}{2(x_1-x_2)^2}\big)\Big), \tag{53a}$$

$$T_\alpha = \frac{1}{c_\beta c_\gamma}\Big(\frac{\sqrt{x_1+j_\alpha}C_1^2}{2p_2^3(x_1-x_2)^2}+\big(\frac{2\sqrt{x_1+j_\beta}\sqrt{x_1+j_\gamma}(x_1+j_\alpha)p_1}{(x_1-x_2)^2p_2^2}-\frac{\sqrt{x_1+j_\beta}\sqrt{x_1+j_\gamma}(x_2+j_\alpha)}{(x_1-x_2)^2p_2}\big)C_1+$$
$$+\sqrt{x_1+j_\alpha}\big(\frac{C_2}{2p_2}+\frac{2(x_1+j_1)(x_1+j_3)(x_1+j_2)p_1^2}{(x_1-x_2)^2p_2}-\frac{2(x_1+j_3)(x_1+j_2)(x_2+j_1)p_1}{(x_1-x_2)^2}+$$
$$+\frac{((j_\beta-j_\alpha+j_\gamma+x_1)x_2^2+(2x_1j_\alpha+2j_\beta j_\gamma)x_2+((j_\beta+j_\gamma))j_\alpha-j_\beta j_\gamma)x_1+j_2j_3j_1)p_2}{2(x_1-x_2)^2}\big)\Big). \tag{53b}$$

11 The Abel-type equations

Using the equations of spectral curves (49), (50) and the formulae (4) it is easy to obtain the Abel-type equations written in the differential form [22]:

$$\frac{x_1 dx_1}{8p_1(x_1+j_1)(x_1+j_2)(x_1+j_3)} + \frac{x_2 p_2^3 dx_2}{2(p_2^4(x_2+j_1)(x_2+j_2)(x_2+j_3) - C_1^2)} = dt_1, \tag{54}$$

$$\frac{dx_1}{8p_1(x_1+j_1)(x_1+j_2)(x_1+j_3)} + \frac{p_2^3 dx_2}{2(p_2^4(x_2+j_1)(x_2+j_2)(x_2+j_3) - C_1^2)} = dt_2, \tag{55}$$

where t_1, t_2 are "times" corresponding to the hamiltonians H and K.

Let us transform them into more standard form. For this purpose we make the change of variables:

$$w = 2\sqrt{(x_1+j_1)(x_1+j_2)(x_1+j_3)}\, p_1, \quad W = \frac{\sqrt{2}C_1}{p_2}, \quad \Phi(x) = (x+j_1)(x+j_2)(x+j_3). \tag{56}$$

In such the notations the separation curves acquire the form:

$$\mathcal{C}:\ w^2 + (C_2 x_1^2 + Hx_1 + K) + 2C_1\sqrt{\Phi(x_1)} = 0. \tag{57}$$

$$\mathcal{K}:\ W^4 + 2(C_2 x_2^2 + Hx_2 + K)W^2 + 4C_1^2\Phi(x_2) = 0. \tag{58}$$

and the Abel-type equations (54)-(55) are written as follows

$$\frac{x_1 dx_1}{w(w^2 + (C_2x_1^2 + Hx_1 + K))} + \frac{\sqrt{2}x_2 dx_2}{W(W^2 + (C_2x_2^2 + Hx_2 + K))} = -\frac{dt_1}{C_1}, \tag{59}$$

$$\frac{dx_1}{2w(w^2 + (C_2x_1^2 + Hx_1 + K))} + \frac{\sqrt{2}dx_2}{W(W^2 + (C_2x_2^2 + Hx_2 + K))} = -\frac{dt_2}{C_1}. \tag{60}$$

12 Remarks on the Abel-Jacobi inversion problem

It is possible to show [26] that the differentials w_1, w_2

$$w_1 = \frac{dx_1}{w(w^2 + (C_2 x_1^2 + H x_1 + K))}, \quad w_2 = \frac{x_1 dx_1}{2w(w^2 + (C_2 x_1^2 + H x_1 + K))} \tag{61}$$

participating in our Abel-type equations are two holomorphic differentials on the curve (57), while the differentials W_1, W_2

$$W_1 = \sqrt{2}\frac{dx_2}{W(W^2 + (C_2 x_2^2 + H x_2 + K))}, \quad W_2 = \sqrt{2}\frac{x_2 dx_2}{W(W^2 + (C_2 x_2^2 + H x_2 + K))} \tag{62}$$

also participating in our Abel-type equations, are two holomorphic differentials on the curve (58), their periods are commensurable and, that is why, the Abel-type equations (59)-(60) can be inverted in terms of meromorphic functions of times. They lead to a well-defined, so called Abel–Prym map from the product $\mathcal{C} \times \mathcal{K}$ to a two-dimensional Prym subvariety of the Jacobian of the curve $\mathcal{K}$ (see [9, 10] for details on algebraic geometry of these curves). Moreover, it can be shown that separated coordinates are expressed in terms of so-called higher order theta-functions on the Prym variety of the curve $\mathcal{K}$ [26].

Important Remark. In contrast to the standard Abel map, the Abel–Prym map is not one-to-one: a full preimage of a point in the Prym variety consists of 8 pairs of points on $\mathcal{C} \times \mathcal{K}$. The coordinates of these 8 pairs of points can be identified with the eight sets of separating variables for the Clebsch system connected with eight solutions $v^{(1)}$, ... , $v^{(8)}$ of the constraint equation (44). This enables one to derive a (new) theta-function solution for the original variables S_α, T_α: they can be expressed as symmetric functions of coordinates of the eight points on the curve $\mathcal{C}$, while the latter functions can be written in terms of the higher order theta-functions which solve the problem of inversion of the Abel–Prym map.

13 The Weber-Neumann interpretation of the assymetric SoV

Finally we would like to present the "Weber-Neumann interpretation" of the obtained assymetric SoV for the Clebsch model [25]. For this purpose we extend the phase space of the Clebsch system $\mathcal{P} = \mathbb{C}^6$, endowed with the Poisson pencil $\{\ ,\}_u = u\{\ ,\}_1 + \{\ ,\ \}_2$.

Let E be an elliptic curve

$$y^2 = (v + j_1)(v + j_2)(v + j_3). \tag{63}$$

Let $\hat{E}$ be its un-ramified $4:1$ covering on which the functions v_α, $\alpha \in \overline{1,3}$ are correctly defined.

Let us prolong the Poisson pencil $\{\ ,\}_u = u\{\ ,\}_1 + \{\ ,\ \}_2$ onto the extended phase space $\hat{\mathcal{P}} = \mathcal{P} \times \hat{E}$ with the coordinates (S_α, T_α, v) by the following formulae:

$$\{S_\alpha, S_\beta\}_{\hat{1}} = \{S_\alpha, S_\beta\}_1, \tag{64a}$$

$$\{S_\alpha, T_\beta\}_{\hat{1}} = \{S_\alpha, T_\beta\}_1, \tag{64b}$$

$$\{T_\alpha, T_\beta\}_{\hat{1}} = \{T_\alpha, T_\beta\}_1, \tag{64c}$$

$$\{S_\alpha, v\}_{\hat{1}} = -(\sum_{\beta=1}^{3} \frac{\partial Z(C_1)}{\partial S_\beta}\{S_\alpha, S_\beta\}_1 + \frac{\partial Z(C_1)}{\partial T_\beta}\{S_\alpha, T_\beta\}_1)(\frac{\partial Z(C_1)}{\partial v})^{-1}, \tag{64d}$$

$$\{T_\alpha, v\}_{\hat{1}} = -(\sum_{\beta=1}^{3} \frac{\partial Z(C_1)}{\partial S_\beta}\{T_\alpha, S_\beta\}_1 + \frac{\partial Z(C_1)}{\partial T_\beta}\{T_\alpha, T_\beta\}_1)(\frac{\partial Z(C_1)}{\partial v})^{-1}. \tag{64e}$$

The brackets $\{\ ,\ \}_{\hat{2}}$ are defined similarly. This explicit representation allows to check the following

- The pencil $u\{\ ,\ \}_{\hat{1}} + \{\ ,\ \}_{\hat{2}}$ is a new Poisson pencil.

- It has two common Casimir functions C_1 and $C_1' = Z(C_1)$, and a quadratic polynomial Casimir function $C_2(u) = C_2 u^2 + Hu + K$.
- The new Poisson pencil defines two vector fields $\hat{X}_H$ and $\hat{X}_K$ on $\hat{P}$, which are integrable Hamiltonian vector fields on $\hat{P}$.
- The vector fields X_H and X_K of Clebsch are the projections of the vector fields $\hat{X}_H$ and $\hat{X}_K$ along the canonical projection $\pi : \hat{\mathcal{P}} \to \mathcal{P}$.

Under such the definition the constructed SoV for the Clebsch model on $\mathcal{P}$ can be interpreted as SoV for the extended Clebsch model defined on the special symplectic leafs in $\hat{\mathcal{P}}$ given by the equation

$$C_1' = Z(C_1) = 0. \tag{65}$$

In such a way the obtained SoV for the Clebsch model is interpreted as Weber-Neumann-type SoV for its extension. The constraint equations (44) is interpreted as zero level set of the Casimir function C_1'. The strange condition that the vector field Z annuls its own components is reduced — modulo the normalization of $\hat{Z}$ — to the requirement that the extended vector field $\hat{Z}$ has the form:

$$\hat{Z} = \sum_{\alpha=1}^{3} A_\alpha(v)\frac{\partial}{\partial S_\alpha} + \sum_{\alpha=1}^{3} B_\alpha(v)\frac{\partial}{\partial T_\alpha} + 0 \cdot \frac{\partial}{\partial v}, \tag{66}$$

i.e. its components depend only on the coordinate v of the proposed one-dimensional extension.

In such a way this view-point provides geometric interpretation of our asymmetric SoV and geometric interpretation of its main constraint (44).

14 Conclusion and discussion

Regardless the interpretation, the proposed SoV exhibits the following qualitatively new properties:

1. It has two *different* curves of separation $\mathcal{C}$ and $\mathcal{K}$, i.e. we are out of the "magic recipe".
2. The genus of separation curves $g = 3$ *is larger* than the number of degrees of freedom $n = 2$.
3. In contrast to the standard Abel map, the corresponding Abel–Prym map from $\mathcal{C} \times \mathcal{K}$ to Prym variety of $\mathcal{K}$ *is not one-to-one*.

At last, two natural questions may arise in the context of the presented results:

1. Is there any "symmetric" SoV for the Clebsch model with two separation curves being equal?
2. Is there any other — except for the Clebsch model — examples of the Lax-integrable models with different separation curves?

The answer to the both questions is positive.

As for the first question: in [24] we have constructed symmetric SoV for the Clebsch model on $\mathcal{C} \times \mathcal{C}$.

As for the second question: using the bi-hamiltonian equivalence of the Clebsch and Shotky-Frahm models we have constructed asymmetric SoV on $\mathcal{C} \times \mathcal{K}$ also for the Shotky-Frahm model [23]. Moreover, we have recently obtained asymmetric SoV for several systems with two and three degrees of freedom that are not equivalent to Clebsch and Shotky-Frahm models. These results will soon be published.

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Thank you for the attention!