Extended Lie point symmetries. The case of inhomogeneous NLS equation

Jan L. Cieśliński

Uniwersytet w Białymstoku, Wydział Fizyki

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General idea

Non-parametric "Lax pair" — $\mathcal{E}'$

compatibility conditions — $\mathcal{E}$

Classical differential geometry

$\mathcal{E}'$ Gauss-Wiengarten eqs.

$\mathcal{E}$ Gauss-Codazzi equations

spectral parameter as a **group parameter** ?

Conjecture:

$$\dim\left(\mathrm{Sym}(\mathcal{E}')\right) < \dim\left(\mathrm{Sym}(\mathcal{E})\right) \quad \Longrightarrow \quad \text{YES.}$$

(non-removable spectral parameter)

Motivation

R.Sasaki (1979): KdV, sine-Gordon, NlS eqs., ...

$$\dim(\mathrm{Sym}(\mathcal{E})) - \dim(\mathrm{Sym}(\mathcal{E}')) = 1$$

spectral parameter is a group parameter (Galilean, Lorentz, scaling,...)

Systematic treatment

J.Krasil'shchik, A.N.Vinogradov (1989)

D.Levi, A.Sym, G.Z.Tu (1989-1990), J.L.C. (1991-1994),

further developments: M.Marvan, J.Krasil'shchik.

alternative technique ("characteristic element'): M.Marvan

Nonlinear Schrödinger equation and its non-parametric “Lax pair”

$$iq_{,t} + q_{,xx} + 2q|q|^2 = 0$$

$$\Psi_{,x} = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix} \Psi , \quad \Psi_{,t} = \begin{pmatrix} i|q|^2 & iq_{,x} \\ i\bar{q}_{,x} & -i|q|^2 \end{pmatrix} \Psi$$

Lie point symmetries				*spectral parameter*
∂_x,	∂_t,	$x\partial_x + 2t\partial_t - q\partial_q$	$iq\partial_q + \boldsymbol{s}\Psi\partial_\Psi$	$2t\partial_x + ixq\partial_q + x\boldsymbol{s}\Psi\partial_\Psi$
$\tilde{x} = x + k$ $\tilde{t} = t$ $\tilde{q} = q$ $\tilde{\Psi} = \Psi$	$\tilde{x} = x$ $\tilde{t} = t + k$ $\tilde{q} = q$ $\tilde{\Psi} = \Psi$	$\tilde{x} = e^k x$ $\tilde{t} = e^{2k} t$ $\tilde{q} = e^{-k} q$ $\tilde{\Psi} = \Psi$	$\tilde{x} = x$ $\tilde{t} = t$ $\tilde{q} = e^{ik} q$ $\tilde{\Psi} = G_k \Psi$	$\tilde{x} = x + 2kt$ $\tilde{t} = t$ $\tilde{q} = e^{ikx + ik^2 t} q$ $\tilde{\Psi} = H_k \Psi$

k – group parameter, $\quad \boldsymbol{s} = \frac{1}{2} i\sigma_3, \quad \sigma_3 = \mathrm{diag}(1, -1),$

$G_k := \exp(k\boldsymbol{s}) \equiv \cos\frac{k}{2} + 2\boldsymbol{s}\sin\frac{k}{2}, \quad H_k := \exp\left((kx + k^2 t)\boldsymbol{s}\right),$

Lie point symmetries of the non-parametric "Lax pair"

Nonlinear eqs. (comp.cond.) $\mathcal{E}$: $\Delta = 0$

$$\mathcal{A} := \{ w : \mathrm{pr}^{(n)} w(\Delta) = 0|_{\Delta=0} \}$$

$$w = \xi(x,t,q)\partial_x + \tau(x,t,q)\partial_t + \eta(x,t,q)\partial_q$$

Non-parametric "Lax pair" $\mathcal{E}'$: $\Delta' = 0$

$$\begin{cases} \Psi_{,x} = U\Psi, \\ \Psi_{,t} = V\Psi \end{cases}$$

$$\mathcal{A}' := \{ w : \mathrm{pr}^{(n)} (w + M(x,t,q)\Psi\partial_\Psi)(\Delta') = 0|_{\Delta=0,\ \Delta'=0} \}$$

Lemma:

$$\left. \begin{cases} D_x M = [U, M] + \mathrm{pr}^{(n)} w(U) + (D_x\xi)U + (D_x\tau)V \\ D_t M = [V, M] + \mathrm{pr}^{(n)} w(V) + (D_t\xi)U + (D_t\tau)V \end{cases} \right|_{\Delta=0}$$

Spectral parameter is a group parameter
in the case of the Nonlinear Schrödinger equation

Non-parametric "Lax pair"

$$\Psi_{,x} = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix} \Psi \, , \qquad \Psi_{,t} = \begin{pmatrix} i|q|^2 & iq_{,x} \\ i\bar{q}_{,x} & -i|q|^2 \end{pmatrix} \Psi$$

vector field (*k* – corresponding group parameter)

$$2t\partial_x + ixq\partial_q + x\boldsymbol{s}\Psi\partial_\Psi \qquad \boldsymbol{s} = \frac{1}{2} i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \lambda := \frac{1}{2}k$$

Lax pair with a spectral parameter

$$\Psi_{,x} = \begin{pmatrix} i\lambda & q \\ -\bar{q} & = i\lambda \end{pmatrix} \Psi \, , \qquad \Psi_{,t} = \begin{pmatrix} -2i\lambda^2 + i|q|^2 & -2\lambda q + iq_{,x} \\ 2\lambda\bar{q} + i\bar{q}_{,x} & 2i\lambda^2 - i|q|^2 \end{pmatrix} \Psi$$

Inhomogeneous Nonlinear Schrödinger system

Given $f = f(x,t)$:

$$\begin{cases} iq_{,t} + (fq)_{,xx} + 2qR = 0 \ , \\ R_{,x} = (f|q|^2)_{,x} + f_{,x} |q|^2 \ . \end{cases} \quad \longleftrightarrow \quad \begin{array}{c} \boldsymbol{S}_{,t} = \boldsymbol{S} \times (f\boldsymbol{S}_{,x})_{,x} \\ \text{Heisenberg ferromagnet} \end{array}$$

Non-parametric "Lax pair" (exists for any $f = f(x,t)$):

$$\Psi_{,x} = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix} \Psi \ , \qquad \Psi_{,t} = \begin{pmatrix} iR & i(fq)_{,x} \\ i(f\bar{q})_{,x} & -iR \end{pmatrix} \Psi$$

Inhomogeneous Nonlinear Schrödinger system

Spectral parameter and Lie point symmetries

Proposition: Non-removable parameter can be inserted by Lie point symmetries if and only if

$$f(x,t) = a(t)(x + c_1 + c_2 A(t)),$$

where $a = a(t)$ is arbitrary and $A(t) := \int_0^t a(\tau) d\tau$

$$\Psi_{,x} = \begin{pmatrix} i\lambda & q \\ -\bar{q} & -i\lambda \end{pmatrix} \Psi \,, \quad \Psi_{,t} = \begin{pmatrix} -2if\lambda^2 + iR & -2\lambda f q + i(fq)_{,x} \\ 2\lambda f\bar{q} + i(f\bar{q})_{,x} & 2if\lambda^2 - iR \end{pmatrix} \Psi$$

$$\lambda = \frac{k}{2 + 2A(t)} \,, \quad k = \text{const}$$

However it is well known that this Lax pair is valid for any $f(x,t) = xa(t) + b(t)$ with any a, b.

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Extended Lie point symmetries

$$\tilde{x} = x + k\,\xi(x, t, q, R) + \ldots$$

$$\tilde{t} = t + k\,\tau(x, t, q, R) + \ldots$$

$$\tilde{q} = q + k\,\eta(x, t, q, R) + \ldots$$

$$\tilde{R} = R + k\,\rho(x, t, q, R) + \ldots$$

$$\tilde{f} = f + k\,\Phi(x, t) + \ldots$$

$$\tilde{\Psi} = \Psi + k\,M(x, t, q, R)\,\Psi + \ldots$$

Non-removable parameter by extended symmetries

Proposition: Non-removable spectral parameter can be inserted by extended Lie point symmetries iff $f(x,t) = xa(t) + b(t)$.

$$\tilde{f} = f + k(\alpha x + \beta) + \ldots$$

Vector field corresponding to the spectral parameter:

$$2(Ax + B)\partial_x + (ix - 2A)q\partial_q + x\boldsymbol{s}\Psi\partial_\Psi + 2aA\partial_a + (4bA - 2aB)\partial_b$$

where $\quad A(t) := \int_0^t a(\tau)d\tau, \quad B(t) := \int_0^t b(\tau)d\tau$,

$$\boldsymbol{s} = \frac{1}{2}i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

An explicit group of nonlocal transformations

$$T_k a = \frac{a}{(1 - kA)^2} \,,$$

$$T_k b = \frac{b}{(1 - kA)^4} - \frac{2ka}{(1 - kA)^2} \int_0^t \frac{b(\tau)\, d\tau}{(1 - kA(\tau))^3} \,,$$

$$T_k A = \frac{A}{1 - kA}$$

Proposition: If $k < ||a||^{-1}$, then $(a, b) \to (T_k a, T_k b)$ is a one-parameter group of transformations mapping $\mathcal{L}^1(\mathbb{R}) \times \mathcal{L}^1(\mathbb{R})$ into $\mathcal{L}^1(\mathbb{R}) \times \mathcal{L}^1(\mathbb{R})$.

$$||a|| := \int_{-\infty}^{\infty} a(t) dt, \quad A(t) = \int_0^t a(\tau) d\tau, \quad B(t) = \int_0^t b(\tau) d\tau$$

Remarks and conclusions

- Extending the space of variables by adding A, we have a local transformation: $(a, A) \to (T_k a, T_k A)$.
- Open problem: to find a similar extension for the space (a, b).
- Extended Lie point symmetries are closely related to equivalence transformations (e.g., L.V. Ovsiannikov, R.Popovych)
- Extended Lie point symmetries can be used to insert the spectral parameter (another case: Bianchi system for hyperbolic surfaces with Gaussian curvature $K = \rho^{-2}, \ \rho_{,xy} = 0$). More examples?
- Find more cases of non-trivial nonlocal transformations derived from extended Lie point symmetries.
- Is it possible to use full group of symmetries of the Lax pair instead of gauge symmetries?

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Thank you for attention

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