# Invertible linear ordinary differential operators and their generalizations

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# Posing the problem

Example 1

$$\Delta = \begin{pmatrix} 1 & -a & 0 \\ -c & ca+1 & -b \\ dc & -dca-d & db+1 \end{pmatrix},$$

$$\Delta^{-1} = \begin{pmatrix} ac+1 & abd+a & ab \\ c & bd+1 & b \\ 0 & d & 1 \end{pmatrix}$$

where

$$a,b,c,d\colon\ f(t) \mapsto \sum_l g_l(t)\frac{d^l f(t)}{dt^l}$$

# Posing the problem

Generalizations:

- Invertible linear difference operators

$$a, b, c, d\colon\ f(t) \mapsto \sum_l g_l(t) f(t - l\tau), \quad g_l(t+\tau) = g_l(t).$$

- Linearizations of invertible nonlinear ordinary differential operators

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# Outline

Description of invertible linear ordinary differential operators

Unicellular operators

Generalizations of invertible linear ordinary differential operators

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# References

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beamericonarticleshaded V C.
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# Linear differential operators

Consider

- a one-dimensional manifold $M$, $A = C^\infty(M)$;
- two vector bundles $\xi, \zeta$, $P = C^\infty(\xi), Q = C^\infty(\zeta)$;
- the mapping $\delta_a : \mathrm{Hom}_{\mathbb{R}}(Q, P) \longrightarrow \mathrm{Hom}_{\mathbb{R}}(Q, P)$, $\delta_a(f)(q) = f(aq) - af(q)$, $a \in A, q \in Q, f \in \mathrm{Hom}_{\mathbb{R}}(Q, P)$.

Define linear differential operators of order $\leq k$

$$\mathrm{Diff}_k^+(Q, P) = \{\nabla \in \mathrm{Hom}_{\mathbb{R}}(Q, P) \mid (\delta_{a_0} \circ \ldots \circ \delta_{a_k})(\nabla) = 0 \ \forall a_0, \ldots, a_k \in A\}$$

and the multiplication $(a^+\Delta)(p) = \Delta(ap)$ in $\mathrm{Diff}_k^+(Q, P)$, $p \in Q$.

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# Sequences

An operator $\Delta \in \mathrm{Diff}_L^+(Q,P)$ is invertible if
$\exists \Delta^{-1} \in \mathrm{Diff}_K(P,Q) \; : \Delta^{-1} \circ \Delta = \mathrm{id}_Q,\ \Delta \circ \Delta^{-1} = \mathrm{id}_P$.
Denote $G_l = \mathrm{Diff}_l^+(A,Q)$ and

$$F_k = \{\Delta \circ \nabla \mid \nabla \in \mathrm{Diff}_k^+(A,P)\}, \quad k,l \geq 0.$$

A point $t \in M$ is d-regular for $\Delta$ if in a neighborhood of $t$
$\dim(F_k \cap G_l) = \mathrm{const}, l = 0,1,\ldots,L, k = 0,1,\ldots,K$.
Consider the sequences

$$d_{k,l} = \dim(F_k \cap G_l), \quad k,l \geq 0,$$

$\varkappa_{k,l} = d_{k,l} - d_{k-1,l} - d_{k,l-1} + d_{k-1,l-1}$,
$d_{k,l} = \varkappa_{k,l} = 0$ if $k < 0$ or $l < 0$,
The sequence $\{\rho_{k,l} = \varkappa_{k,l} - \varkappa_{k-1,l-1}\}$ is the table of $\Delta$.

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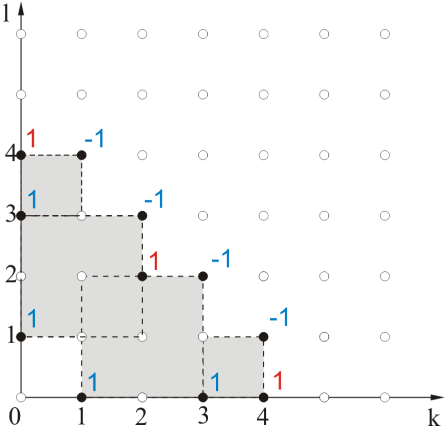
# Example 3

The table of the operator from Example 1, when $\operatorname{ord} a = \operatorname{ord} d = 1$ and $\operatorname{ord} b = \operatorname{ord} c = 2$

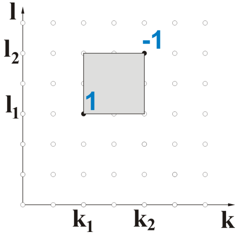


# Schemes of squares

Consider a finite set $\mathcal{S}$ of squares in the first quarter of the $(k, l)$-plane such that

- corners of the squares have integer coordinates;
- the sides of the squares are parallel to the coordinate axes.

Define the table $\{\rho^{s}_{k,l}\}$ of a square $s \in \mathcal{S}$



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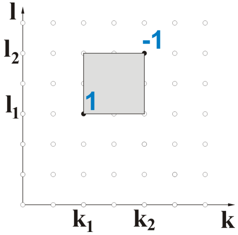

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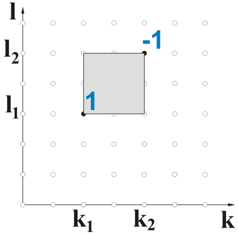

# Schemes of squares

Suppose there exist $m \in \mathbb{N}$ and a sequence $\{a_{k,l} \geq 0\}$ such that the sequence $\{\rho_{k,l} = \sum_{s \in \mathcal{S}} \rho^s_{k,l} + a_{k,l}\}_{k,l \geq 0}$ satisfies

$$\sum_{i=0}^{\infty} \rho_{i,l} = 0, \qquad \sum_{i=0}^{\infty} \rho_{i,0} = m, \qquad \sum_{i=0}^{k-1} \rho_{i,l} \geq z_{k,l},$$

$$\sum_{j=0}^{\infty} \rho_{k,j} = 0, \qquad \sum_{j=0}^{\infty} \rho_{0,j} = m, \qquad \sum_{j=0}^{l-1} \rho_{k,j} \geq z_{k,l},$$

for all $k, l \in \mathbb{N}$ (here $z_{k,l}$ is the number of the squares with the upper right corner at the point $(k, l)$). Then the set $\mathcal{S}$ is a d-scheme, the sequence $\{\rho_{k,l}\}$ is an $m$-table of the d-scheme.

# Schemes of squares

Theorem 1.

(a) The table of any invertible operator $\Delta\colon A^m \to A^m$ over a neighborhood of its d-regular point coincides with an $m$–table of some d-scheme.

(b) If a d-scheme has an $m$–table, then there is an invertible operator of $A^m$ to $A^m$ such that its table coincides with the given $m$-table.

# Constructing invertible operators

Consider

- a d-scheme with $\lambda$ squares;
- its $m$-table $\{\rho_{k,l}\}$;
- the corresponding sequence $\{a_{k,l}\}$;
- two finite sets $Z$ and $B$ consisting of $\lambda$ and $\lambda + m$ elements respectively.

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# Constructing invertible operators

Denote by $Z_{k,l}$ (by $B_{k,l}$) the set of all elements of $Z$ (respectively, $B$) with coordinates $(k,l)$.

Let the map $\psi\colon Z \to B$ takes each upper right corner of a square to its lower left corner.

Define the relation on $\mathbb{Z}^2$:

$$(n_1,n_2) \prec (m_1,m_2) \iff n_1 \le m_1,\ n_2 \le m_2,\ n_1+n_2 < m_1+m_2.$$

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# Constructing invertible operators

Consider $\square_{\xi,\beta} \in \mathrm{Diff}_l^+(A, A), \xi \in Z, \beta \in B$ such that

(1°) $\beta \prec \xi \;\Rightarrow\; \operatorname{ord} \square_{\xi,\beta} \le \min(\xi_1 - \beta_1, \xi_2 - \beta_2)$,
$\beta \not\prec \xi \;\Rightarrow\; \square_{\xi,\beta} = 0$;

(2°) $\beta = \psi(\xi) \;\Rightarrow\; \operatorname{ord} \square_{\xi,\beta} = \xi_1 - \beta_1$,
$\beta \ne \psi(\xi), \xi_1 - \beta_1 = \xi_2 - \beta_2 \;\Rightarrow\; \operatorname{ord} \square_{\xi,\beta} < \xi_1 - \beta_1$;

(3°) the matrix $(\square_{\xi,\beta}), \xi \in Z_{k,j}, j > 0, \beta \in B_{k,i}, i \ge 0$ is nonsingular at any point $t \in M$ for all $k > 0$;

(4°) the matrix $(\square_{\xi,\beta}),\ \xi \in Z_{j,l},\ j > 0,\ \beta \in B_{i,l},\ i \ge$ is nonsingular at any point $t \in M$ for all $l > 0$.

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# Constructing invertible operators

Denote
by $\beta_1^1, \dots, \beta_m^1$ elements of $B$ with the zero first coordinate,
by $\beta_1^2, \dots, \beta_m^2$ elements of $B$ with the zero second coordinate.

To each element $\beta \in B$ assign a differential operator $\nabla_\beta \in \mathrm{Diff}_*^+(A, Q)$ such that

(A) the elements $\nabla_{\beta_1^2}, \dots, \nabla_{\beta_m^2}$ form a basis for the module $\mathrm{Diff}_0^+(A, Q)$;

(B) for any $\xi \in Z_{k,l}, k, l > 0$ we have

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# Constructing invertible operators

From (1°)-(4°), (A) and (B) it follows that

$$\nabla_{\beta_i^1} = \sum_{j=1}^{m} \nabla_{\beta_j^2} \circ \Delta_{ji}, \qquad i = 1, \dots, m, \qquad \Delta_{ji} \in \mathrm{Diff}_l^+(A, A).$$

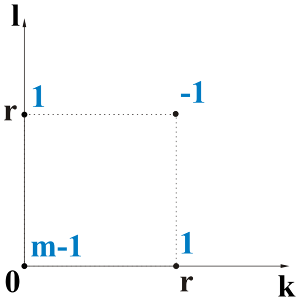
Theorem 2.

(a) For any d-scheme, for its $m$–table, for operators $\Box_{\xi,\beta}, \xi \in Z, \beta \in B$ satisfying to conditions (1°)–(4°), and for bases for modules $P$ and $Q$ the matrix $(\Delta_{ji})$ defines an invertible operator of $P$ to $Q$.

(b) For any invertible differential operator $\Delta\colon\ P \to Q$ there exist scalar linear differential operators $\Box_{\xi,\beta}, \xi \in Z, \beta \in B$ satisfying to conditions (1°)–(4°) and bases for modules $P$ and $Q$ such that over a neighborhood of a generic point $t \in M$ the operator $\Delta$ has the matrix $(\Delta_{ji})$.

# Unicellular operators

An invertible operator is unicellular if its d-scheme consists of a single square



# Unicellular operators

Theorem 3. For any unicellular operator $\Delta\colon P \to Q$ there exist bases of modules $P$ and $Q$ in which the matrix of $\Delta$ has the block form

$$\begin{pmatrix} 1 & \square \\ 0 & E \end{pmatrix},$$

where $\square$ is a row of scalar operators, $0$ is the column of zero operators, $E$ is a unit matrix. The maximal order of operators from the row $\square$ coincides with the size of the square of the d-scheme for $\Delta$.

# Representation of invertible operators as compositions of unicellular operators

Theorem 4. Any invertible differential operator $\Delta$ in the neighborhood of a generic point $t \in M$ is a composition of unicellular operators

$$\Delta = U_1 \circ \ldots \circ U_\lambda,$$

where the minimal $\lambda$ is equal to the number of squares of the d-scheme for the operator $\Delta$.

# Example 4.

$$\begin{pmatrix} D & D^2 - t & D^3 - tD + t + 1 \\ 1 & D & D^2 \\ D^2 & D^3 - tD & D^4 - tD^2 + (t+1)D \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 0 & -(t+1) \\ 1 & 0 & 0 \\ 0 & 1 & -(t+1)D \end{pmatrix} \circ \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{t+1}D & 1 & 0 \\ -\frac{1}{t+1}D & 0 & 1 \end{pmatrix} \circ$$

$$\begin{pmatrix} 1 & D & 0 \\ 0 & \frac{1}{t+1} & 0 \\ 0 & \frac{t}{t+1} & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & D \\ 0 & 0 & -1 \end{pmatrix}$$

for $t \neq -1$.

# Generalizations

Consider

- embedded finite-dimensional vector bundles $\xi_0 \subset \ldots \subset \xi_l \subset \xi_{l+1} \subset \ldots$ on $M$;
- embedded submodules $G_0 \subset \ldots \subset G_l \subset G_{l+1} \subset \ldots$, $G_l = C^\infty(\xi_l)$;
- a mapping $D : G \longrightarrow G, G = \cup_{l=0}^{\infty} G_l$, such that $D(G_l) \subset G_{l+1}, l \geq 0$, and

$$D : \frac{G_l}{G_{l-1}} \longrightarrow \frac{G_{l+1}}{G_l}, \qquad l \geq 0, \quad G_{-1} = 0$$

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# Examples of $D$-generated modules:

- $G_l = \mathrm{Diff}_l^+(A, P)$ with generating operator $D\colon\ \beta \mapsto \beta \circ \dfrac{d}{dt}$;
- $G_l = \mathrm{diff}_l^+(A, P) = \{\sum_{i=0}^{l} a_i\delta^i \mid a_i \in P\}$, where $\delta\colon\ f(t) \mapsto f(t-\tau)$ is the delay operator, $D\colon\ \beta \mapsto \delta \circ \beta \circ (\mathrm{id}_A - \delta^{-1})$ is a generating operator;
- the module of Cartan forms of a jet space $J^\infty(\mathbb{R}, \mathbb{R}^m)$:

$$G_l = \mathcal{C}\Lambda_l^1 = \mathrm{span}\{\, dx_i^j - x_{i+1}^j dt \mid i = 0, 1, \ldots, l, j = 1, \ldots, m\,\}.$$

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# $D$-operators

Consider

- two $D$-generated modules $G = \cup_{l=0}^{\infty} G_l, F = \cup_{k=0}^{\infty} F_k$;
- their generating operators $D_G, D_F$.

A homomorphism $\Delta\colon\ G \to F$ is a $D$-operator of order $\leq L$ if $\Delta \circ D_F = D_G \circ \Delta$, $\Delta(F_k) \subset G_{k+L}$ for all $k \geq 0$.

A D-operator $\Delta\colon\ G \to F$ is invertible if there exists a D-operator $\Delta^{-1}\colon\ F \to G$ such that $\Delta^{-1} \circ \Delta = \mathrm{id}_G$, $\Delta \circ \Delta^{-1} = \mathrm{id}_F$.

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# Examples of $D$-operators

- $\beta \in \mathrm{Diff}_k^+(A, P) \mapsto \Delta \circ \beta \in \mathrm{Diff}_{k+l}^+(A, Q)$, where $\Delta \in \mathrm{Diff}_l^+(P, Q)$;
- $\beta \in \mathrm{diff}_k^+(A, P) \mapsto \Delta \circ \beta \in \mathrm{diff}_{k+l}^+(A, Q)$, where $\Delta \in \mathrm{diff}_l^+(P, Q)$, the elements of the algebra $A$ and the coefficients of $\Delta$ are periodic functions with period $\tau$ (a delay);
- $\omega \in \mathcal{C}\Lambda_k^1 \mapsto (F^\infty)^*(\omega) \in \mathcal{C}\Lambda_{k+L}^1$, where $F^\infty\colon\ J^\infty(\mathbb{R}, \mathbb{R}^m) \to J^\infty(\mathbb{R}, \mathbb{R}^m)$ is the infinite prolongation of a nonlinear ordinary differential operator $F\colon\ J^L(\mathbb{R}, \mathbb{R}^m) \to J^0(\mathbb{R}, \mathbb{R}^m)$ .

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## THANK YOU

FOR YOUR ATTENTION