Nontrivial 1-parameter families of zero-curvature representations obtained via symmetry actions

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An outline

- On the problem of constructing 1-parameter families of zero-curvature representations (ZCRs)
- Preliminaries
- Main results and some examples

A typical property

A typical property of integrable PDEs is that of admitting ZCRs.

Particularly important are parameter dependent ZCRs α_λ
(λ *is a spectral parameter*).

The presence of a parameter is crucial from several point of view:

- search of exact solutions;
- existence of parametric Bäcklund transformations;
- existence of hierarchies of conservation laws.

Nontriviality of a parameter

A cohomological obstruction to triviality:

Michal Marvan, On the horizontal gauge cohomology and non-removability of the spectral parameter, Acta Appl. Math. 72 (2002) 51-65

Using horizontal gauge cohomology:

- we know when a parameter is trivial;
- we know how to remove a parameter.

Embedding α in a parameter dependent family α_λ

Often we only know a nonparametric ZCR α.

Can we insert a nontrivial parameter?

- Many attempts using classical symmetries:
  - Lund-Regge (1976);
  - Sasaki (1979);
  - Levi-Sym-Tu (1990);
  - Cieslinski-Goldstein-Sym (1994).
- A cohomology-based method:
  - Marvan (2010)

About the symmetry-method

Main problem with symmetry-method:

- Often one obtains a trivial 1-parameter family of ZCRs α_λ.
  How to recognize “good” symmetries? (if any)

An unproved conjecture (Cieslinski-Goldstein-Sym 1994):

"Good" symmetries can be identified by a mismatch of the symmetry algebras of $\mathscr{E}$ and that of the covering defined by the ZCR.

Cohomology-based infinitesimal criterion

We found an infinitesimal criterion which allows one to solve the problem of identifying "good" symmetries, if any.

Relatively to a given ZCR α:

We are able to distinguish between "good" and "bad" symmetries.

In particular we found that "bad" symmetries form a sub-algebra of the Lie algebra of classical symmetries, which is invariantly associated to any ZCR α.

Notations

$$\{\mathbf{F}(\mathbf{x},\mathbf{u}_\sigma)=0,\quad |\sigma|\le k\} =: \mathscr{E}\subset J^k(\pi)$$

$$\pi: \begin{array}{ccc} E & \to & M \\ (\mathbf{x},\mathbf{u}) & \mapsto & (\mathbf{x}) \end{array}$$

$$M\leftarrow \ldots \leftarrow J^k(\pi)\leftarrow \ldots \leftarrow J^l(\pi)\leftarrow \ldots \leftarrow J^\infty(\pi)$$

$$\pi_{\infty,k}: J^\infty(\pi)\to J^k(\pi),$$

$$\pi_\infty: J^\infty(\pi)\to M$$

$$\mathscr{E}\leftarrow \mathscr{E}^{(1)}\leftarrow \mathscr{E}^{(2)}\leftarrow \ldots \leftarrow \mathscr{E}^{(\infty)}\subset J^\infty(\pi)$$

$\mathscr{E}$ formally integrable

$$\mathscr{E}^{(\infty)}=\{\mathbf{D}_\tau(\mathbf{F})=0:\ |\tau|\ge 0\}$$

$$\ldots\leftarrow \mathscr{C}^k(\mathscr{E})\leftarrow \mathscr{C}^{k+1}(\mathscr{E}^{(1)})\leftarrow \ldots \leftarrow \mathscr{C}(\mathscr{E})$$

$\Lambda^*(\mathscr{E})$ forms over $\mathscr{E}^{(\infty)}$

Bicomplex structure of $\Lambda^*(\mathscr{E})$

$\mathscr{T}(\mathscr{E}) = \mathscr{V}(\mathscr{E}) \oplus \mathscr{C}(\mathscr{E})$ $\qquad$ $\bar{\pi}_\infty := \pi_\infty|_{\mathscr{E}^{(\infty)}} : \mathscr{E}^{(\infty)} \to M$

$\Lambda^1(\mathscr{E}) = \Lambda^{(1,0)}(\mathscr{E}) \oplus \Lambda^{(0,1)}(\mathscr{E})$

$\Lambda^{(1,0)}(\mathscr{E}) := Ann(\mathscr{V}(\mathscr{E}))$ $\qquad$ (loc. gen. by $\{dx^i\}$)

$\Lambda^{(0,1)}(\mathscr{E}) := Ann(\mathscr{C}(\mathscr{E}))$ $\qquad$ (loc. gen. by $\left\{\bar{\theta}^j_\sigma := \theta^j|_{\mathscr{E}^{(\infty)}}\right\}$)

$\Lambda^{(p,0)}(\mathscr{E})$ $\qquad$ (p-horizontal)

$\Lambda^{(0,q)}(\mathscr{E})$ $\qquad$ (q-vertical)

$\Lambda^r(\mathscr{E}) = \bigoplus_{p+q=r} \Lambda^{(p,q)}(\mathscr{E})$

Bicomplex structure of $\Lambda^*(\mathscr{E})$

$$\boxed{\bar{d} := d|_{\mathscr{E}^{(\infty)}} = \bar{d}_H + \bar{d}_V}, \qquad \boxed{\bar{d}_H^2 = \bar{d}_V^2 = 0}, \qquad \bar{d}_H \circ \bar{d}_V = -\bar{d}_V \circ \bar{d}_H$$

$$\dots \longrightarrow \Lambda^{(p,q)}(\mathscr{E}) \xrightarrow{\bar{d}_H} \Lambda^{(p+1,q)}(\mathscr{E}) \longrightarrow \dots \qquad \text{(horizontal complex)}$$

$$\dots \longrightarrow \Lambda^{(p,q)}(\mathscr{E}) \xrightarrow{\bar{d}_V} \Lambda^{(p,q+1)}(\mathscr{E}) \longrightarrow \dots \qquad \text{(vertical complex)}$$

The action on $\Lambda^*(\mathscr{E})$ is completely determined by

$$\bar{d}_H(\omega_1 \wedge \omega_2) = \bar{d}_H(\omega_1) \wedge \omega_2 + (-1)^{\omega_1} \omega_1 \wedge \bar{d}_H(\omega_2),$$
$$\bar{d}_V(\omega_1 \wedge \omega_2) = \bar{d}_V(\omega_1) \wedge \omega_2 + (-1)^{\omega_1} \omega_1 \wedge \bar{d}_V(\omega_2).$$

and by the action on functions

$$\bar{d}_H f := \bar{D}_i(f) dx^i, \qquad \bar{d}_V f := \frac{\partial f}{\partial u^j_\sigma} \bar{\theta}^j_\sigma$$

with $\bar{D}_i$ denoting the total derivatives restricted to $\mathscr{E}^{(\infty)}$.

$\mathfrak{g}$-valued horizontal forms

$\bar{\Lambda}^q(\mathscr{E}) := \Lambda^{(q,0)}(\mathscr{E})$

$\mathfrak{g}$ Lie algebra of a matrix Lie group G, $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{R})$ or $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{C})$

$\mathfrak{g} \otimes \bar{\Lambda}^*(\mathscr{E}) = \bigoplus_q \mathfrak{g} \otimes \bar{\Lambda}^q(\mathscr{E})$ ($\mathfrak{g}$-valued horizontal forms)

$$[A_1 \omega_1, A_2 \omega_2] := [A_1, A_2]\, \omega_1 \wedge \omega_2,$$

[,] can be extended by linearity to $\mathfrak{g} \otimes \bar{\Lambda}^*(\mathscr{E})$

$$\begin{cases} [\rho, \sigma] = -(-1)^{rs}[\sigma, \rho] \\ (-1)^{rt}[\rho, [\sigma, \tau]] + (-1)^{sr}[\sigma, [\tau, \rho]] + (-1)^{ts}[\tau, [\rho, \sigma]] = 0 \\ \bar{d}_H[\rho, \sigma] = [\bar{d}_H \rho, \sigma] + (-1)^r [\rho, \bar{d}_H \sigma] \end{cases}$$

$\rho \in \mathfrak{g} \otimes \bar{\Lambda}^r(\mathscr{E})$, $\sigma \in \mathfrak{g} \otimes \bar{\Lambda}^s(\mathscr{E})$, $\tau \in \mathfrak{g} \otimes \bar{\Lambda}^t(\mathscr{E})$

ZCRs

Definition (ZCRs)

A $\mathfrak{g}$-valued zero curvature representation (ZCR) for an equation $\mathscr{E}$ is a 1-form $\alpha \in \mathfrak{g} \otimes \bar{\Lambda}^1(\mathscr{E})$ such that

$$\bar{d}_H \alpha = \frac{1}{2}[\alpha, \alpha].$$

Gauge transformations and removable parameters

Any G-valued smooth function S on $\mathscr{E}^{(\infty)}$, defines a gauge transformation:

$$\alpha \quad (\mathrm{ZCR}) \quad \mapsto \quad \alpha^S := \bar{d}_H S \cdot S^{-1} + S \cdot \alpha \cdot S^{-1} \quad (\mathrm{ZCR})$$

$$\alpha \quad \rightsquigarrow \quad \alpha_\lambda := \alpha^{M_\lambda},\ \lambda \in]a,b[\quad \rightsquigarrow \quad (\alpha_\lambda)^{M_\lambda^{-1}} = \left(\alpha^{M_\lambda}\right)^{M_\lambda^{-1}} = \alpha$$

$$\left(\alpha^{S_1}\right)^{S_2} = \alpha^{S_2 S_1} \quad \Rightarrow \quad (\alpha_\lambda)^{\left(M_{\lambda_0} M_\lambda^{-1}\right)} = \left(\alpha_\lambda^{M_\lambda^{-1}}\right)^{M_{\lambda_0}} = \alpha^{M_{\lambda_0}} = \alpha_{\lambda_0},$$

Definition (removable parameters)

Let α_λ be a 1-parameter family of $\mathfrak{g}$-valued ZCRs of $\mathscr{E}$, with $\lambda \in]a,b[\subset \mathbb{R}$.

$$\left[\ \lambda \text{ is removable from } \alpha_\lambda\ \right] \iff \left[\ \forall \lambda_0 \in]a,b[\text{ exists a G -valued smooth function } S_\lambda \text{ such that } S_{\lambda_0} = \mathbb{I} \text{ (identity) and } \alpha_{\lambda_0} = \alpha_\lambda^{S_\lambda^{-1}}.\ \right].$$

If λ is not removable, then α_λ is called **nontrivial**.

Marvan's gauge complex of a $\mathfrak{g}$-valued ZCR

Let α be a ZCR of $\mathscr{E}$

$$\bar{d}_H\alpha = \frac{1}{2}[\alpha,\alpha]. \tag{1}$$

Marvan's horizontal gauge complex of a ZCR α:

$$0 \to \mathfrak{g}\otimes\bar{\Lambda}^0(\mathscr{E}) \xrightarrow{\bar{\partial}_\alpha} \mathfrak{g}\otimes\bar{\Lambda}^1(\mathscr{E}) \xrightarrow{\bar{\partial}_\alpha} \mathfrak{g}\otimes\bar{\Lambda}^2(\mathscr{E}) \longrightarrow \dots \longrightarrow \mathfrak{g}\otimes\bar{\Lambda}^n(\mathscr{E}) \longrightarrow 0$$

$$\bar{\partial}_\alpha := \bar{d}_H - ad_\alpha : \quad \begin{array}{ccc} \mathfrak{g}\otimes\bar{\Lambda}^p(\mathscr{E}) & \longrightarrow & \mathfrak{g}\otimes\bar{\Lambda}^{p+1}(\mathscr{E}) \\ \omega & \mapsto & \bar{d}_H\omega - [\alpha,\omega] \end{array}$$

In view of (1) one has:

$$\bar{\partial}_\alpha^2 = 0.$$

The obstruction to the removability of a parameter

Theorem. (Marvan 2002)

If α_λ is a 1-parameter family of $\mathfrak{g}$-valued ZCRs for $\mathscr{E}$, with $\lambda \in]a, b[$, then:

1. $\dot{\alpha}_\lambda := \frac{d}{d\lambda}\alpha_\lambda$ is a 1-cocycle with respect to $\bar{\partial}_{\alpha_\lambda}$, i.e., $\bar{\partial}_{\alpha_\lambda}(\dot{\alpha}_\lambda) = 0$;
2. the parameter λ is removable if, and only if, there exists a solution $K \in \mathfrak{g} \otimes \bar{\Lambda}^0(\mathscr{E})$ of the equation

$$\dot{\alpha}_\lambda = \bar{\partial}_{\alpha_\lambda}(K). \tag{2}$$

For any solution K of (2) and $\lambda_0 \in]a, b[$, the G-valued matrix S_λ such that $\alpha_{\lambda_0} = \alpha_\lambda^{S_\lambda^{-1}}$ is the solution of the Cauchy problem

$$\begin{cases} \dot{S}_\lambda = K\, S_\lambda, \\ S_{\lambda_0} = \mathbb{I}. \end{cases}$$

The first gauge cohomology group $\bar{H}^1_\alpha(\mathscr{E}, \mathfrak{g})$ is the obstruction to removability of a parameter from a ZCR.

Action of symmetries on ZCRs

$F : J^{(\infty)}(\pi) \to J^{(\infty)}(\pi)$ $\qquad F = A^{(\infty)}$ $\quad$ (contact transf.)

$$\begin{array}{ccc} \mathfrak{g}\otimes\Lambda^{(a+1,b)}(\pi) & \overset{\pi^{(a+\mathbf{1},b)}\circ F^*}{\longrightarrow} & \mathfrak{g}\otimes\Lambda^{(a+1,b)}(\pi) \\ d_H\uparrow & \circlearrowleft & \uparrow d_H \\ \mathfrak{g}\otimes\Lambda^{(a,b)}(\pi) & \underset{\pi^{(a,b)}\circ F^*}{\longrightarrow} & \mathfrak{g}\otimes\Lambda^{(a,b)}(\pi) \end{array}$$

Using (a,b)-projections: $\qquad \pi^{(a,b)} : \mathfrak{g}\otimes\Lambda^*(\pi) \to \mathfrak{g}\otimes\Lambda^{(a,b)}(\pi)$

$\mathscr{E},\mathscr{Y} \subset J^k(\pi)$ $\quad$ (form. int.), $\qquad A(\mathscr{E}) = \mathscr{Y}$, $\qquad \bar{F} := F|_{\mathscr{E}^{(\infty)}}$

$$\begin{array}{ccc} \mathfrak{g}\otimes\Lambda^{(a+1,b)}(\mathscr{Y}) & \overset{\bar{\pi}_{\mathscr{E}}^{(a+\mathbf{1},b)}\circ \bar{F}^*}{\longrightarrow} & \mathfrak{g}\otimes\Lambda^{(a+1,b)}(\mathscr{E}) \\ \bar{d}_{H,\mathscr{Y}}\uparrow & \circlearrowleft & \uparrow \bar{d}_{H,\mathscr{E}} \\ \mathfrak{g}\otimes\Lambda^{(a,b)}(\mathscr{Y}) & \underset{\bar{\pi}_{\mathscr{E}}^{(a,b)}\circ \bar{F}^*}{\longrightarrow} & \mathfrak{g}\otimes\Lambda^{(a,b)}(\mathscr{E}) \end{array}$$

Using (a,b)-projections: $\qquad \bar{\pi}_{\mathscr{E}}^{(a,b)} : \mathfrak{g}\otimes\Lambda^*(\mathscr{E}) \to \mathfrak{g}\otimes\Lambda^{(a,b)}(\mathscr{E})$

Action of symmetries on ZCRs

Proposition.

If F is the infinite prolongation of a point or contact transformation, which maps a formally integrable equation $\mathscr{E} \subset J^k(\pi)$ to a formally integrable equation $\mathscr{Y} \subset J^k(\pi)$, then

$$\bar{F}^{\#} := \bar{\pi}_{\mathscr{E}}^{(1,0)} \circ \bar{F}^{*} : \quad \begin{array}{ccc} \mathfrak{g} \otimes \bar{\Lambda}^1(\mathscr{Y}) & \rightarrow & \mathfrak{g} \otimes \bar{\Lambda}^1(\mathscr{E}) \\ \beta & \mapsto & \alpha = \bar{F}^{\#}(\beta) \end{array}$$

maps ZCRs of $\mathscr{Y}$ to ZCRs of $\mathscr{E}$.
In particular, if $\bar{F}$ is the restriction to $\mathscr{E}^{(\infty)}$ of a classical symmetry of a formally integrable equation $\mathscr{E}$, then $\bar{F}^{\sharp}$ maps any ZCR α of $\mathscr{E}$ to a ZCR $\bar{F}^{\#}(\alpha)$.

Projected Lie derivatives

Let $Z \in \mathscr{D}(J^\infty(\pi))$ and $\omega \in \mathfrak{g} \otimes \Lambda^{(p,q)}(\pi)$:

$$Z(\omega) := \pi^{(p,q)}(L_Z(\omega)) \qquad (\pi^{(p,q)}\text{-projected Lie derivative}).$$

If Z is a generalized symmetry of $\mathscr{E}$ and $\omega \in \mathfrak{g} \otimes \Lambda^{(p,q)}(\mathscr{E})$:

$$\bar{Z}(\omega) := \bar{\pi}^{(p,q)}(L_{\bar{Z}}(\omega)) \qquad (\overline{\pi}^{(p,q)}\text{-projected Lie derivative}),$$

with $\bar{Z} := Z|_{\mathscr{E}^{(\infty)}}$.

Commutation formula for $\bar{Z}$ and $\bar{d}_H$

If Z is a generalized symmetry of $\mathscr{E}$, for any $\omega \in \mathfrak{g} \otimes \Lambda^{(p,q)}(\mathscr{E})$ one has:

$$\bar{Z}(\bar{d}_H(\omega)) = \bar{d}_H(\bar{Z}(\omega)).$$

Then for a ZCR α

$$\bar{d}_H \alpha - \frac{1}{2}[\alpha, \alpha] = 0,$$

one gets

$$0 = \bar{Z}\left(\bar{d}_H \alpha - \tfrac{1}{2}[\alpha, \alpha]\right) = \bar{d}_H\left(\bar{Z}(\alpha)\right) - \tfrac{1}{2}[\bar{Z}(\alpha), \alpha] - \tfrac{1}{2}[\alpha, \bar{Z}(\alpha)] = \bar{d}_H\left(\bar{Z}(\alpha)\right) - [\alpha, \bar{Z}(\alpha)] = \bar{\partial}_\alpha \bar{Z}(\alpha).$$

$\bar{Z}(\alpha)$ is closed w.r.t. $\bar{\partial}_\alpha$

Proposition.

$\bar{Z}(\alpha)$ is a 1-cocycle with respect to $\bar{\partial}_\alpha$, i.e.,

$$\bar{\partial}_\alpha \bar{Z}(\alpha) = 0, \tag{3}$$

for any (generalized) symmetry Z of $\mathscr{E}$ and any ZCR $\alpha \in \mathfrak{g} \otimes \bar{\Lambda}^1(\mathscr{E})$.

Construction of 1-parameter family of ZCRs α_λ

For a classical symmetry Z of $\mathscr{E}$, with associated flow $\{A_\lambda\}$:

$$\alpha_\lambda := A_\lambda^{\#}(\alpha) \qquad \text{(1 -parameter family of ZCRs).}$$

Infinitesimal obstruction to removability

Theorem (infinitesimal obstruction to removability)

The parameter λ in $\alpha_\lambda = A^{\#}_\lambda(\alpha)$ is removable if, and only if, $\bar{Z}(\alpha)$ is a coboundary with respect to $\bar{\partial}_\alpha$, i.e.,

$$\bar{Z}(\alpha) = \bar{\partial}_\alpha K,$$

for some $\mathfrak{g}$-valued smooth function K on $\mathscr{E}^{(\infty)}$.

Definition (gauge-like symmetries)

Z is a generalized (or classical) **gauge-like** symmetry for the ZCR $\alpha \in \mathfrak{g} \otimes \bar{\Lambda}^1(\mathscr{E})$ iff

$$\bar{Z}(\alpha) = \bar{\partial}_\alpha K,$$

for some $\mathfrak{g}$-valued smooth function K on $\mathscr{E}^{(\infty)}$.

Gauge-like symmetries form a Lie algebra

Proposition (algebra of gauge-like symmetries)

If Z_1 and Z_2 are gauge-like for α with

$$\bar{Z}_1(\alpha) = \bar{\partial}_\alpha K_1, \quad \bar{Z}_2(\alpha) = \bar{\partial}_\alpha K_2, \tag{4}$$

then

$$\overline{[Z_1, Z_2]}(\alpha) = \bar{\partial}_\alpha (K_{12})$$

with

$$K_{12} = \bar{Z}_1(K_2) - \bar{Z}_2(K_1) - [K_1, K_2].$$

Therefore, gauge-like symmetries of a ZCR α form a Lie sub-algebra of the Lie algebra of generalized symmetries of $\mathscr{E}$.

Modulo contact transformations, such an algebra is invariantly associated to the ZCR α.

KDV: $u_t = u_{xxx} + 6uu_x$

Classical symmetries are generated by "prolongations" of

$$Y_1 = \partial_x, \quad Y_2 = t\partial_x + \frac{1}{6}\partial_u, \quad Y_3 = \partial_t, \quad Y_4 = 3t\partial_t + x\partial_x - 2u\partial_u$$

A ZCR is

$$\alpha = \begin{pmatrix} 0 & u-1 \\ -1 & 0 \end{pmatrix} \mathrm{dx} + \begin{pmatrix} u_x & u_{xx} + 2u + 2u^2 - 4 \\ -4 - 2u & -u_x \end{pmatrix} \mathrm{dt}.$$

The prolonged flow of Y_4, up to order 2, is such that

$$(t, x, u, u_x, u_{xx}) \mapsto \left(e^{3\lambda} t,\ e^{\lambda} x,\ e^{-2\lambda} u,\ e^{-3\lambda} u_x,\ e^{-4\lambda} u_{xx}\right)$$

$$\alpha_\lambda = \begin{pmatrix} 0 & -e^{\lambda} + e^{-\lambda} u \\ -e^{\lambda} & 0 \end{pmatrix} dx + \begin{pmatrix} u_x & e^{-\lambda} u_{xx} + 2e^{\lambda} u + 2e^{-\lambda} u^2 - 4e^{3\lambda} \\ -4e^{3\lambda} - 2e^{\lambda} u & -u_x \end{pmatrix} dt.$$

Chen-Lee-Liu system:

$$\begin{cases} u_t + u_{xx} - 2uvu_x = 0 \\ v_t + v_{xx} - 2uvv_x = 0 \end{cases}$$

Classical symmetries are generated by "prolongations" of

$$Y_1 = \partial_x, \quad Y_2 = \partial_t, \quad Y_3 = -u\partial_u + v\partial_v, \quad Y_4 = x\partial_x + 2t\partial_t - v\partial_v.$$

$$\alpha := \begin{pmatrix} \frac{1}{2}uv - \frac{1}{2} & u \\ v & -\frac{1}{2}uv + \frac{1}{2} \end{pmatrix} dx + \begin{pmatrix} 2\left(\frac{1}{2}uv - \frac{1}{2}\right)^2 + \frac{1}{2}u_x v - \frac{1}{2}uv_x & u^2v - u + u_x \\ uv^2 - v - v_x & -2\left(\frac{1}{2}uv - \frac{1}{2}\right)^2 - \frac{1}{2}u_x v + \frac{1}{2}uv_x \end{pmatrix} dt$$

The prolonged flow of Y_4 , up to order 1, is such that

$$(t, x, u, v, u_x, v_x) \mapsto \left(e^{2\lambda}t,\ e^{\lambda}x,\ u,\ e^{-\lambda}v\ e^{-\lambda}u_x,\ e^{-2\lambda}v_x\right)$$

Chen-Lee-Liu system:

One has that

$$\alpha_\lambda = \begin{pmatrix} \frac{1}{2}uv - \frac{1}{2}e^\lambda & e^\lambda u \\ v & -\frac{1}{2}uv + \frac{1}{2}e^\lambda \end{pmatrix} dx +$$
$$\begin{pmatrix} \frac{1}{2}\left(uv - e^\lambda\right)^2 + \frac{1}{2}\left(u_x v - u v_x\right) & e^\lambda u^2 v - e^{2\lambda} u + e^\lambda u_x \\ uv^2 - e^\lambda v - v_x & -\frac{1}{2}\left(uv - e^\lambda\right)^2 - \frac{1}{2}\left(u_x v - u v_x\right) \end{pmatrix} dt.$$

Burgers: $u_t = u_{xx} + uu_x$

The algebra of classical symmetries is generated by

$$Y_1 = \partial_x, \qquad Y_2 = \partial_t, \qquad Y_3 = x\partial_x + 2t\partial_t - u\partial_u,$$
$$Y_4 = t\partial_x - \partial_u, \qquad Y_5 = -xt\partial_x - t^2\partial_t + (x + tu)\,\partial_u.$$

Consider for instance the following two ZCRs:

$$\alpha = \begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix} dx + \begin{pmatrix} 0 & 0 \\ u_x + u^2 & 0 \end{pmatrix} dt,$$

$$\beta = \begin{pmatrix} \frac{u}{4} & 0 \\ -\frac{1}{2} & -\frac{u}{4} \end{pmatrix} dx + \begin{pmatrix} \frac{u_x}{4} + \frac{u^2}{8} & 0 \\ -\frac{u}{4} & -\frac{u_x}{4} - \frac{u^2}{8} \end{pmatrix} dt.$$

With respect to α, the algebra of classical symmetries is gauge-like. Whereas for β only Y_1, Y_2 and Y_3 are gauge-like.

An example with a non-projectable symmetry:

For $u_{xt} = \sin u$, the field $X = x\partial_x - t\partial_t$ generates a projectable symmetry which is non gauge-like w.r.t.

$$\alpha := \begin{pmatrix} 1 & -\frac{u_x}{2} \\ \frac{u_x}{2} & -1 \end{pmatrix} dx + \frac{1}{4}\begin{pmatrix} cos(u) & sin(u) \\ sin(u) & -cos(u) \end{pmatrix} dt.$$

Under the point transformation

$$\tau = \quad t - u, \qquad \xi = x, \qquad v = u,$$

the equation, the ZCR and the symmetry transform to

$$v_{\xi\tau} = \frac{1}{v_\tau - 1}\left(v_\xi v_{\tau\tau} + v_\tau^3 sin(v) - 3v_\tau^2 sin(v) + 3v_\tau sin(v) - sin(v)\right),$$

$$\beta = \begin{pmatrix} 1 - \frac{v_\xi\, cos(v)}{4} & \frac{v_\xi}{2(v_\tau - 1)} - \frac{v_\xi\, sin(v)}{4} \\ \frac{v_\xi}{2(1 - v_\tau)} - \frac{v_\xi\, sin(v)}{4} & \frac{v_\xi\, cos(v)}{4} - 1 \end{pmatrix} d\xi + \frac{1 - v_\tau}{4}\begin{pmatrix} cos(v) & sin(v) \\ sin(v) & -cos(v) \end{pmatrix} d\tau,$$

$$Y = \xi\partial_\xi + (v - \tau)\partial_\tau \quad \text{(non-projectable and non gauge-like w.r.t. } \beta\text{).}$$

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