Scalar differential invariants of 2-dimensional Killing foliations

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Alexandre Mikhailovich Vinogradov

Born: 18 February 1938
Novorossiysk, Russia

Died: 20 September 2019
(aged 81)
Lizzano in Belvedere, Italy

Vinogradov was my professor in Salerno.

He was an outstanding professor, and anybody here knows how charismatic and visionary he was.

The main purpose of his life, I believe, was doing mathematics. He couldn't stay away from it, and from sharing his ideas with students.

Hence I am not surprised by the anecdote told by his daughter Katya, according to which the first thing Vinogradov asked to his doctor was whether he would have enough time to finish his last book. The same I would say for the other anecdote, from Katya, which has been reported in the Foreword of second edition of Jet Nestruev.

I remember with particular sadness the moment when I learned of his death. It was by an email from Michal Marvan.

With Michal Marvan we decided to dedicate our last paper to him.

Recognizing when two metrics are the same

Equivalence problem:

"Given two metrics g and g', on an n-dimensional manifold, is there any diffeomorphism ϕ such that $\phi^*(g') = g$?"

It is an important geometrical problem.

Also in physics, e.g. in General Relativity, one often faces with questions like

- When two spacetimes are the same?
- How to provide evidence that a given solution of Einstein equations is new?

Invariants play a key role

When we have a class of geometric objects, which is left invariant by some group of transformations, then up to these transformations we can study and characterize these objects by means of their invariants.

With the invariants we can:

- study the equivalence problem;
- study invariant properties;
- describe and caracterize special cases and explicit examples.

Cartan-Karlhede algorithm for the metric equivalence problem

Equivalence problem:

"Given two metrics g and g', on an n-dimensional manifold, is there any diffeomorphism ϕ such that $\phi^*(g') = g$?"

- Christoffel and Lipschitz (1870) first studied this problem.
- E. Cartan solved it by using the method of moving-frames (computing up to $n(n+1)/2$ order covariant derivatives of R).
- A. Karlhede simplified Cartan's solution when $n = 4$ (computing only up to 7th order covariant derivatives of R).

A. Karlhede, A review of the geometrical equivalence of metrics in General Relativity, Gen. Rel. Grav., Vol. 12, No. 9 (1980)

Our problem is a particular instance of the general problem

On a 4-dimensional manifold $\mathscr{M}$, we will consider the pseudo-Riemannian metrics $\mathbf{g}$ such that:

1. $\mathfrak{Kill}(\mathbf{g}) = \mathscr{G}_2$, with $\mathscr{G}_2 := < \xi_{(1)}, \xi_{(2)} >$ being a 2-dimensional Abelian algebra of vector fields on $\mathscr{M}$.

2. $\mathbf{g}$ has non-null Killing leaves (i.e., $\mathbf{g}$ does not degenerate on the 2-dimenisional Killing leaves).

In view of 1

$$\Xi := span\{\xi_{(1)}, \xi_{(2)}\}$$

is a 2-dimensional integrable distribution. We call killing leaves (or Killing orbits) its 2-dimensional integral manifolds.

The property 2, on the other hand, is equivalent to say that $\Xi^{\perp}$ is transversal to Ξ.

We consider the local equivalence problem for these metrics, with respect to the Lie pseudogroup $\mathfrak{G}$ of local diffeomorphisms of $\mathscr{M}$ which preserve $\mathscr{G}_2$.

A particular instance with lower order invariants

The Cartan-Karlhede solution is general but leads to invariants that may be too much complicated, due to their order (which is 7 when $n = 4$). Indeed, in practice, by reducing the problem to a more specific situation, there tipically appear lower order invariants which are more simple and even more useful for practical applications.

This happens in our particular instance, where one can distinguish two main cases:

- orthogonally transitive case: $\Xi^{\perp}$ <u>integrable</u>

  here one has a fundamental system of 4 functionally independent <u>first order</u> scalar differential invariants.

- orthogonally intransitive case: $\Xi^{\perp}$ <u>not integrable</u>

  here one has a fundamental system of 6 functionally independent <u>first order</u> scalar differential invariants.

References

The orthogonally transitive case, with a solution of the equivalence problem, has been considered in the paper:

- M. Marvan and O. Stolín, On local equivalence problem of spacetimes with two orthogonally transitive commuting Killing fields, Journal of Mathematical Physics, vol 49 (2008)

The orthogonally intransitive case has been considered in the paper:

- D. Catalano Ferraioli and M. Marvan, The equivalence problem for generic four-dimensional metrics with two commuting Killing vector, Annali di Matematica Pura ed Applicata, vol 199 (2020)

Where the equivalence problem has been solved with the only exception of a special case. Neverthless, we use the relative simplicity of our invariants to explicitly describe Lorentzian Einstein metrics belonging to this special case.

Plan of the talk

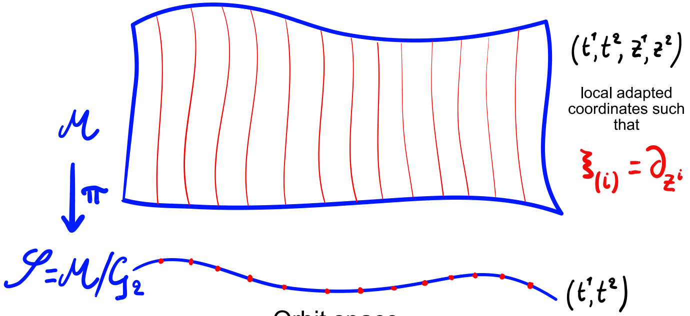
- Introduction of adapted coordinates (t^1, t^2, z^1, z^2)
- Definition of the metric $\tilde{\mathbf{g}}$ on the orbit space $\mathscr{S} = \mathscr{M}/\mathscr{G}_2$, for the pseudo-Riemannian submersion

$$\pi : \quad \mathscr{M} \quad \to \quad \mathscr{S} \quad = \mathscr{M}/\mathscr{G}_2 \qquad (t^1, t^2, z^1, z^2) \mapsto (t^1, t^2)$$

- First scalar invariants $C_\rho, C_\chi, Q_\chi, Q_\gamma$
- Semi-invariant horizontal frame ($\rightsquigarrow$ invariant differentiations)
- When $\Xi^\perp$ is not integrable
  - Semi-invariant vertical frame
  - Additional scalar invariants $\ell_{\mathscr{C}}$ and $\Theta_{\mathscr{C}}$
- Solution of the equivalence problem (with the exception of orthogonally intransitive metrics with $C_\rho \ell_{\mathscr{C}} = 0$)
- A discussion of the special case: by imposing the first order condition $C_\rho \ell_{\mathscr{C}} = 0$ to Lorentzian Λ-vacuum Einstein metrics

Local adapted coordinates

$\mathcal{M}$ 4-dim manifold | $\mathcal{G}_2 = \langle \xi_{(1)}, \xi_{(2)} \rangle$ 2-dim Abelian algebra of vector fields



Orbit space
(locally a 2-dim manifold)

$\mathfrak{G}$-transformations in adapted coordinates

Proposition. In adapted coordinates, $\mathfrak{G}$-transformations (i.e., the transformations of $\mathfrak{G}$) $\mathscr{M} \to \mathscr{M}$, $(t^1, t^2, z^1, z^2) \mapsto (\bar{t}^1, \bar{t}^2, \bar{z}^1, \bar{z}^2)$, have the form

$$\bar{t}^{\,i} = \phi^i(t^1, t^2), \qquad \bar{z}^{\,i} = \alpha^i_j z^j + \psi^i(t^1, t^2),$$

where $\phi^i(t^1, t^2)$ and $\psi^i(t^1, t^2)$ are arbitrary differentiable functions and

$$\alpha^i_j \in \mathbb{R}, \qquad J_\phi = \begin{vmatrix} \partial_{t^1}\phi^1 & \partial_{t^2}\phi^1 \\ \partial_{t^1}\phi^2 & \partial_{t^2}\phi^2 \end{vmatrix} \neq 0, \qquad (\alpha^i_j) \in \mathrm{GL}(2, \mathbb{R}).$$

Natural extension of $\mathfrak{G}$ to the bundle τ of considered metrics

In adapted coordinates, any metric $\mathbf{g}$ of the type considered above takes the form

$$\mathbf{g} = b_{ij}(t^1,t^2)\,dt^i\,dt^j + 2f_{ik}(t^1,t^2)\,dt^i\,dz^k + h_{kl}(t^1,t^2)\,dz^k\,dz^l.$$

Thus if one considers

$$\bar{\mathbf{g}} = \bar{b}_{mn}(\bar{t}^1,\bar{t}^2)\,d\bar{t}^m\,d\bar{t}^n + 2\bar{f}_{mr}(\bar{t}^1,\bar{t}^2)\,d\bar{t}^m\,d\bar{z}^r + \bar{h}_{rs}(\bar{t}^1,\bar{t}^2)\,d\bar{z}^r\,d\bar{z}^s,$$

that transforms to $\mathbf{g}$, under pull-back of $\mathfrak{G}$-transformations, one has that

$$\begin{aligned}
b_{ij} &= \bar{b}_{mn}\frac{\partial\phi^m}{\partial t^i}\frac{\partial\phi^n}{\partial t^j} + 2\bar{f}_{mr}\frac{\partial\phi^m}{\partial t^i}\frac{\partial\psi^r}{\partial t^j} + \bar{h}_{rs}\frac{\partial\psi^r}{\partial t^i}\frac{\partial\psi^s}{\partial t^j},\\
f_{ik} &= \bar{f}_{mr}\alpha^r_k\frac{\partial\phi^m}{\partial t^i} + \bar{h}_{rs}\alpha^r_k\frac{\partial\psi^s}{\partial t^i},\\
h_{kl} &= \bar{h}_{rs}\alpha^r_k\alpha^s_l.
\end{aligned}$$

In particular

$$\det\bar{\mathbf{g}} = (\det\alpha^i_j)^2\,\left(J_\phi\right)^2\det\mathbf{g} \neq 0.$$

Natural extension of $\mathfrak{G}$ to the bundles $J^m\tau$, $m = 0, 1, 2, \ldots$

The pseudogroup $\mathfrak{G}$ naturally extends to

$$\tau : E \to \mathscr{M}$$

the bundle of metrics satisfying above assumptions (1)-(2).
The extension of $\mathfrak{G}$ to τ will be denoted by $\mathfrak{G}_\tau$.

It is in this bundle that we consider the equivalence problem.

We solved the problem by using scalar differential invariants:

functions on the jet prolongations $J^m\tau$, $m = 0, 1, 2, \ldots$, that are invariant with respect to the action of the corresponding prolonged pseudogroups $\mathfrak{G}_\tau^{(m)}$ on $J^m\tau$.

Generic dimensions of orbit spaces $J^m\tau/\mathfrak{G}_\tau^{(m)}$, and the number of functionally independent differential invariants

By using the infinitesimal generators of $\mathfrak{G}_\tau^{(m)}$, one gets the following

Proposition. The generic dimension $N_m = \dim\left(J^m\tau/\mathfrak{G}_\tau^{(m)}\right)$, for $m = 0, 1, 2, \ldots$, varies as follows:

- when $\Xi^\perp$ is not integrable (orthogonally intransitive case), one has

m	0	1	2	$\cdots$
N_m	0	6	20	$\cdots$

- when $\Xi^\perp$ is integrable (orthogonally transitive case), one has

m	0	1	2	$\cdots$
N_m	0	4	14	$\cdots$

Pseudo-Riemannian submersion

$$\boxed{\mathbf{g} = b_{ij}(t^1,t^2)\,dt^i\,dt^j + 2f_{ik}(t^1,t^2)\,dt^i\,dz^k + h_{kl}(t^1,t^2)\,dz^k\,dz^l}$$

- $\mathbf{h} = h_{kl}dz^k dz^l$ is the metric (possibly non Riemannian) on Killing leaves.

On the other hand, one can also rearrange $\mathbf{g}$ in the form (Geroch)

$$\boxed{\mathbf{g} = \tilde{g}_{ij}(t^1,t^2)\,dt^i\,dt^j + h_{kl}(t^1,t^2)\left(dz^k + f_i^{\,k}\,dt^i\right)\left(dz^l + f_j^{\,l}\,dt^j\right)}$$

$$\tilde{g}_{ij} := b_{ij} - f_{ik}f_{jl}h^{kl}, \qquad\qquad f_j^{\,k} := f_{js}h^{sk}$$

- $\tilde{\mathbf{g}} = \tilde{g}_{ij}(t^1,t^2)\,dt^i\,dt^j$ defines a metric (possibly non Riemannian) on $\mathscr{S} = \mathscr{M}/\mathscr{G}_2$, which will be called "orbit metric".

Pseudo-Riemannian submersion between $(\mathscr{M},\mathbf{g})$ and $(\mathscr{S},\tilde{\mathbf{g}})$:

$$\pi : \mathscr{M} \to \mathscr{S}, \quad (t^1,t^2,z^1,z^2) \mapsto (t^1,t^2)$$

First 4 scalar invariants: C_ρ, C_χ, Q_χ, Q_γ

Remark: For any invariant symmetric $(0,2)$ tensor $\underline{\mu = \mu_{ij}\,dt^i\,dt^j \text{ on } \mathscr{S}}$, we denote by $\underline{\hat{\mu} = \mu_i^j\,dt^i \otimes \partial_{tj}}$ ($\mu_i^j = \mu_{is}\tilde{\mathbf{g}}^{sj}$) the associated invariant self-adjoint $(1,1)$-tensor field

$$\mu(Z_1, Z_2) = \tilde{\mathbf{g}}(Z_1, \hat{\mu}(Z_2)), \qquad Z_i \in \mathscr{D}(\mathscr{S}).$$

Then the trace C_μ and the determinant Q_μ of $\hat{\mu}$ are scalar invariants:

$$C_\mu = \mu_{ij}\tilde{g}^{ij}, \qquad Q_\mu = \frac{\det \mu_{ij}}{\det \tilde{\mathbf{g}}}.$$

Application: the pseudogroup action leaves invariant the 1-form σ and the symmetric $(0,2)$-tensor χ

$$\sigma = d\,(\ln(\det \mathbf{h})) = \frac{d(\det \mathbf{h})}{\det \mathbf{h}}, \qquad \chi := \frac{1}{(\det \mathbf{h})}(dh_{11}\,dh_{22} - dh_{12}\,dh_{12})$$

then we get the invariant symmetric $(0,2)$-tensors

$$\rho := \sigma^2 \text{ (symmetric product)}, \qquad \gamma := \chi - \tfrac{1}{4}\rho \quad \text{(Cosgrove)}$$

and the nontrivial scalar invariants $C_\rho, C_\chi, Q_\chi, Q_\gamma$, whereas $C_\gamma = Q_\rho = 0$.

Coordinate expressions of C_ρ, C_χ, Q_χ, Q_γ

In coordinates:

1. $C_\rho = \dfrac{1}{(\det \mathbf{h})^2}(\det \mathbf{h})_{,i}(\det \mathbf{h})_{,j}\,\tilde{g}^{ij}$;
2. $C_\chi = \dfrac{1}{(\det \mathbf{h})}\,\tilde{g}^{ij}\begin{vmatrix} h_{11,i} & h_{12,j} \\ h_{12,i} & h_{22,j} \end{vmatrix}$;
3. $Q_\chi = \dfrac{\det \chi_{ij}}{\det \tilde{\mathbf{g}}}, \qquad \chi_{ij} = \dfrac{1}{2\det \mathbf{h}}\begin{vmatrix} h_{11,i} & h_{12,j} \\ h_{21,i} & h_{22,j} \end{vmatrix} + \dfrac{1}{2\det \mathbf{h}}\begin{vmatrix} h_{11,j} & h_{12,i} \\ h_{21,j} & h_{22,i} \end{vmatrix}$;
4. $Q_\gamma = \dfrac{\det \gamma_{ij}}{\det \tilde{\mathbf{g}}} = \dfrac{1}{4(\det \mathbf{h})^3 \det \tilde{\mathbf{g}}}\begin{vmatrix} h_{11} & h_{12} & h_{22} \\ h_{11,1} & h_{12,1} & h_{22,1} \\ h_{11,2} & h_{12,2} & h_{22,2} \end{vmatrix}^2$.

In the orthogonally transitive case, it is a fundamental system of functionally independent scalar differential invariants.
In the orthogonally intransitive case, we need 2 additional scalar invariants.

Semi-invariant orthogonal frame on $\mathscr{S}$, when $C_\rho \neq 0$

Let

$$\pi_1 := \pi \circ \tau_1 : J^1\tau \to \mathscr{S}, \qquad \sigma = d\left(\ln(\det \mathbf{h})\right), \qquad \mathrm{vol}_{\tilde{\mathbf{g}}} = \sqrt{|\det \tilde{\mathbf{g}}|}\, dt^1 \wedge dt^2,$$

we can consider the π_1-relative vector fields $\mathscr{X}$ and $\mathscr{X}^\perp$ on $\mathscr{S}$:

$$\sigma = \tilde{\mathbf{g}}(\mathscr{X}, -), \qquad \sigma = \mathscr{X}^\perp \lrcorner\, \mathrm{vol}_{\tilde{\mathbf{g}}}\,.$$

These fields are linearly independent iff $C_\rho \neq 0$. Moreover

$$\mathscr{X} = \tilde{\mathbf{g}}^{is} \frac{(\det \mathbf{h})_{,s}}{\det \mathbf{h}} \partial_{t^i}, \qquad \mathscr{X}^\perp = \frac{(\det \mathbf{h})_{,2}}{(\det \mathbf{h})\sqrt{|\det \tilde{\mathbf{g}}|}} \partial_{t^1} - \frac{(\det \mathbf{h})_{,1}}{(\det \mathbf{h})\sqrt{|\det \tilde{\mathbf{g}}|}} \partial_{t^2}$$

$$\tilde{\mathbf{g}}(\mathscr{X}, \mathscr{X}) = C_\rho, \quad \tilde{\mathbf{g}}(\mathscr{X}, \mathscr{X}^\perp) = 0, \quad \tilde{\mathbf{g}}(\mathscr{X}^\perp, \mathscr{X}^\perp) = \pm_{\tilde{\mathbf{g}}} C_\rho, \quad \pm_{\tilde{\mathbf{g}}} := \mathrm{sgn} \det \tilde{\mathbf{g}},$$

$$\mathscr{X} \mapsto \mathscr{X}, \qquad \mathscr{X}^\perp \mapsto \mathrm{sgn}(J_\phi)\, \mathscr{X}^\perp$$

$\{\mathscr{X}, \mathscr{X}^\perp\}$ is a semi-invariant orthogonal frame on $\mathscr{S}$, when $C_\rho \neq 0$.

Riemannian submersion

For the pseudo-Riemannian submersion $\pi : \mathscr{M} \to \mathscr{S}$ we have:

- Ξ is the *vertical distribution;*
- $\Xi^\perp$ is the *horizontal distribution* generated by vector fields

$$\mathbf{e}_j = \frac{\partial}{\partial t^j} - f_j^k \frac{\partial}{\partial z^k}, \qquad j = 1, 2 \qquad \text{(where } f_j^k = f_{js} h^{sk});$$

- $T\mathscr{M} = \Xi \oplus \Xi^\perp$;
- $\mathrm{ver} = \mathrm{pr}_\Xi : TM \to \Xi$ such that

$$\mathrm{ver}\left(\tfrac{\partial}{\partial t^j}\right) = \mathrm{ver}\left(\tfrac{\partial}{\partial t^j} - f_j^k \tfrac{\partial}{\partial z^k} + f_j^k \tfrac{\partial}{\partial z^k}\right) = f_j^k \tfrac{\partial}{\partial z^k},$$
$$\mathrm{ver}\left(\tfrac{\partial}{\partial z^k}\right) = \tfrac{\partial}{\partial z^k};$$

- $\mathrm{hor} = \mathrm{pr}_{\Xi^\perp} : TM \to \Xi^\perp$ such that

$$\mathrm{hor}\left(\tfrac{\partial}{\partial t^j}\right) = \mathrm{hor}\left(\tfrac{\partial}{\partial t^j} - f_j^k \tfrac{\partial}{\partial z^k} + f_j^k \tfrac{\partial}{\partial z^k}\right) = \tfrac{\partial}{\partial t^j} - f_j^k \tfrac{\partial}{\partial z^k},$$
$$\mathrm{hor}\left(\tfrac{\partial}{\partial z^k}\right) = 0.$$

O'Neill tensors A and T

O'Neill tensors **A** and **T** of the submersion π:

$$\mathbf{A}(W_1, W_2) = \mathbf{O}(W_1, W_2) + \mathbf{E}(W_1, W_2),$$

$$\mathbf{T}(W_1, W_2) = \mathbf{N}(W_1, W_2) + \mathbf{L}(W_1, W_2),$$

where

$$\mathbf{O}(W_1, W_2) = \mathrm{ver}\left(\nabla_{\mathrm{hor}\, W_1} \mathrm{hor}\, W_2\right), \quad \mathbf{E}(W_1, W_2) = \mathrm{hor}\left(\nabla_{\mathrm{hor}\, W_1} \mathrm{ver}\, W_2\right),$$

$$\mathbf{N}(W_1, W_2) = \mathrm{ver}\left(\nabla_{\mathrm{ver}\, W_1} \mathrm{hor}\, W_2\right), \quad \mathbf{L}(W_1, W_2) = \mathrm{hor}\left(\nabla_{\mathrm{ver}\, W_1} \mathrm{ver}\, W_2\right),$$

for arbitrary vector fields W_1, W_2 on $\mathscr{M}$.

Lift of vector fields

Every vector field X on $\mathscr{S}$ can be uniquely lifted to an horizontal vector field $\widehat{X}$ on $\mathscr{M}$ which is π-related to X.

In particular every invariant vector field on $\mathscr{S}$ can be uniquely lifted to an invariant vector field on $\mathscr{M}$.

Moreover, the lift preserves the scalar product.

In coordinates,

$$\widehat{\frac{\partial}{\partial t^i}} = \frac{\partial}{\partial t^i} - f_i^k \frac{\partial}{\partial z^k} = \mathbf{e}_i, \quad i = 1, 2.$$

Lifting $\{\mathscr{X}, \mathscr{X}^\perp\}$ to a semi-invariant horizontal frame

The *mean-curvature vector field* $\mathscr{H} := \sum_{s=1}^{2} \mathbf{T}(\mathbf{v}_s, \mathbf{v}_s)$, where $\{\mathbf{v}_1, \mathbf{v}_2\}$ is any vertical orthonormal frame, has components $\mathscr{H}^a = \mathbf{g}^{kl}\mathbf{T}^a_{kl}$.
We have that:

$$\mathscr{H} = -\frac{1}{2}\hat{\mathscr{X}} = -\frac{1}{2}\mathscr{X}^i\left(\frac{\partial}{\partial t^i} - f_i^k\frac{\partial}{\partial z^k}\right), \qquad \mathscr{H}^\perp := -\frac{1}{2}\hat{\mathscr{X}}^\perp.$$

By invariance of scalar product under the lift:

$$\mathbf{g}(\mathscr{H}, \mathscr{H}^\perp) = 0, \qquad \ell_{\mathscr{H}} = \mathbf{g}(\mathscr{H}, \mathscr{H}) = \frac{1}{4}\tilde{\mathbf{g}}(\mathscr{X}, \mathscr{X}) = \frac{1}{4}C_\rho$$

$$\ell_{\mathscr{H}^\perp} = \mathbf{g}(\mathscr{H}^\perp, \mathscr{H}^\perp) = \frac{1}{4}\tilde{\mathbf{g}}(\mathscr{X}^\perp, \mathscr{X}^\perp) = \frac{1}{4}\left(\pm_{\tilde{\mathbf{g}}} C_\rho\right) = \pm_{\tilde{\mathbf{g}}}\,\ell_{\mathscr{H}}.$$

Therefore $\{\mathscr{H}, \mathscr{H}^\perp\}$ is a semi-invariant horizontal orthogonal frame, when $C_\rho \neq 0$.

The Ehresmann curvature $\mathbf{c}$

The *Ehresmann curvature* is the skew-symmetric tensor

$$\mathbf{c} : \mathscr{D}(\mathscr{M}) \otimes \mathscr{D}(\mathscr{M}) \to \mathscr{D}(\mathscr{M})$$

$$\mathbf{c}(W_1, W_2) = \mathrm{ver}\,[\mathrm{hor}\, W_1, \mathrm{hor}\, W_2]\,, \qquad W_1, W_2 \in \mathscr{D}(\mathscr{M}).$$

$\mathbf{c}$ is traceless (i.e., $\mathbf{c}^a_{ab} = 0$) and its nonzero components, in adapted coordinates, are

$$\mathbf{c}^{k^*}_{ij} = \partial_j f^k_i - \partial_i f^k_j\,, \qquad k^* = k + 2.$$

Of course, $\mathbf{c} = 0$ if, and only if, $\Xi^\perp$ is involutive.

Semi-invariant vertical frame $\{\mathscr{C}, \mathscr{C}^\perp\}$, when $\ell_{\mathscr{C}} \neq 0$

The curvature vector field

$$\mathscr{C} := \frac{\mathbf{c}(\partial_{t_1}, \partial_{t_2})}{\sqrt{|\det \tilde{\mathbf{g}}|}} = \mathscr{C}^k \frac{\partial}{\partial z^k}, \qquad \mathscr{C}^k = \frac{\partial_{t^2} f_1^k - \partial_{t^1} f_2^k}{\sqrt{|\det \tilde{\mathbf{g}}|}}, \qquad k = 1, 2$$

transforms as $\mathscr{C} \mapsto (\operatorname{sgn} J_\phi)\mathscr{C}$.

Hence we get the scalar invariant

$$\ell_{\mathscr{C}} = \mathbf{g}(\mathscr{C}, \mathscr{C}) = h_{kl}\mathscr{C}^k\mathscr{C}^l = \frac{h_{kl}(\partial_{t^2} f_1^k - \partial_{t^1} f_2^k)(\partial_{t^2} f_1^l - \partial_{t^1} f_2^l)}{|\det \tilde{\mathbf{g}}|}.$$

Moreover, when $\ell_{\mathscr{C}} \neq 0$, one can define another vertical field $\mathscr{C}^\perp$ as

$$\mathbf{g}(\mathscr{C}, \mathscr{C}^\perp) = 0, \qquad \mathrm{vol}_{\mathbf{h}}(\mathscr{C}, \mathscr{C}^\perp) > 0, \qquad \ell_{\mathscr{C}^\perp} = \mathbf{g}(\mathscr{C}^\perp, \mathscr{C}^\perp) = \pm_{\mathbf{h}} \ell_{\mathscr{C}},$$

where $\mathrm{vol}_h = \sqrt{|\det h|}\, dz^1 \wedge dz^2$ and $\pm_{\mathbf{h}} = \operatorname{sgn} \det \mathbf{h}$.

It turns out that

$$\mathscr{C}^\perp = \frac{h_{s2}\mathscr{C}^s}{\sqrt{|\det \mathbf{h}|}} \frac{\partial}{\partial z^1} - \frac{h_{s1}\mathscr{C}^s}{\sqrt{|\det \mathbf{h}|}} \frac{\partial}{\partial z^2}, \qquad \mathscr{C}^\perp \mapsto (\operatorname{sgn} J_\phi)(\operatorname{sgn} \det \alpha_j^i)\mathscr{C}^\perp.$$

Thus, the pair $\{\mathscr{C}, \mathscr{C}^\perp\}$ defines a semi-invariant vertical orthogonal frame on Ξ.

The sixth scalar differential invariant

Using O'Neill tensors **T** and **A** we get further invariants.

The simplest sixth invariant is

$$\Theta_{\mathscr{C}} = \det \mathbf{T}_{\mathscr{C}}$$

where $\mathbf{T}_{\mathscr{C}}$ is the $(1,1)$-tensor field on $\mathscr{M}$ defined as

$$\mathbf{T}_{\mathscr{C}}(\boldsymbol{.}) := \mathbf{T}(\mathscr{C}, \boldsymbol{.})\,.$$

In coordinates:

$$\Theta_{\mathscr{C}} = \frac{1}{16}\Theta_{\mathrm{I}}^2$$

with

$$\Theta_{\mathrm{I}} = \frac{1}{|\det \mathbf{h}|^{1/2}\,|\det \tilde{\mathbf{g}}|^{1/2}} \begin{vmatrix} h_{11,1} & h_{12,1} & h_{22,1} \\ h_{11,2} & h_{12,2} & h_{22,2} \\ (\mathscr{C}^2)^2 & -\mathscr{C}^1\mathscr{C}^2 & (\mathscr{C}^1)^2 \end{vmatrix}.$$

Describing higher order invariants

Recall that

$$\mathscr{X} \mapsto \mathscr{X}, \qquad \mathscr{X}^{\perp} \mapsto \operatorname{sgn}(J_\phi)\mathscr{X}^{\perp}$$

These field lift to the "invariant" differentiations

$$X = \tilde{\mathbf{g}}^{is}\frac{(\det\mathbf{h})_{,s}}{\det\mathbf{h}}D_{t^i}, \qquad X^{\perp} = \frac{(\det\mathbf{h})_{,2}}{(\det\mathbf{h})\sqrt{|\det\tilde{\mathbf{g}}|}}D_{t^1} - \frac{(\det\mathbf{h})_{,1}}{(\det\mathbf{h})\sqrt{|\det\tilde{\mathbf{g}}|}}D_{t^2}$$

$$X \mapsto X, \qquad X^{\perp} \mapsto \operatorname{sgn}(J_\phi)X^{\perp}$$

By chosing two invariants $\mathscr{I}^1, \mathscr{I}^2$

$$\Delta = \begin{vmatrix} X\mathscr{I}^1 & X\mathscr{I}^2 \\ X^{\perp}\mathscr{I}^1 & X^{\perp}\mathscr{I}^2 \end{vmatrix}$$

we have the following invariant differentiation operators

$$\mathscr{I} \longmapsto \frac{1}{\Delta}\begin{vmatrix} X\mathscr{I} & X\mathscr{I}^2 \\ X^{\perp}\mathscr{I} & X^{\perp}\mathscr{I}^2 \end{vmatrix} \qquad \mathscr{I} \longmapsto \frac{1}{\Delta}\begin{vmatrix} X\mathscr{I}^1 & X\mathscr{I} \\ X^{\perp}\mathscr{I}^1 & X^{\perp}\mathscr{I} \end{vmatrix}$$

Thus one can obtain higher order invariants by iterated applications of these invariants. Other invariants are described in the paper.

Solution of the equivalence problem in the generic (i.e., $\ell_{\mathscr{C}} C_\rho \neq 0$) orthogonally intransitive case

For any given metric there exist a pair of functionally independent invariants among:

$$I_1 = C_\rho, \quad I_2 = C_\chi, \quad I_3 = Q_\chi, \quad I_4 = Q_\gamma, \quad I_5 = \ell_{\mathscr{C}} \quad I_6 = (\Theta_{\mathrm{I}})^2.$$

Theorem. Consider two generic orthogonally intransitive metrics

$$\mathbf{g} = b_{ij}(t^1,t^2)\,dt^i\,dt^j + 2f_{ik}(t^1,t^2)\,dt^i\,dz^k + h_{kl}(t^1,t^2)\,dz^k\,dz^l$$

and

$$\bar{\mathbf{g}} = \bar{b}_{mn}(\bar{t}^1,\bar{t}^2)\,d\bar{t}^m\,d\bar{t}^n + 2\bar{f}_{mr}(\bar{t}^1,\bar{t}^2)\,d\bar{t}^m\,d\bar{z}^r + \bar{h}_{rs}(\bar{t}^1,\bar{t}^2)\,d\bar{z}^r\,d\bar{z}^s,$$

with two pairs of functionally independent invariants

$$\{\mathscr{I}^1(t^1,t^2), \mathscr{I}^2(t^1,t^2)\} \qquad \{\bar{\mathscr{I}}^1(\bar{t}^1,\bar{t}^2), \bar{\mathscr{I}}^2(\bar{t}^1,\bar{t}^2)\}$$

formed by the same fundamental invariants for $\mathbf{g}$ and $\bar{\mathbf{g}}$, respectively. These metrics **are equivalent if, and only if,** the remaining 4 fundamental scalar invariants of $\mathbf{g}$ depend on $(\mathscr{I}^1, \mathscr{I}^2)$ in the same way as the corresponding ones of $\bar{\mathbf{g}}$ depend on $(\bar{\mathscr{I}}^1, \bar{\mathscr{I}}^2)$.

Solution of the equivalence problem in the generic (i.e., $\ell_{\mathscr{C}} C_\rho \neq 0$) orthogonally intransitive case

The equivalence class of a generic orthogonally intransitive metric $\mathbf{g}$ is completely characterised by the way the six fundamental first-order scalar differential invariants $I_1 = C_\rho, I_2 = C_\chi,\ I_3 = Q_\chi,\ I_4 = Q_\gamma,\ I_5 = \ell_{\mathscr{C}},$ $I_6 = (\Theta_{\mathrm{I}})^2$ depend on a pair of functionally independent first-order scalar differential invariants $(\mathscr{I}^1, \mathscr{I}^2)$.

Van den Berg metric

Van den Bergh metric

$$\mathbf{g} = \cosh\left(\sqrt{6}\,t^1\right)\left\{\sinh^4\left(t^2\right)\left[\left(dt^1\right)^2 - \left(dt^2\right)^2\right] + 2\sinh^2\left(t^2\right)\left[dz^2 + \cosh\left(t^2\right)dt^1\right]^2\right\}$$

$$+\frac{12}{\cosh\left(\sqrt{6}\,t^1\right)}\left[dz^1 + \cosh\left(t^2\right)dz^2 + 1/2\cosh^2\left(t^2\right)dt^1\right]^2.$$

is a Ricci-flat metric with two Killing vector fields ∂_{z^1} and ∂_{z^2} and orthogonally intransitive Ξ.

Van den Berg metric

In this case:

$$C_\rho = -4\,\frac{\cosh^2\left(t^2\right)}{\cosh\left(\sqrt{6}\,t^1\right)\sinh^6\left(t^2\right)},$$

$$C_\chi = -6\,\frac{\sinh^2\left(\sqrt{6}\,t^1\right)-1}{\cosh^3\left(\sqrt{6}\,t^1\right)\sinh^4\left(t^2\right)},$$

$$Q_\chi = 6\,\frac{\sinh^2\left(\sqrt{6}\,t^1\right)\left[-6\sinh^2\left(t^2\right)+\cosh^2\left(t^2\right)\cosh^2\left(\sqrt{6}\,t^1\right)\right]}{\cosh^6\left(\sqrt{6}\,t^1\right)\sinh^{10}\left(t^2\right)},$$

$$Q_\gamma = -36\,\frac{\sinh^2\left(\sqrt{6}\,t^1\right)}{\cosh^6\left(\sqrt{6}\,t^1\right)\sinh^8\left(t^2\right)},$$

$$\ell_{\mathscr{C}} = 2\,\frac{1}{\cosh\left(\sqrt{6}\,t^1\right)\sinh^4\left(t^2\right)},$$

$$(\Theta_1)^2 = 144\,\frac{\sinh^2\left(\sqrt{6}\,t^1\right)}{\sinh^{16}\left(t^2\right)\cosh^8\left(\sqrt{6}\,t^1\right)}.$$

Hence we can take the following two functionally independent invariants

$$\mathscr{I}^1 = C_\rho, \qquad \mathscr{I}^2 = \ell_{\mathscr{C}}.$$

Van den Berg metric

By choosing

$$\mathscr{I}^1 = C_\rho, \qquad \mathscr{I}^2 = \ell_{\mathscr{C}},$$

one can write

$$t^1 = \frac{1}{\sqrt{6}} \operatorname{arccosh}\left(\frac{2}{\ell_{\mathscr{C}}}\left(\frac{C_\rho}{2\ell_{\mathscr{C}}}+1\right)^2\right), \qquad t^2 = \operatorname{arctanh}\left(\frac{1}{\sqrt{1-\frac{1}{2}\frac{C_\rho}{\ell_{\mathscr{C}}}}}\right).$$

Then the remaining 4 fundamental invariants read

$$\begin{cases} C_\chi = \dfrac{-3\,\ell_{\mathscr{C}}\left(-8\,\ell_{\mathscr{C}}^6 + \left(C_\rho^2 + 4\,C_\rho \ell_{\mathscr{C}} + 4\ell_{\mathscr{C}}^2\right)^2\right)}{\left(C_\rho + 2\,\ell_{\mathscr{C}}\right)^4}, \\ Q_\chi = \dfrac{-3\,\ell_{\mathscr{C}}\left(48\,\ell_{\mathscr{C}}^7 + C_\rho\left(C_\rho^2 + 4\,C_\rho \ell_{\mathscr{C}} + 4\ell_{\mathscr{C}}^2\right)^2\right)\left(\left(C_\rho^2 + 4\,C_\rho \ell_{\mathscr{C}} + 4\ell_{\mathscr{C}}^2\right)^2 - 4\ell_{\mathscr{C}}^6\right)}{4\left(C_\rho + 2\,\ell_{\mathscr{C}}\right)^8}, \\ Q_\gamma = \dfrac{-36\,\ell_{\mathscr{C}}^8\left(\left(C_\rho^2 + 4\,C_\rho \ell_{\mathscr{C}} + 4\ell_{\mathscr{C}}^2\right)^2 - 4\ell_{\mathscr{C}}^6\right)}{\left(C_\rho + 2\,\ell_{\mathscr{C}}\right)^8}, \\ (\Theta_{\mathrm{I}})^2 = -\ell_{\mathscr{C}}^2\, Q_\gamma. \end{cases}$$

This gives an invariant characterisation of the class represented by Van den Bergh metric.

Solution of the equivalence problem when $\Xi^{\perp}$ is integrable

The equivalence class of a orthogonally transitive metric $\mathbf{g}$ is completely characterised by the signature of its restriction to the Killing leaves and by the way the four fundamental first-order scalar differential invariants $I_1 = C_\rho,\ I_2 = C_\chi,\ I_3 = Q_\chi,\ I_4 = Q_\gamma$ depend on two functionally independent first-order scalar differential invariants $(\mathscr{I}^1, \mathscr{I}^2)$.

A discussion of the special case $C_\rho \ell_{\mathscr{C}} = 0$

We can distinguish two sub-cases:

- $C_\rho = 0$;
- $\ell_{\mathscr{C}} = 0$.

When $C_\rho \ell_{\mathscr{C}} = 0$ the equivalence problem is not solved but, since C_ρ and $\ell_{\mathscr{C}}$ are first order invariants, in some particular cases we can use them to explicitly describe metrics satisfying the special condition $C_\rho \ell_{\mathscr{C}} = 0$.

The particular case of Lorentzian Einstein metrics

Aiming at possible applications in General Relativity, we considered the case of <u>Lorentzian Λ-vacuum Einstein metrics</u>:

$$\mathbf{R}_{\mu\nu} - \Lambda \mathbf{g}_{\mu\nu} = 0, \qquad \Lambda \in \mathbb{R}$$

with

$$\ell_{\mathscr{C}} C_\rho = 0.$$

We distinguished the following two main sub-cases:

(I) $C_\rho = 0$, with the further sub-cases:

- (I-1) $\det \tilde{\mathbf{g}} < 0$ (i.e., with Lorentzian orbit metric);
- (I-2) $\det \tilde{\mathbf{g}} > 0$ (i.e., with Riemannian orbit metric).

(II) $\ell_{\mathscr{C}} = 0$.

Lorentzian Einstein metrics with $C_\rho = 0$ and $\det \tilde{\mathbf{g}} < 0$

Theorem. All Lorentzian Λ-vacuum Einstein metrics with 2-dimensional Abelian Killing algebra, non-null Killing leaves, $C_\rho = 0$ and $\det \tilde{\mathbf{g}} < 0$ (hence $\det \mathbf{h} > 0$), satisfy $\Lambda = 0$ and there exist adapted coordinates (t^1, t^2, z^1, z^2) such that

$$\mathbf{g} = dt^1\, dt^2 + R^2 (dz^1 + W\, dz^2)^2 + S^2 (dz^2)^2,$$

with R, W and S differentiable functions of t^1 such that $R\,S \neq 0$ and

$$(W')^2 = \frac{2S^2}{R^2}\left(\frac{R''}{R} + \frac{S''}{S}\right).$$

In particular these metrics are such that $\mathscr{C} = 0$ (then $\ell_{\mathscr{C}} = 0$), hence are orthogonally transitive and, in addition, are pp-waves since ∂_{t^2} is a null Killing vector field such that $\nabla \partial_{t^2} = 0$.

Lorentzian Λ-vacuum Einstein metrics with $C_\rho = 0$ and $\det \tilde{\mathbf{g}} > 0$

Theorem. All Lorentzian Λ-vacuum Einstein metrics with 2-dimensional Abelian Killing algebra, non-null Killing leaves, $C_\rho = 0$ and $\det \tilde{\mathbf{g}} > 0$ (hence $\det \mathbf{h} < 0$), satisfy $\Lambda = 0$ and there exist adapted coordinates (t^1, t^2, z^1, z^2) such that $\mathbf{g}$ has either the form

$$\mathbf{g} = (dt^1)^2 + (dt^2)^2 + \psi(dz^1)^2 + 2\left(c\, t^1 dt^2 + dz^2\right) dz^1,$$

with $c \in \mathbb{R}$ and $\psi = \psi(t^1, t^2)$ a differentiable function such that $\psi_{,11} + \psi_{,22} = c^2$, or the form

$$\mathbf{g} = e^{t^1}(dt^1)^2 + e^{t^1}(dt^2)^2 + \psi(dz^1)^2 + 2\left(c\, e^{t^1} dt^2 + dz^2\right) dz^1,$$

with $c \in \mathbb{R}$ and $\psi = \psi(t^1, t^2)$ a differentiable function such that $\psi_{,11} + \psi_{,22} = e^{t^1} c^2$.

In particular, these metrics have $\ell_{\mathscr{C}} = 0$ and are orthogonally transitive if, and only if, $c = 0$. Moreover, they are pp-waves, since ∂_{z^2} is a null Killing vector field such that $\nabla \partial_{z^2} = 0$.

Lorentzian Λ-vacuum Einstein metrics with $\ell_{\mathscr{C}} = 0$

Theorem. Every Lorentzian Λ-vacuum metric with 2-dimensional Abelian Killing algebra, non-null Killing leaves and $\ell_{\mathscr{C}} = 0$ belongs to one of the following families of metrics:

1. *pp*-waves described above for $C_\rho = 0$;

2. Kundu orthogonally intransitive vacuum metrics ($\Lambda = 0$)

$$\frac{1}{\sqrt{x}}\big(dx^2 + dy^2 + (x^{3/2}\psi + 1)\, du^2 + 2\, dy\, du - 2x^{3/2} du dv\big),$$

$$\psi_{xx} + \frac{1}{x}\psi_x + \psi_{yy} = 0,$$

with $\psi = \psi(x, y)$;

Lorentzian Λ-vacuum Einstein metrics with $\ell_{\mathscr{C}} = 0$

3. Λ-vacuum orthogonally intransitive metrics (generalized Kundu)

$$-\frac{3}{\Lambda}\frac{c^2x}{c^2x^3+1}\,dx^2+\frac{c^2x^3+1}{x}\,dy^2+\frac{2}{x}\,dy\,du+\frac{x^3\psi+1}{x}\,du^2+2x^2\,du\,dv,\qquad c,\Lambda\in\mathbb{R}-\{0\}$$

$$\frac{(c^2x^3+1)^2}{x^2}\,\psi_{xx}+\frac{(c^2x^3+1)(4c^2x^3+1)}{x^3}\,\psi_x-3\frac{c^2}{\Lambda}\,\psi_{yy}=0,\qquad \psi=\psi(x,y);$$

with $c,\Lambda\in\mathbb{R}-\{0\}$ and $\psi=\psi(x,y)$;

4. Λ-vacuum orthogonally intransitive metrics (generalized Kundu)

$$-\frac{3}{\Lambda x^2}\,dx^2+\frac{1}{x^2}dy^2+2x\,dy\,du+\frac{2}{x^2}\,du\,dv+\frac{x^6+\psi}{2x^2}\,du^2,$$

$$\psi_{xx}-\frac{2}{x}\,\psi_x-\frac{3}{\Lambda}\,\psi_{yy}=0,$$

with $\Lambda\in\mathbb{R}-\{0\}$ and $\psi=\psi(x,y)$.

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