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A DARBOUX–GETZLER THEOREM FOR DIFFERENCE HAMILTONIAN OPERATORS

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Local and nonlocal geometry of PDEs and Integrability

SISSA, Trieste, 9 October 2018

The general picture

Hamiltonian Equations

$$u_t = P\,\delta H$$

Hamiltonian PDEs		Hamiltonian differential-difference equations
$u_t = uu_x + u_{xxx}$		$u_t = uu_1 - uu_{-1}$
∂	Hamiltonian structure	$u(\mathcal{S} - \mathcal{S}^{-1})u$
$\int \frac{u^3}{6} - \frac{u_x^2}{2}$	Hamiltonian (functional)	$\int u$

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Outline

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An integrable discrete equation (Kac and Van Moerbeke, 1975)

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From Volterra's system to Volterra chain

Lotka-Volterra for predator-prey x, y: $\begin{cases} \dot{x} &= -\alpha x + \beta xy \\ \dot{y} &= -\gamma xy + \delta y \end{cases}$

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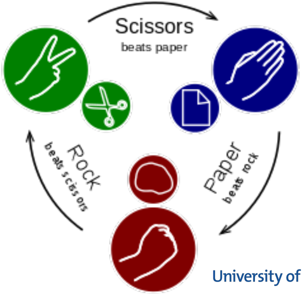
$$\dot{u}_i = \beta_i u_i u_{i+1} - \beta_{i-1} u_i u_{i-1}$$



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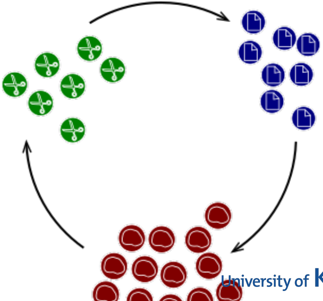

Hamiltonian formulation

$$\frac{\partial u}{\partial t} = uu_1 - uu_{-1}$$

Let $\mathcal{S}$ be the *shift operator*: $\mathcal{S}u_j = u_{j+1}$, $\mathcal{S}(fg) = \mathcal{S}f\mathcal{S}g$, $\mathcal{S}1 = 1$.
Then we can write

$$\frac{\partial u}{\partial t} = u(\mathcal{S} - \mathcal{S}^{-1})u \cdot 1$$

or $u_t = P\delta H/\delta u$ with

$$P = u\left(\mathcal{S} - \mathcal{S}^{-1}\right)u \qquad H = \int u.$$

Local densities

Difference polynomials: Ring of polynomials in the variables u^i_n, $i \in \{1, \dots, \ell\}$, $n \in \mathbb{Z}$, endowed with automorphism $\mathcal{S}$ such that

$$\mathcal{S} u^i_n = u^i_{n+1}, \qquad \mathcal{S}(\alpha f + \beta g) = \alpha \mathcal{S} f + \beta \mathcal{S} g, \qquad \mathcal{S}(fg) = \mathcal{S} f \mathcal{S} g.$$

Difference functions: Extension $(\mathcal{A}, \mathcal{S})$ of the algebra of differential polynomials such that

$$\frac{\partial f}{\partial u^i_n} = 0 \qquad \forall\, f \in \mathcal{A} \text{ for all but finitely many } (i, n)$$

$$\mathcal{S} \frac{\partial f}{\partial u^i_n} = \frac{\partial f}{\partial u^i_{n+1}} \mathcal{S} \qquad \forall\, f \in \mathcal{A}, (i, n)$$

$\mathcal{A}$ is the space of densities of local functionals

Local functionals

Kuperschmidt (1980): *Discrete Lax equations and differential-difference calculus*

We write $F = \int f$, $f \in \mathcal{A}$. But what does $\int$ mean? Recall that, as a function, $u = u(m, t)$, $m \in L$ and $L \to \infty$

$$\int f = \int \mathcal{S} f \quad \Rightarrow \int (\mathcal{S} - 1) f = 0.$$

In analogy with formal calculus of variations for differential calculus, where $\int \partial f = 0$ and $\mathcal{F} = \mathcal{A}/\partial\mathcal{A}$, we define

$$\mathcal{F} = \frac{\mathcal{A}}{(\mathcal{S} - 1)\mathcal{A}}, \quad \int : \mathcal{A} \twoheadrightarrow \mathcal{F}$$

$\mathcal{F}$ Space of local functionals

A *variational derivative* such that $\delta (\mathcal{S} - 1) f = 0$.

$$\frac{\delta f}{\delta u^i} := \sum_{n \in \mathbb{Z}} \mathcal{S}^{-n} \frac{\partial f}{\partial u^i_n}$$

Hamiltonian operators

Local difference operator

$$K = \sum_{s=S}^{T} A_{(s)} \mathcal{S}^s$$

with $A_{(s)} \in \mathcal{A}$. (S, T) is the *order* of the operator.

Hamiltonian operator

The operator K (of order $(-S, S)$) is Hamiltonian if the bracket

$$\{F, G\} := \int \frac{\delta F}{\delta u} K \left(\frac{\delta G}{\delta u} \right)$$

is a Lie algebra bracket, namely

$$\{F, G\} = -\{G, F\} \qquad \{F, \{G, H\}\} + \text{p.c.} = 0$$

Proposition: K skewsymmetric $\Leftrightarrow$ $K = \sum_{s=1}^{S} A_{(s)} \mathcal{S}^s - \mathcal{S}^{-s} A_{(s)}$.

Normal form of local Hamiltonian oper's

Scalar case ($\ell = 1$)

De Sole, Kac, Valeri, Wakimoto (2018). Poisson λ brackets for differential-difference equations.
Classification results up to order $(-5, 5)$
For $(-1, 1)$ order operators:

$$K_g = g(u)g(u_1)\mathcal{S} - g(u)g(u_1)\mathcal{S}^{-1}$$

with $g(u)$ function of single variable.

Normal form

With the change of coordinates $v = \int \frac{1}{g}$

$$K_1 = \mathcal{S} - \mathcal{S}^{-1}$$

constant operator, analogue of ∂ in differential case.

Outline

The cochain complex

Multivectors (cl. mech.)	Local difference multivectors
$\Lambda^0 = C^\infty(M)$	$\Lambda^0 = \mathcal{F}$
$\Lambda^1 = \mathfrak{X}(M) = \Gamma(TM)$	$\Lambda^1 = \mathrm{Der}^{(\mathcal{S}-1)}(\mathcal{A})$ (evolutionary difference v.f.)
$\vdots$	Λ^2 = Skewadjoint differential operators (skewsymmetric brackets)
Λ^k = k-vectors ($\Gamma(TM^{\wedge k})$)	Λ^k = Local k-vector fields ($\mathrm{Alt}(\mathcal{F}^k, \mathcal{F})$)

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$$\hat{X} = \left(\mathcal{S}^n X\right) \frac{\partial}{\partial u_n} \qquad \mathcal{X}(\textstyle\int f) = \displaystyle\int \hat{X}(f) = \int X \frac{\delta \int f}{\delta u}$$

The θ formalism

Idea from supergeometry ($\Lambda(M) \cong T^*[1]M$) used by Getzler in the differential case

Define $\hat{\mathcal{A}} = \mathcal{A}[\{\theta_n,\ n \in \mathbb{Z}\}]$ with $\theta_n\theta_m = -\theta_m\theta_n$ and $\theta_n^2 = 0$. Let $\mathcal{S}\theta_n = \theta_{n+1}$, $\frac{\delta F}{\delta\theta} := \mathcal{S}^{-n}\frac{\partial f}{\partial\theta_n}$. Then

$$\hat{\mathcal{F}} = \frac{\hat{\mathcal{A}}}{(\mathcal{S}-1)\hat{\mathcal{A}}} \cong \Lambda^\bullet, \qquad \hat{\mathcal{F}}^p \cong \Lambda^p$$

where $\hat{\mathcal{F}}^p$ is the component with $\deg_\theta \hat{\mathcal{F}}^p = p$.
For instance,

$$\int X\frac{\delta}{\delta u} \rightsquigarrow \int X\theta \qquad \sum_{s=1}^{S} A_{(s)}\mathcal{S}^s - \mathcal{S}^{-s}A_{(s)} \rightsquigarrow \int \sum_{s=1}^{S} A_{(s)}\theta\theta_s$$

The Schouten bracket

Definition

$$[\ ,\ ]\colon \Lambda^p \times \Lambda^q \to \Lambda^{p+q-1}$$

such that:

1. Coincides with commutator of vector fields for $p = q = 1$
2. Graded commutativity: $[Q, P] = (-1)^{pq}[P, Q]$
3. Graded Jacobi:
   $(-1)^{pr}[[P, Q], R] + (-1)^{qp}[[Q, R], P] + (-1)^{rq}[[R, P], Q] = 0$

Formula

Using the θ formalism, let $P \in \hat{\mathcal{F}}^p$ and $Q \in \hat{\mathcal{F}}^q$:

$$[P, Q] = \int \left(\frac{\delta P}{\delta \theta} \frac{\delta Q}{\delta u} + (-1)^p \frac{\delta P}{\delta u} \frac{\delta Q}{\delta \theta} \right)$$

The Poisson cohomology: definition

Theorem

A local bivector P is Hamiltonian if and only if

$$[P, P] = 0$$

Then $\mathrm{d}_P = [P, \cdot\,]\colon \Lambda^k \to \Lambda^{k+1}$ *is such that* $\mathrm{d}_P^2 = 0$.

Cochain complex

$$0 \longrightarrow \mathcal{F} \xrightarrow{\mathrm{d}_P} \Lambda^1 \xrightarrow{\mathrm{d}_P} \Lambda^2 \xrightarrow{\mathrm{d}_P} \cdots \xrightarrow{\mathrm{d}_P} \Lambda^k \xrightarrow{\mathrm{d}_P} \cdots$$

Definition (Poisson cohomology)

$$H^k(P) = \frac{\ker \mathrm{d}_P\colon \Lambda^k \to \Lambda^{k+1}}{\operatorname{Im} \mathrm{d}_P\colon \Lambda^{k-1} \to \Lambda^k}$$

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The Poisson cohomology

Interpretation of the lowest cohomology groups

- H^0 Casimirs
- H^1 not Hamiltonian symmetries
- H^2 nontrivial infinitesimal deformations
- H^3 obstruction to deformations

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$F \in \mathcal{F}$ such that $\{F, G\} = 0 \ \forall G \in \mathcal{F}$

The Poisson cohomology

Interpretation of the lowest cohomology groups

H^0 Casimirs

H^1 not Hamiltonian symmetries

H^2 nontrivial infinitesimal deformations

H^3 obstruction to deformations

Symmetry $X \in \Lambda^1$ such that $X(\{F, G\}) = \{X(F), G\} + \{F, X(G)\}$ for any $F, G \in \mathcal{F}$

Hamiltonian vector field $X \in \Lambda^1$ such that $X(F) = \{F, H\}$ for any F and one $H \in \mathcal{F}$

The Poisson cohomology

Interpretation of the lowest cohomology groups

H^0 Casimirs

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H^3 obstruction to deformations

$Q \in \Lambda^2$ such that $[P + \epsilon Q, P + \epsilon Q] = O(\epsilon^2)$, but Q is not $[X, P]$.

The Poisson cohomology

Interpretation of the lowest cohomology groups

H^0 Casimirs
H^1 not Hamiltonian symmetries
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$$[P + \epsilon Q, P + \epsilon Q] = \epsilon^2 T$$

Look for R such that $[P + \epsilon Q + \epsilon^2 R, —] = O(\epsilon^3)$.
To do so, one must solve

$$T = -2[P, R] \qquad (*)$$

Since $T \in \ker \mathrm{d}_P$, the equation $(*)$ is always solvable if $H^3 = 0$.

Second cohomology group and classification

Let $P + \epsilon Q$ be an (infinitesimal) deformation of P.
If $H^2(P) = 0$, then there exists $X \in \hat{\mathcal{F}}^1$ such that $Q = \mathrm{ad}_X P$

$$\left(e^{\mathrm{ad}_{\epsilon X}} - 1\right) P = \epsilon Q + \epsilon^2 R + \dots \qquad \text{and} \qquad R = \mathrm{ad}_Y P$$

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$$\left(e^{\mathrm{ad}_{\epsilon X - \epsilon^2 Y}} - 1\right) P = \epsilon Q + \epsilon^3 S + \dots \qquad \text{and} \qquad S = \mathrm{ad}_Z P$$

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and so on...

$$\phi^*(P) = \epsilon Q$$

The differential case: Getzler's theorem

Theorem (*A Darboux theorem for Hamiltonian operators in the formal calculus of variations*, 2002)

For any N, $D = 1$ and P a Hamiltonian operator of first order ($P = \eta^{ij}\partial$) (hydrodynamic type),

$$H^k(\mathrm{d}_P) = 0 \qquad \forall k > 1$$

Moreover H^0_0 and H^1_0 are the only nonvanishing groups.

All the compatible Hamiltonian structures ($[P, Q] = 0$) are trivial deformations of P. There exists a change of coordinates which sends P to Q, and P is constant: "Darboux's" coordinates.

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A (Darboux)-Getzler's theorem for the difference case

The normal form

$$K = \mathcal{S} - \mathcal{S}^{-1}$$

Cohomology: "difference" Getzler

$$H^p(\mathrm{d}_K) = 0 \qquad \forall\, p > 1$$

Moreover,

$$H^0(\mathrm{d}_K) = \left\{ \int (\alpha + \beta u)\,, (\alpha, \beta) \in \mathbb{C}^2 \right\} \quad H^1(\mathrm{d}_K) = \{\text{constant v.f.}\} \cong \mathbb{C}$$

Sketch of the proof – 1

Exact sequences

We have two cochain complexes defined by $K = \int \theta\theta_1 \in \hat{\mathcal{F}}^2$.

On $\hat{\mathcal{F}}$: $\mathrm{d}_K = \int (\theta_1 - \theta_{-1}) \frac{\delta}{\delta u}$

On $\hat{\mathcal{A}}$: $D_K = \sum_n (\theta_{n+1} - \theta_{n-1}) \frac{\partial}{\partial u_n}$

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{F} & \xrightarrow{\mathrm{d}_K} & \hat{\mathcal{F}}^1 & \xrightarrow{\mathrm{d}_K} & \cdots \\
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Short exact sequence:

$$0 \longrightarrow \hat{\mathcal{A}}/\mathbb{C} \xrightarrow{(\mathcal{S}-1)} \hat{\mathcal{A}} \xrightarrow{\int} \hat{\mathcal{F}} \longrightarrow 0$$

Long exact sequence:

$$\cdots \longrightarrow H^p(\hat{\mathcal{A}}/\mathbb{C}) \longrightarrow H^p(\hat{\mathcal{A}}) \longrightarrow H^p(\hat{\mathcal{F}}) \longrightarrow H^{p+1}(\hat{\mathcal{A}}/\mathbb{C}) \longrightarrow \cdots$$

Sketch of the proof – 2

The cohomology $H(\hat{\mathcal{A}})$

We investigate $H(\hat{\mathcal{A}}, D_K)$, with the idea that $(D_K, \hat{\mathcal{A}}) \cong (\mathrm{d}_{DR}, \mathcal{M})$ (*Lichnerowitz*), and (because $\mathcal{M}$ topologically trivial)

$$H(\hat{\mathcal{A}}, D_K) \cong H(\mathcal{M}, \mathrm{d}_{DR}) \cong H^0(\mathcal{M}, \mathrm{d}_{DR}) \cong \text{const. 0-forms on } \mathcal{M}$$

We have

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$$D_K = \sum_n \mathrm{d}u_n \frac{\partial}{\partial u_n} \quad \Rightarrow \quad H(\hat{\mathcal{A}}) = \{\text{consts. in } u_n, \text{ no } \mathrm{d}u_n\}$$

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No θ's such that θ_p and θ_{p+2}. Hence

$$H(\hat{\mathcal{A}}, D_K) = \mathbb{C}[\theta_0, \theta_1] = \langle 1, \theta, \theta_1, \theta\theta_1 \rangle$$

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$$H^p(\hat{\mathcal{A}}, D_K) = 0\ \forall\, p > 2 \qquad\qquad H^p(\hat{\mathcal{F}}, \mathrm{d}_K) = 0\ \forall\, p > 2$$

Sketch of the proof – 3

The lowest groups, explicitly

$H^0(\hat{\mathcal{F}}) \cong \mathbb{C}^2$

From exact sequence, $H^0(\hat{\mathcal{F}}) \cong H^0(\hat{\mathcal{A}}) \oplus \ker(\mathcal{S} - 1)|_{H^1(\hat{\mathcal{A}})}$

1. $H^p(\hat{\mathcal{A}}) \hookrightarrow H^p(\hat{\mathcal{F}})$ by $\int \Rightarrow \int \alpha$
2. The kernel is $\beta(\theta + \theta_1)$, monodimensional $\Rightarrow \int \beta u$

$H^1(\hat{\mathcal{F}}) \cong \mathbb{C}$

$\mathcal{S} - 1 = -2\mathrm{Id}$ on $H^2(\hat{\mathcal{A}})$, whence $\ker(\mathcal{S} - 1) = 0$.
Then $H^1(\hat{\mathcal{F}}) = \int H^1(\hat{\mathcal{A}}) = \int(\alpha\theta + \beta\theta_1) = (\alpha + \beta) \int \theta$.

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Then $H^1(\hat{\mathcal{F}}) = \int H^1(\hat{\mathcal{A}}) = \int(\alpha\theta + \beta\theta_1) = (\alpha+\beta)\int\theta$.

And $H^2(\hat{\mathcal{F}})$?

$$0 \longrightarrow H^2(\hat{\mathcal{A}}) \xrightarrow{-2\mathrm{Id}} H^2(\hat{\mathcal{A}}) \xrightarrow{\int} H^2(\hat{\mathcal{F}}) \longrightarrow 0 \quad \text{and}$$

$$H^2(\hat{\mathcal{F}}) = H^2(\hat{\mathcal{A}}) \Big/ H^2(\hat{\mathcal{A}}) \cong 0.$$

Outline

BiHamiltonian structure of Volterra chain

The Volterra chain $u_t = u(u_1 - u_{-1})$ is in fact biHamiltonian

$$u_t = K_u \frac{\delta H_1}{\delta u} \qquad u_t = \tilde{K}_u \frac{\delta H_0}{\delta u}$$

$$K_u = u(\mathcal{S} - \mathcal{S}^{-1})u \qquad \tilde{K}_{2,u} = uu_1 u_2{}^{v_1}\mathcal{S}^2 + uu_1(u + u_1)\mathcal{S} - \text{skew part}$$

$$H_1 = \int u \qquad H_0 = \int \log u$$

$$[K_u, \tilde{K}_{2,u}] = 0$$

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From the theorem: $\exists\, X$ s.t. $\tilde{K}_{2,1} = [X, K]$ and $X = -\int (e^u + e^{u_1})\,\theta$
Explicit change of coordinates: $v = F(u) = -\log(u) - \log(u_1)$

$$F'_* K F'^{\dagger}_* = \tilde{K}_{2,1}|_{F(u)}$$

Low order pairs

From De Sole, Kac et al.

List up to order $(-5, 5)$ of compatible pairs (we already eliminate $g(u)$ dependance)

i Constant form: $\sum_{k=1}^{t} c_k K_k = \sum_{k=1}^{t} c_k \mathcal{S}^k - \mathcal{S}^{-k}$

ii General form:
- K and $\tilde{K}_{2,1}$
- K and $\tilde{K}_{3,1} = e^{u_1+iu_2}\mathcal{S}^3 + i(e^{u+iu_1} - e^{u_1+iu_2})\mathcal{S}^2 + e^{u+iu_1}\mathcal{S} -$ skew

iii 4-th special: $\tilde{K}_{2,1}$ and $Q_4 = e^{u_1-u_2+u_3}\mathcal{S}^4 + (e^{u-u_1+u_2} + e^{u_1-u_2+u_3})\mathcal{S}^3 + e^{u-u_1+u_2}\mathcal{S}^2 -$ skew

iv 5-th special: $K + K_2$ and $Q_5 = e^{\varepsilon u_2+u_3}\mathcal{S}^5 - \left(\varepsilon e^{\varepsilon u_1+u_2} + \varepsilon^{-1}e^{\varepsilon u_2+u_3}\right)\mathcal{S}^4 + \left(\varepsilon^{-1}e^{\varepsilon u+u_1} + e^{\varepsilon u_1+u_2} + \varepsilon e^{\varepsilon u_2+u_3}\right)\mathcal{S}^3 - \left(\varepsilon e^{\varepsilon u+u_1} + \varepsilon^{-1}e^{\varepsilon u_1+u_2}\right)\mathcal{S}^2 + e^{\varepsilon u+u_1}\mathcal{S} -$ skew
where $\varepsilon^3 = 1$

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where $\varepsilon^3 = 1$

Compatible pairs and vector fields

$(-3, 3)$ order operator

The operator $\tilde{K}_{3,1} = e^{u_1+iu_2}\mathcal{S}^3 + i(e^{u+iu_1} - e^{u_1+iu_2})\mathcal{S}^2 + e^{u+iu_1}\mathcal{S}$ − skew is compatible with K, hence there exist X s.t. $\tilde{K}_{3,1} = [X, K]$.

$$X = -\int (e^{u+iu_1} - ie^{u_1+iu_2})\theta$$

A constant higher order operator

Let us consider $K_4 = \mathcal{S}^4 - \mathcal{S}^{-4}$. It is compatible with K, hence $K_4 = [Y, K]$.

$$X = -\int (u_1 + u_3)\theta \qquad v = F(u) = u - iu_1 + iu_2 + u_3$$

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Thanks for the attention!

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Happy birthday!