Cohomology of Lie algebroids on schemes

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LOCAL an NONLOCAL GEOMETRY
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Why Lie algebroids on schemes?

- A Lie algebroid on a C^∞ manifold is completely determined by a Lie-Rinehart algebra (its global sections). So the theory of Lie algebroids on C^∞ manifolds is inherently local in character
- Moving to the holomorphic or algebraic category allows for a nontrivial mix of the (local) Lie-Rinehart algebra structure with the cohomology of the base variety
- Using schemes (or complex spaces) one can allow for singularities

So the introduction of schemes as "parameter spaces" can be seen as the next step of an algebraization programme, as started for instance in Joseph's papers

- Calculus over commutative algebras: a coincise user guide, Acta Appl. Math. **49** (1997) 235–248
- Poincaré δ-Lemma for smooth algebras, ibid. **49** (1997) 249–255
- Characteristics of linear differential operators over commutative algebras, ibid. **49** (1997) 257–269

Lie algebroids

X: a differentiable manifold, or complex manifold, or a complex space, or a noetherian separated scheme over an algebraically closed field $\Bbbk$

Lie algebroid: a vector bundle/coherent sheaf $\mathscr{C}$ with a morphism of $\mathscr{O}_X$-modules $a\colon \mathscr{C} \to \Theta_X$ and a $\Bbbk$-linear Lie bracket on the sections of $\mathscr{C}$ satisfying

$$[s, ft] = f[s, t] + a(s)(f)\, t$$

for all sections s, t of $\mathscr{C}$ and f of $\mathscr{O}_X$.

- a is a morphism of sheaves of Lie $\Bbbk$-algebras
- $\ker a$ is a bundle of Lie $\mathscr{O}_X$-algebras

Examples

- A sheaf of Lie algebras, with $a = 0$
- Θ_X, with $a = \mathrm{id}$
- More generally, foliations, i.e., a is injective
- Poisson structures $\Omega^1_X \xrightarrow{\pi} \Theta_X$,

  Poisson-Nijenhuis bracket

  $$\{\omega, \tau\} = \mathrm{Lie}_{\pi(\omega)}\tau - \mathrm{Lie}_{\pi(\tau)}\omega - d\pi(\omega, \tau)$$

  Jacobi identity $\Leftrightarrow [\![\pi, \pi]\!] = 0$

- Atiyah algebroid of a vector bundle/coherent sheaf $\mathscr{E}$

$$0 \longrightarrow End(\mathscr{E}) \longrightarrow \mathscr{D}_{\mathscr{E}} \xrightarrow{\sigma} \Theta_X \dashrightarrow 0$$

$\mathscr{D}_{\mathscr{E}}$: sheaf of 1-st order diff. operators on $\mathscr{E}$ with scalar symbol. If $\mathscr{E}$ is a vector bundle a diff. op. D can be locally written

$$D(s)^\alpha = \sum_{i,\beta} A(z)^{\alpha i}_\beta \frac{\partial s^\beta}{\partial z^i} + \sum_\beta B(z)^\alpha_\beta\, s^\beta$$

D has scalar symbol if

$$A(z)^{\alpha i}_\beta = \delta^\alpha_\beta\, v^i(z)$$

$$\sigma(D) = v \quad \text{or} \quad \sigma_\xi(D) = \xi(v)$$

Lie algebroid morphisms

$f\colon \mathscr{C} \to \mathscr{C}'$ a morphism of $\mathscr{O}_X$-modules & sheaves of Lie k-algebras

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{\ f\ } & \mathscr{C}' \\ & {\scriptstyle a}\searrow & \downarrow{\scriptstyle a'} \\ & & \Theta_X \end{array}$$

$\Rightarrow$ $\ker f$ is a bundle of Lie algebras

Lie-Rinehart algebras

A a finitely generated commutative, associative unital algebra over a field $\mathbb{k}$

Lie-Rinehart algebra over $(\mathbb{k}, A)$: a pair (L, a) where

- L is an A-module equipped with a $\mathbb{k}$-linear Lie algebra bracket $\{\,,\,\}$
- $a\colon L \to \mathrm{Der}_{\mathbb{k}}(A)$ is a representation of L in $\mathrm{Der}_{\mathbb{k}}(A)$ (the anchor) that satisfies the Leibniz rule

$$\{s, ft\} = f\{s,t\} + a(s)(f)\,t$$

where $s, t \in L$ and $f \in A$.

If A is a commutative $\mathbb{k}$-algebra and $X = \mathrm{Spec}(A)$, then

$$\textbf{Lie-Alg}_X \simeq \textbf{Lie-Rin}_A$$

Derived functors

$\mathfrak{A}$ an abelian category, $A \in \mathrm{Ob}(\mathfrak{A})$; then

$$\mathrm{Hom}(-, A) \colon \mathfrak{A} \to \mathfrak{Ab}$$

is a (contravariant) left exact functor, i.e., if $0 \to B' \to B \to B'' \to 0$ is exact, then

$$0 \to \mathrm{Hom}(B'', A) \to \mathrm{Hom}(B, A) \to \mathrm{Hom}(B', A)$$

is exact as well

Definition

$I \in \mathrm{Ob}(\mathfrak{A})$ *is injective if* $\mathrm{Hom}(-, I)$ *is exact, i.e., if*

$$0 \to \mathrm{Hom}(B'', I) \to \mathrm{Hom}(B, I) \to \mathrm{Hom}(B', I) \to 0$$

is exact whenever $0 \to B' \to B \to B'' \to 0$ *is exact*

Definition

The category $\mathfrak{A}$ has enough injectives if every object in $\mathfrak{A}$ has an injective resolution

$$0 \to A \to I^0 \to I^1 \to I^2 \to \dots$$

$\mathfrak{A}$ abelian category with enough injectives

$$F\colon \mathfrak{A} \to \mathfrak{B} \quad \text{left exact functor}$$

Derived functors $R^iF\colon \mathfrak{A} \to \mathfrak{B}$

$$R^iF(A) = H^i(F(I^\bullet))$$

Example: Sheaf cohomology. X topological space, $\mathfrak{A} = \mathfrak{Sh}_X$, $\mathfrak{B} = \mathfrak{Ab}$, $F = \Gamma$ (global sections functor)

$$R^i\Gamma(\mathscr{F}) = H^i(X, \mathscr{F})$$

Hyperfunctors

$\mathfrak{A}$ category with enough injectives, $F\colon \mathfrak{A} \to \mathfrak{B}$ left exact functor

$\mathscr{K}^\bullet$ complex of objects in $\mathfrak{A}$, $\mathscr{I}^\bullet$ quasi-isomorphic injective complex

(i.e. there is a morphism $\mathscr{K}^\bullet \to \mathscr{I}^\bullet$ which is an isomorphism in cohomology)

$$\boxed{\mathbb{R}^i F(\mathscr{K}^\bullet) = H^i(F(\mathscr{I}^\bullet))}$$

Example (Hypercohomology): $\mathfrak{A} = \mathfrak{Sh}_X$, $\mathfrak{B} = \mathfrak{Ab}$, $F = \Gamma$ (global sections functor)

$\mathscr{K}^\bullet \in K_+(\mathfrak{Sh}_X)$

$$\boxed{\mathbb{H}^i(X, \mathscr{K}^\bullet) = H^i(\Gamma(\mathscr{I}^\bullet))}$$

(Hyper)cohomology of a Lie algebroid

$\mathscr{C}$ Lie algebroid over a scheme, $(\rho, \mathscr{M})$ a representation, i.e., $\mathscr{M}$ is a coherent $\mathscr{O}_X$-module, and

$$\rho \colon \mathscr{C} \to \mathscr{D}_{\mathscr{M}}$$

is a Lie algebroid morphism

$$\Omega(\mathscr{C},\mathscr{M})^\bullet = \mathscr{M} \otimes_{\mathscr{O}_X} \Lambda^\bullet_{\mathscr{O}_X} \mathscr{C}^*, \qquad \partial_{\mathscr{C},\mathscr{M}} \colon \Omega(\mathscr{C},\mathscr{M})^\bullet \to \Omega(\mathscr{C},\mathscr{M})^{\bullet+1}$$

$$\begin{aligned}(\partial_{\mathscr{C},\mathscr{M}}\xi)(s_1,\dots,s_{p+1}) &= \sum_{i=1}^{p+1}(-1)^{i-1}\rho(s_i)(\xi(s_1,\dots,\hat{s}_i,\dots,s_{p+1})) \\ &+ \sum_{i<j}(-1)^{i+j}\xi([s_i,s_j],\dots,\hat{s}_i,\dots,\hat{s}_j,\dots,s_{p+1})\end{aligned}$$

for $s_1,\dots,s_{p+1}$ sections of $\mathscr{C}$, and ξ a section of $\Omega(\mathscr{C},\mathscr{M})^p$

$\Rightarrow$ hypercohomology $\mathbb{H}^\bullet(\Omega(\mathscr{C},\mathscr{M})^\bullet, \partial_{\mathscr{C},\mathscr{M}}) =: \mathbb{H}^\bullet(\mathscr{C};\mathscr{M})$

In the previous examples this reduces to

- Cartain-Eilenberg Lie algebra cohomology
- de Rham cohomology
- foliated de Rham cohomology
- Lichnerowicz-Poisson cohomology

The Lie algebroid cohomology of the Atiyah algebroid of a vector bundle was studied in

B., V. Rubtsov, Cent. Eur. J. Math. **10** (2012) 1442–1454.

An even remoter predecessor (in the C^∞ category) is

V. N. Rubtsov, Cohomology of Der-complex, Russian Math. Surv., **35** (4) (1980), pp. 190–191.

The category Rep($\mathscr{C}$)

From now on, X will be a scheme (with the previous hypotheses)

Given a Lie algebroid $\mathscr{C}$ there is a notion of enveloping algebra $\mathfrak{U}(\mathscr{C})$

It is a sheaf of associative $\mathscr{O}_X$-algebras with a $\mathbb{k}$-linear augmentation $\mathfrak{U}(\mathscr{C}) \to \mathscr{O}_X$

$$\boxed{\text{Rep}(\mathscr{C}) \simeq \mathfrak{U}(\mathscr{C})\text{-}\mathbf{mod}}$$

$\Rightarrow$ Rep($\mathscr{C}$) has enough injectives

A $\Bbbk$-algebra with an algebra monomorphism $\imath\colon A \to \mathfrak{U}(L)$ and a $\Bbbk$-module morphism $\jmath\colon L \to \mathfrak{U}(L)$, such that

$$[\jmath(s), \jmath(t)] - \jmath([s,t]) = 0\,, \quad s, t \in L\,,$$
$$[\jmath(s), \imath(f)] - \imath(a(s)(f)) = 0\,, \quad s \in L, f \in A \qquad (*)$$

Construction: standard enveloping algebra $U(A \rtimes L)$ of the semi-direct product $\Bbbk$-Lie algebra $A \rtimes L$

$$\mathfrak{U}(L) = U(A \rtimes L)/V, \qquad V = \langle f(g,s) - (fg, fs)\rangle$$

- $\mathfrak{U}(L)$ is an A-module via the morphism $\imath$
- due to (*) the left and right A-module structures are different
- morphism $\varepsilon\colon \mathfrak{U}(L) \to \mathfrak{U}(L)/I = A$ (the augmentation morphism) where I is the ideal generated by $\jmath(L)$. Note that ε is a morphism of $\mathfrak{U}(L)$-modules but not of A-modules, as $\varepsilon(fs) = a(s)(f)$ when $f \in A$, $s \in L$.

Lie alg. cohomology as derived functor

Given a representation $(\rho, \mathscr{M})$ of $\mathscr{M}$ define

$$\mathscr{M}^{\mathscr{C}}(U) = \{m \in \mathscr{M}(U) \,|\, \rho(\mathscr{C})(m) = 0\}$$

and a left exact functor

$$\begin{aligned} I^{\mathscr{C}} \colon \ \mathrm{Rep}(\mathscr{C}) \ &\to \ \mathbb{k}\textbf{-mod} \\ \mathscr{M} \ &\mapsto \ \Gamma(X, \mathscr{M}^{\mathscr{C}}) \end{aligned}$$

Theorem (B 2016[1])

If $\mathscr{C}$ is locally free

$$\mathbb{H}^{\bullet}(\mathscr{C}; \mathscr{M}) \simeq R^{\bullet} I^{\mathscr{C}}(\mathscr{M})$$

([1]) J. of Algebra **483** (2017) 245–261

Proof

A δ-functor is a collection of functors $\{S^i : \mathfrak{A} \to \mathfrak{B}\}$ such that for every exact sequence $0 \to A \to B \to C \to 0$ in $\mathfrak{A}$ there are morphisms $\sigma^i : S^i(C) \to S^{i+1}(A)$ giving rise to a long exact sequence

$$0 \to S^0(A) \to S^0(B) \to S^0(C) \xrightarrow{\sigma^0} S^1(A) \\ \to S^1(B) \to S^1(C) \xrightarrow{\sigma^1} S^1(A) \to \dots$$

functorial w.r.t. morphisms of exact sequences

Theorem

If $\{S^\bullet\}$, $\{T^\bullet\}$ are δ-functors $\mathfrak{A} \to \mathfrak{B}$ such that

- $S^i(I) = T^i(I) = 0$ *for all* $i > 0$ *when* I *is an injective object*
- $S^0 \simeq T^0$

then $S^i \simeq T^i$ *for all* $i \geq 0$.

We apply this to the functors $I^{\mathscr{C}}$ and

$$\mathbb{H}^i(\mathscr{C}; -)\colon \operatorname{Rep}(\mathscr{C}) \to \Bbbk\text{-}\mathbf{mod}$$

When $\mathscr{C}$ is not locally free this method only provides morphisms

$$R^i I^{\mathscr{C}}(\mathscr{M}) \to H^i(\mathscr{C}; \mathscr{M})$$

Grothendieck's thm about composition of derived functors

$$\mathfrak{A} \xrightarrow{F} \mathfrak{B} \xrightarrow{G} \mathfrak{C}$$

$\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, abelian categories

$\mathfrak{A}$, $\mathfrak{B}$ with enough injectives

F and G left exact, F sends injectives to G-acyclics (i.e., $R^i G(F(I)) = 0$ for $i > 0$ when I is injective)

Theorem

For every object A in $\mathfrak{A}$ there is a spectral sequence abutting to $R^\bullet(G \circ F)(A)$ whose second page is

$$E_2^{pq} = R^p F(R^q G(A))$$

Local to global

$$\begin{array}{ccc} \mathrm{Rep}(\mathscr{C}) & \xrightarrow{(-)^{\mathscr{C}}} & \mathbb{k}_X\textbf{-mod} \\ & {\scriptstyle I^{\mathscr{C}}} \searrow & \downarrow{\scriptstyle \Gamma} \\ & & \mathbb{k}\textbf{-mod} \end{array}$$

Grothendieck's theorem on the derived functors of a composition of functors implies:

Theorem (Local to global spectral sequence)

There is a spectral sequence, converging to $\mathbb{H}^\bullet(\mathscr{C};\mathscr{M})$, whose second term is

$$E_2^{pq} = H^p(X, \mathscr{H}^q(\mathscr{C};\mathscr{M}))$$

Hochschild-Serre

Extension of Lie algebroids

$$0 \to \mathscr{K} \to \mathscr{E} \to \mathscr{Q} \to 0$$

$\mathscr{K}$ is a sheaf of Lie $\mathscr{O}_X$-algebras

$$\begin{array}{ccc} \mathrm{Rep}(\mathscr{E}) & \xrightarrow{(-)^{\mathscr{K}}} & \mathrm{Rep}(\mathscr{Q}) \\ & {\scriptstyle I^{\mathscr{E}}} \searrow & \downarrow {\scriptstyle I^{\mathscr{Q}}} \\ & & \Bbbk\text{-}\mathbf{mod} \end{array}$$

Moreover, the sheaves $\mathscr{H}^q(\mathscr{K};\mathscr{M})$ are representations of $\mathscr{Q}$

Theorem (Hochschild-Serre type spectral sequence)

For every representation $\mathscr{M}$ of $\mathscr{E}$ there is a spectral sequence E converging to $\mathbb{H}^\bullet(\mathscr{E};\mathscr{M})$, whose second page is

$$E_2^{pq} = \mathbb{H}^p(\mathscr{Q};\mathscr{H}^q(\mathscr{K};\mathscr{M})).$$

Also proved in

B, Mencattini, Rubtsov, Tortella, Nonabelian holomorphic Lie algebroid extensions, Internat. J. Math. **26** (2015) 1550040

generalizing Hochschild-Serre's paper, i.e., considering the filtration $F^p\Omega^\bullet_{\mathscr{E}}$

$$F^p\Omega^q_{\mathscr{E}} = \begin{array}{l}\text{Sheaf of sections of } \Omega(\mathscr{E})^q \text{ that are annihilated by}\\ \text{the inner product with } q-p+1 \text{ sections of } \mathscr{K}\end{array}$$

The extension problem

An extension

$$0 \to \mathscr{K} \to \mathscr{E} \xrightarrow{\pi} \mathscr{Q} \to 0 \tag{1}$$

defines a morphims

$$\begin{aligned} \alpha\colon \mathscr{Q} &\to \mathscr{O}ut(Z(\mathscr{K})) \\ \alpha(x)(y) &= \{y, x'\} \qquad \text{where} \quad \pi(x') = x \end{aligned} \tag{2}$$

The extension problem is the following:

Given a Lie algebroid $\mathscr{Q}$, a coherent sheaf of Lie $\mathscr{O}_X$-algebras $\mathscr{K}$, and a morphism α as in (2), does there exist an extension as in (1) which induces the given α?

We assume $\mathscr{Q}$ is locally free

Abelian extensions

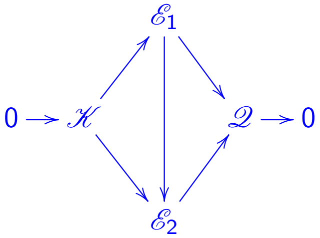
If $\mathscr{K}$ is abelian, $(\mathscr{K}, \alpha)$ is a representation of $\mathscr{Q}$ on $\mathscr{K}$, and one can form the semidirect product

$$\mathscr{E} = \mathscr{K} \rtimes_\alpha \mathscr{Q},$$

$$\mathscr{E} = \mathscr{K} \oplus \mathscr{Q} \quad \text{as } \mathscr{O}_X\text{-modules},$$
$$\{(\ell, x), (\ell', x')\} = (\alpha(x)(\ell') - \alpha(x')(\ell), \{x, x'\})$$

Theorem ([2])

If $\mathscr{K}$ is abelian, the extension problem is unobstructed; extensions are classified up to equivalence by the hypercohomology group $\mathbb{H}^2(\mathscr{Q}; \mathscr{K})^{(1)}_\alpha$



([2]) U.B., I. Mencattini, V. Rubtsov, and P. Tortella, Nonabelian holomorphic Lie algebroid extensions, op. cit.

$\mathscr{M}$ a representation of a Lie algebroid $\mathscr{C}$. Sharp truncation of the Chevalley-Eilenberg complex $\sigma^{\geq 1}\Lambda^\bullet\mathscr{C}^* \otimes \mathscr{M}$ defined by

$$0 \longrightarrow \mathscr{C}^* \otimes \mathscr{M} \longrightarrow \Lambda^2\mathscr{C}^* \otimes \mathscr{M} \longrightarrow \cdots$$

We denote $\mathbb{H}^i(\mathscr{C};\mathscr{M})^{(1)} := \mathbb{H}^i(X, \sigma^{\geq 1}\Lambda^\bullet\mathscr{C}^* \otimes \mathscr{M})$

Derivation of $\mathscr{C}$ in $\mathscr{M}$: morphism $d\colon \mathscr{C} \to \mathscr{M}$ such that

$$d(\{x,y\}) = x(d(y)) - y(d(x))$$

Proposition

The functors $\mathbb{H}^i(\mathscr{C};-)^{(1)}$ are, up to a shift, the derived functors of

$$\begin{aligned} \mathrm{Der}(\mathscr{C};-)\colon\ \mathrm{Rep}(\mathscr{C}) &\to \mathbb{k}\text{-}\mathbf{mod} \\ \mathscr{M} &\mapsto \mathrm{Der}(\mathscr{C},\mathscr{M}) \end{aligned}$$

i.e.,

$$R^i\,\mathrm{Der}(\mathscr{C};-) \simeq \mathbb{H}^{i+1}(\mathscr{C};-)^{(1)}$$

Realize the hypercohomology using Čech cochains: if $\mathscr{U}$ is an affine cover of X, and $\mathscr{F}^\bullet$ a complex of sheaves on X, then $\mathbb{H}^\bullet(X, \mathscr{F}^\bullet)$ is isom. to the cohomology of the total complex T of

$$K^{p,q} = \check{C}^p(\mathscr{U}, \mathscr{F}^q)$$

$$0 \longrightarrow \mathscr{K}_{|U_i} \longrightarrow \mathscr{E}_{|U_i} \underset{s_i}{\overset{\pi}{\rightleftarrows}} \mathscr{Q}_{|U_i} \longrightarrow 0 \tag{3}$$

If $U_i \in \mathscr{U}$, $\mathrm{Hom}(\mathscr{Q}_{|U_i}, \mathscr{E}_{|U_i}) \to \mathrm{Hom}(\mathscr{Q}_{|U_i}, \mathscr{Q}_{|U_i})$ is surjective, so that one has splittings s_i

$$\{\phi_{ij} = s_i - s_j\} \in \check{C}^1(\mathscr{U}, \mathscr{K} \otimes \mathscr{Q}^*)$$

This is a 1-cocycle, which describes the extension as an extension of $\mathscr{O}_X$-modules

$$0 \to \mathscr{K}(U_i) \to \mathscr{E}(U_i) \to \mathscr{Q}(U_i) \to 0$$

is an exact sequence of Lie-Rinehart algebras (over $(\mathbb{k}, \mathscr{O}_X(U_i))$)

which is described by a 2-cocycle ψ_i in the Chevalley-Eilenberg (-Rinehart) cohomology of $\mathscr{Q}(U_i)$ with coefficients in $\mathscr{K}(U_i)$

$$(\phi, \psi) \in \check{C}^1(\mathscr{U}, \mathscr{K} \otimes \mathscr{Q}^*) \oplus \check{C}^0(\mathscr{U}, \mathscr{K} \otimes \Lambda^2 \mathscr{Q}^*) = T^2$$

$$\delta\phi = 0, \qquad d\phi + \delta\psi = 0, \qquad d\psi = 0$$

$\Rightarrow$ cohomology class in $\mathbb{H}^2(\mathscr{Q}; \mathscr{K})^{(1)}_\alpha$

The nonabelian case

Theorem ([2,3])

If $\mathscr{K}$ is nonabelian, the extension problem is obstructed by a class $\mathbf{ob}(\alpha)$ *in* $\mathbb{H}^3(\mathscr{Q}; Z(\mathscr{K}))^{(1)}_\alpha$.

If $\mathbf{ob}(\alpha) = 0$, *the space of equivalence classes of extensions is a torsor on* $\mathbb{H}^2(\mathscr{Q}; Z(\mathscr{K}))^{(1)}_\alpha$.

Proof

$\mathscr{Q}$ can be written as a quotient of a free Lie algebroid $\mathscr{F}$

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & \mathscr{J} & & & & \\
 & & & & \downarrow & & & & \\
0 & \to & \mathscr{N} & \to & \mathfrak{U}(\mathscr{F}) & \to & \mathfrak{U}(\mathscr{Q}) & \to & 0 \\
 & & & & \downarrow & & & & \\
 & & & & \mathscr{O}_X & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}
$$

([3]) E. Aldrovandi, U.B., V. Rubtsov, Lie algebroid cohomology and Lie algebroid extensions, J. of Algebra **505** (2018) 456–481

$$\widetilde{\mathscr{N}}^i = \mathscr{N}^i/\mathscr{N}^{i+1}, \qquad \widetilde{\mathscr{J}}^i = \mathscr{N}^i\mathscr{J}/\mathscr{N}^{i+1}\mathscr{J}, \quad \text{for } i = 0, \dots$$

Locally free resolution

$$\dots \to \widetilde{\mathscr{N}}^2 \to \widetilde{\mathscr{J}}^1 \to \widetilde{\mathscr{N}}^1 \to \widetilde{\mathscr{J}}^0 \to \mathscr{J} \to 0$$

As $\mathrm{Hom}_{\mathfrak{U}(\mathscr{Q})}(\mathscr{J}, Z(\mathscr{K})) \simeq \mathrm{Der}(\mathscr{Q}, Z(\mathscr{K}))$, applying the functor $\mathrm{Hom}_{\mathfrak{U}(\mathscr{Q})}(-, Z(\mathscr{K}))$ we obtain

$$0 \to \mathrm{Der}(\mathscr{Q}, Z(\mathscr{K})) \to \mathrm{Hom}_{\mathfrak{U}(\mathscr{Q})}(\widetilde{\mathscr{J}}^0, Z(\mathscr{K})) \xrightarrow{d_1} \mathrm{Hom}_{\mathfrak{U}(\mathscr{Q})}(\widetilde{\mathscr{K}}^1, Z(\mathscr{K})) \xrightarrow{d_2} \mathrm{Hom}_{\mathfrak{U}(\mathscr{Q})}(\widetilde{\mathscr{J}}^1, Z(\mathscr{K})) \xrightarrow{d_3} \mathrm{Hom}_{\mathfrak{U}(\mathscr{Q})}(\widetilde{\mathscr{K}}^2, Z(\mathscr{K})) \to \dots$$

The cohomology of this complex is isomorphic to $\mathbb{H}^{\bullet+1}(\mathscr{Q}; Z(\mathscr{K}))$.

Pick a lift $\tilde{\alpha}\colon \mathscr{F} \to \mathscr{D}er(\mathscr{K})$ of α and get commutative diagram

$$\begin{array}{ccccccccccc} & & 0 & \longrightarrow & \mathscr{T} & \longrightarrow & \mathscr{F} & \longrightarrow & \mathscr{Q} & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\tilde{\alpha}} & & \downarrow{\scriptstyle\alpha} & & \\ 0 & \longrightarrow & Z(\mathscr{K}) & \longrightarrow & \mathscr{K} & \xrightarrow{\text{ad}} & \mathscr{D}er(\mathscr{K}) & \longrightarrow & \mathscr{O}ut(\mathscr{K}) & \longrightarrow & 0 \end{array}$$

where β is the induced morphism.

Define a morphism

$$o\colon \widetilde{\mathscr{J}}^{1} \to Z(\mathscr{K}) \tag{4}$$

It is enough to define o on an element of the type yx, where x is a generator of $\mathscr{F}$, and y is a generator of $\mathscr{T}$

$$o(yx) = \beta(\{x, y\}) - \tilde{\alpha}(x)(\beta(y)).$$

Note that $o \in \mathrm{Hom}_{\mathfrak{U}(\mathscr{Q})}(\widetilde{\mathscr{J}}^{1}, Z(\mathscr{K}))$.

Lemma

$d_3(o) = 0$. *Moreover, the cohomology class of* $[o] \in \mathbb{H}^3(\mathscr{Q}; Z(\mathscr{K}))^{(1)}$ *only depends on* α.

Part I of the proof: if an extension exists consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{T} & \longrightarrow & \mathscr{F} & \longrightarrow & \mathscr{Q} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \| & & \\
0 & \longrightarrow & \mathscr{K} & \longrightarrow & \mathscr{E} & \longrightarrow & \mathscr{Q} & \longrightarrow & 0
\end{array}$$

Define

$$\tilde{\alpha} \colon \mathscr{F} \to \mathscr{D}er(\mathscr{K}, \mathscr{K}), \qquad \tilde{\alpha} = -\operatorname{ad} \circ \gamma$$

Then $\tilde{\alpha}$ is a lift of α, and for all sections t of $\mathscr{T}$ and x of $\mathscr{F}$

$$\beta(\{x, t\}) - \tilde{\alpha}(x)(\beta(t)) = 0 \qquad (5)$$

so that the obstruction class $\mathbf{ob}(\alpha)$ vanishes.

Conversely, assume that $\mathbf{ob}(\alpha) = 0$, and take a lift $\tilde{\alpha}\colon \mathscr{F} \to \mathscr{D}er(\mathscr{K}, \mathscr{K})$. The corresponding cocycle lies in the image of the morphism d_2, so it defines a morphism $\beta\colon \mathscr{T} \to \mathscr{K}$, which satisfies the equation (5). Again, we consider the extension

$$0 \to \mathscr{T} \to \mathscr{F} \to \mathscr{Q} \to 0.$$

Note that $\mathscr{K}$ is an $\mathscr{F}$-module via $\mathscr{F} \to \mathscr{Q}$. The semidirect product $\mathscr{K} \rtimes \mathscr{F}$ contains the sheaf of Lie algebras

$$\mathscr{H} = \{(\ell, x) \,|\, x \in \mathscr{T},\ \ell = \beta(x)\}.$$

The quotient $\mathscr{E} = \mathscr{K} \rtimes \mathscr{F} / \mathscr{H}$ provides the desired extension

Part II of the proof: reduction to the abelian case

Proposition

Once a reference extension $\mathscr{E}_0$ *has been fixed, the equivalence classes of extensions of* $\mathscr{Q}$ *by* $\mathscr{K}$ *inducing* α *are in a one-to-one correspondence with equivalence classes of extensions of* $\mathscr{Q}$ *by* $Z(\mathscr{K})$ *inducing* α*, and are therefore in a one-to-one correspondence with the elements of the group* $\mathbb{H}^2(\mathscr{Q}; Z(\mathscr{K}))^{(1)}$

$\mathscr{C}_1$, $\mathscr{C}_2$ Lie algebroids with surjective morphisms $f_i \colon \mathscr{C}_i \to \mathscr{Q}$. Assuming $Z(\ker f_1) \simeq Z(\ker f_2) = \mathscr{Z}$ define

$$\mathscr{C}_1 \star \mathscr{C}_2 = \mathscr{C}_1 \times_{\mathscr{Q}} \mathscr{C}_2 / \mathscr{Z},$$

where $\mathscr{Z} \to \mathscr{C}_1 \times_{\mathscr{Q}} \mathscr{C}_2$ by $z \mapsto (z, -z)$

Fix a reference extension $\mathscr{E}_0$ of $\mathscr{Q}$ by $\mathscr{K}$

Lemma

(1) Any extension $\mathscr{E}$ of $\mathscr{Q}$ by $\mathscr{K}$ is equivalent to a product $\mathscr{E}_0 \star \mathscr{D}$ where $\mathscr{D}$ is an extension of $\mathscr{Q}$ by $Z(\mathscr{K})$

(2) Given two extensions $\mathscr{D}_1$, $\mathscr{D}_2$ of $\mathscr{Q}$ by $Z(\mathscr{K})$, the extensions $\mathscr{E}_1 = \mathscr{E}_0 \star \mathscr{D}_1$ and $\mathscr{E}_2 = \mathscr{E}_0 \star \mathscr{D}_2$ are equivalent if and only if $\mathscr{D}_1$ and $\mathscr{D}_2$ are equivalent

Congratulations Joseph!!