Matrix Painlevé equations

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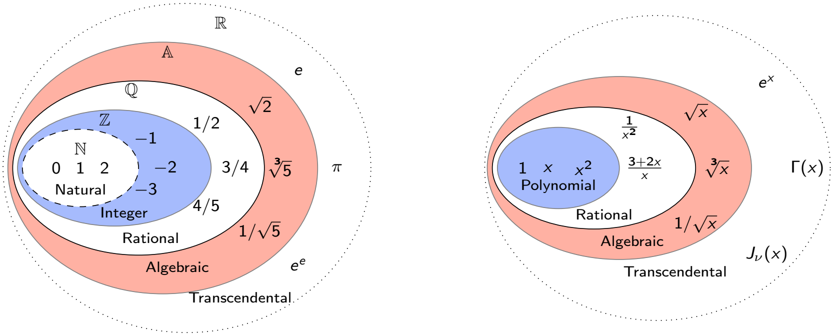
Historical remarks



Number systems and function classes [Joshi, 2019]

- Problem: define new functions by an ODE of the m^{th} order with properties that generalize those of elliptic functions. [L. Fuchs], [H. Poincaré]
- Painlevé property: the general solution of an ODE has no critical movable points.
- $m = 2$: the Painlevé transcendents. [Painlevé, 1902], [Gambier, 1910]

The Painlevé-1 equation: scalar case

$$y'' = 6y^2 + z, \qquad y(z),\ z \in \mathbb{C}. \qquad \mathrm{P}_1$$

- Symmetry reduction of the KdV equation:

$$w_t + 12ww_x + w_{xxx} = 0 \quad \Rightarrow \quad \left| \begin{array}{c} w(x,t) = -y(z) + t, \\ z(x,t) = x - 6t^2 \end{array} \right| \quad \Rightarrow \quad y'' = 6y^2 + z$$

the KdV equation | **the Galilean transformation** | **the P_1 equation**

- System:

$$\begin{cases} u' = v, \\ v' = 6u^2 + z, \end{cases} \quad \Leftrightarrow \quad \mathrm{P}_1 \quad \text{for} \quad y(z) = u(z).$$

- Isomonodromic property:

$$dY(\lambda,z) = (A(\lambda,z)d\lambda + B(\lambda,z)dz)\, Y(\lambda,z) \underset{\text{the flatness condition}}{\Longrightarrow} \partial_z A - \partial_\lambda B = [B,A]$$

with matrices A and B [Jimbo and Miwa, 1981]

$$A(\lambda,z) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\lambda^2 + \begin{pmatrix} 0 & u \\ 4 & 0 \end{pmatrix}\lambda + \begin{pmatrix} -v & u^2 + \frac{1}{2}z \\ -4u & v \end{pmatrix}, \quad B(\lambda,z) = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix}\lambda + \begin{pmatrix} 0 & u \\ 4 & 0 \end{pmatrix}.$$

- The Painlevé-Kovalevskaya (PK) test:

$$y(z) = (z-z_0)^{-2} - \frac{z_0}{10}(z-z_0)^2 - \frac{1}{6}(z-z_0)^3 + \sigma\,(z-z_0)^4 + \dots .$$

The Painlevé-1 equation: matrix case

$$y'' = 6y^2 + z\,\mathbb{I} + a, \qquad y(z),\ a \in \mathrm{Mat}_n(\mathbb{C}), \qquad z \in \mathbb{C}. \qquad \mathrm{P}_1^0$$

- Symmetry reduction of the matrix KdV equation:

$$w_t + 6ww_x + 6w_xw + w_{xxx} = 0 \quad \Rightarrow \quad \begin{array}{c} w(x,t) = -y(z) + t\,\mathbb{I}, \\ z(x,t) = x - 6t^2 \end{array} \quad \Rightarrow \quad y'' = 6y^2 + z\,\mathbb{I} + a$$

the ${}_{\text{mat}}$KdV equation | **the Galilean transformation** | **the P_1^0 equation**

- System:

$$\begin{cases} u' = v, \\ v' = 6u^2 + z\,\mathbb{I} + a, \end{cases} \quad \Leftrightarrow \quad \mathrm{P}_1^0 \quad \text{for} \quad y(z) = u(z).$$

- Isomonodromic representation with matrices A and B

$$A(\lambda, z) = \begin{pmatrix} 0 & \mathbb{I} \\ 0 & 0 \end{pmatrix} \lambda^2 + \begin{pmatrix} 0 & u \\ 4\,\mathbb{I} & 0 \end{pmatrix} \lambda + \begin{pmatrix} -v & u^2 + \frac{1}{2} z\,\mathbb{I} + \frac{1}{2} a \\ -4u & v \end{pmatrix},$$

$$B(\lambda, z) = \begin{pmatrix} 0 & \frac{1}{2}\mathbb{I} \\ 0 & 0 \end{pmatrix} \lambda + \begin{pmatrix} 0 & u \\ 4\,\mathbb{I} & 0 \end{pmatrix}.$$

- The matrix PK test [Balandin and Sokolov, 1998]:

$$y'' = 6y^2 + z\,b + a, \qquad a,\ b \in \mathrm{Mat}_n(\mathbb{C}), \qquad \text{iff} \qquad b = \beta\,\mathbb{I}, \quad \beta \in \mathbb{C}. \tag{1}$$

Classification problems

The Painlevé-2 equation

Scalar case

$$y'' = 2y^3 + zy + \left(\theta - \tfrac{1}{2}\right), \qquad y(z),\ z,\ \theta \in \mathbb{C}. \qquad \mathrm{P}_2$$

- Self-similar reduction of the modified KdV equation:

$$w_t - 6w^2 w_x + w_{xxx} = 0 \quad \Rightarrow \quad \begin{array}{c} w(x,t) = y(z)\,(3t)^{-1/3}, \\ z(x,t) = x\,(3t)^{-1/3} \end{array} \quad \Rightarrow \quad y'' = 2y^3 + zy + \tilde{\theta}$$

the mKdV equation — the transformation — the P_2 equation

- System:

$$\left\{ \begin{array}{lcl} u' & = & -u^2 + v - \frac{1}{2}z, \\ v' & = & 2uv + \theta, \end{array} \right. \quad \Leftrightarrow \quad \mathrm{P}_2 \quad \text{for} \quad y(z) = u(z).$$

- Lax pairs: [Flaschka and Newell, 1980], [Jimbo and Miwa, 1981], [Harnad et al., 1993].

Matrix case [Adler and Sokolov, 2021]

- Ansatz for matrix generalizations:

$$\left\{ \begin{array}{lcl} u' & = & -u^2 + v - \frac{1}{2}z\,\mathbb{I} - b_1, \\ v' & = & 2vu + \beta\,[v,u] + b_2 u + u b_3 + b_4, \end{array} \right. \qquad u(z),\ v(z),\ b_i \in \mathrm{Mat}_n(\mathbb{C}), \quad z,\ \beta \in \mathbb{C}.$$

- There are three non-equivalent systems that pass the matrix PK test: P_2^0, P_2^1, P_2^2.
- Symmetry reductions: ${}_{\mathrm{mat}}\mathrm{NLS} \to \mathrm{P}_2^0$, ${}_{\mathrm{mat}}\mathrm{mKdV}^1 \to \mathrm{P}_2^1$, ${}_{\mathrm{mat}}\mathrm{mKdV}^2 \to \mathrm{P}_2^2$.
- Degeneracies: $\mathrm{P}_2^i \to \mathrm{P}_1^0$ for any $i = 0, 1, 2$.

The Painlevé-4 equation

Scalar case

$$y'' = \frac{1}{2} y^{-1} y'^2 + \frac{3}{2} y^3 + 4zy^2 + 2(z-\gamma)y + \delta y^{-1}, \qquad y(z),\ z,\ \gamma,\ \delta \in \mathbb{C}. \qquad \mathrm{P}_4$$

- Self-similar reduction of the modified NLS equation: mNLS $\to$ P_4.
- System:

$$\begin{cases} u' &= -u^2 + 2uv - 2zu + \theta_1, \\ v' &= -v^2 + 2uv + 2zv + \theta_2, \end{cases} \quad \Leftrightarrow \quad \mathrm{P}_4 \quad \text{for} \quad y(z) = u(z).$$

- Isomonodromic Lax pairs: [Jimbo and Miwa, 1981], [Kitaev, 1985].

Matrix case [Bobrova and Sokolov, 2021a], [Bobrova and Sokolov, 2021b]

- Ansatz for matrix generalizations:

$$\begin{cases} u' &= -u^2 + 2uv + \alpha\,[u, v] - 2zu + b_1 u + u b_2 + b_3 v + v b_4 + b_5, \\ v' &= -v^2 + 2vu + \beta\,[v, u] + 2zv + c_1 v + v c_2 + c_3 u + u c_4 + c_5, \end{cases}$$

  where $u(z),\ v(z),\ b_i,\ c_i \in \mathrm{Mat}_n(\mathbb{C}), \quad z,\ \alpha,\ \beta \in \mathbb{C}$.
- There are three non-equivalent systems that pass the matrix PK test: P_4^0, P_4^1, P_4^2.
- Each of the systems has a non-abelian version of the isomonodromic JM-pair.
- Degeneracies: $\mathrm{P}_4^i \to \mathrm{P}_2^i, \quad i = 0, 1, 2$.

The Painlevé-Kovalevskaya test

The Painlevé-Kovalevskaya test: scalar case

- Suppose that we have a scalar ODE of the m^{th} order for $y = y(z)$, $z \in \mathbb{C}$.
- Identify all possible dominant balances of $y(z)$ near a singular point z_0:

$$y(z) \sim c_k (z - z_0)^k \quad \text{as} \quad z \to z_0.$$

- If all k are integer, then for every **negative** k find possible resonances r.
- If all r are integer, then verify that all resonant conditions are satisfied.
- If for every k there are m arbitrary constants (maximal solution), then the test is passed.

Example

- Consider the scalar system

$$\begin{cases} u' = -u^2 + 2uv - 2zu + c_1, \\ v' = -v^2 + 2uv + 2zv + c_2, \end{cases}$$

- One of the dominant balances of the general formal solution near z_0,

$$u \sim p\,(z - z_0)^{k_1}, \qquad v \sim q\,(z - z_0)^{k_2}, \qquad z \to z_0,$$

is given by $k_1 = k_2 = -1$. Then the residues p, q are defined by

$$p(-p + 2q + 1) = 0, \qquad q(-q + 2p + 1) = 0.$$

- Therefore, there exist three types of the general formal solution:

$$\mathbf{1}: \quad u = -(z - z_0)^{-1} + O(1), \quad v = -(z - z_0)^{-1} + O(1);$$

$$\mathbf{2}: \quad u = (z - z_0)^{-1} + O(1), \quad v = O(1); \quad \mathbf{3}: \quad u = O(1), \quad v = (z - z_0)^{-1} + O(1).$$

- Each of them is maximal. For instance,

$$\mathbf{1}: \quad u = -(z - z_0)^{-1} + \sigma (z - z_0)^2 + \dots, \quad v = -(z - z_0)^{-1} - \sigma (z - z_0)^2 + \dots.$$

The Painlevé-Kovalevskaya test: matrix case

$$y^{(m)}(z) = P\left(z, y(z), y'(z), \dots, y^{(m-1)}(z)\right), \quad y(z) \in \mathrm{Mat}_n(\mathbb{C}), \quad z \in \mathbb{C}, \tag{2}$$

where $P\left(z, y(z), y'(z), \dots, y^{(m-1)}(z)\right)$ is a homogeneous polynomial in all its variables with scalar coefficients.

- Identify all possible dominant balances k in a neighborhood of a singular point z_0:

$$y(z) \sim c_k (z - z_0)^k \qquad \text{as} \qquad z \to z_0, \qquad c_k \in \mathrm{Mat}_n(\mathbb{C}).$$

- If all k are integer, then for every **negative** k determine the form of the leading matrix coefficient c_k. Since c_k can be conjugated by an element $G \in GL_n(\mathbb{C})$, the number of arbitrary constants is equal to $\dim \mathcal{O}_{c_k}$.
- By substitution of the Laurent expansion with matrix coefficients $c_l \in \mathrm{Mat}_n(\mathbb{C})$, $l \geq k$, into (2), for every k write recurrence relations for the matrix coefficients c_l in the form

$$(\mathcal{L} - j\,\mathbb{I})(c_l) = f(c_{l-1}), \quad j, l \in \mathbb{Z},$$

where the linear operator $\mathcal{L} : \mathrm{Mat}_n(\mathbb{C}) \to \mathrm{Mat}_n(\mathbb{C})$ depends only on c_k.
- Possible resonances r can appear in the case when the operator $(\mathcal{L} - j\,\mathbb{I})$ is not invertible. If all r are integer, then for every r verify the solvability of the resonance conditions.

Concl. If for every k there exist $\mathrm{m} \cdot \mathrm{n}^2$ arbitrary constants, then the test is satisfied.

Example 1: a polynomial ODE

$$y'(z) = -y^2(z), \qquad y(z) \in \mathrm{Mat}_n(\mathbb{C}), \qquad z \in \mathbb{C}. \tag{3}$$

- General formal solution:

$$y(z) = \sum_{l \geq -1} c_l (z - z_0)^l, \quad c_l \in \mathrm{Mat}_n(\mathbb{C}). \tag{4}$$

- The substitution of expansion (4) into (3) leads to the system of recurrence relations

$$c_{-1} = c_{-1}^2; \qquad (\mathcal{L} - l\,\mathbb{I})\,(c_l) = f(c_{l-1}), \quad l \in \mathbb{Z}_{\geq 0}, \tag{5}$$

  where the linear operator $\mathcal{L} : \mathrm{Mat}_n(\mathbb{C}) \to \mathrm{Mat}_n(\mathbb{C})$ and the function $f(c_l)$ are given by

$$\mathcal{L}(c_l) = -c_{-1}c_l - c_l c_{-1}; \qquad f(c_{-1}) \equiv 0, \quad f(c_l) = \frac{1}{2}\sum_{m=0}^{l}(c_m c_{l-m} + c_{l-m} c_m).$$

- $c_{-1} = \mathrm{diag}\{\mathbb{I}_{n_1}, 0_{n_2}\}$, with $n_1 + n_2 = n$. Hence, $\dim \mathcal{O}_{c_{-1}} = n^2 - (n_1^2 + n_2^2)$.
- The dimensions d_λ of the eigenspaces corresponding to the eigenvalues λ of the operator $\mathcal{L}$:

$$d_{-2} = n_1^2, \qquad d_{-1} = 2n_1 n_2, \qquad d_0 = n_2^2.$$

- The resonance appears at $l = 0$. Thus, the number of arbitrary constants equals

$$d_0 + \dim \mathcal{O}_{c_{-1}} = n^2 - n_1^2 = n^2 - 1 \qquad \Leftrightarrow \qquad n_1 = 1.$$

- All resonance conditions are satisfied, as $f(c_{-1}) = 0$.
- Therefore, the general formal solution (4) is maximal, and the test is passed.

Example 2: a system of polynomial ODEs

$$\left\{\begin{array}{lcl} u' &=& -u^2+2\,uv+\alpha[u,v], \\ v' &=& -v^2+2\,vu+\beta[v,u], \end{array}\right. \qquad u(z),\ v(z)\in \mathrm{Mat}_n(\mathbb{C}), \qquad z,\ \alpha,\ \beta\in\mathbb{C}. \tag{6}$$

- General formal solution:

$$u=\frac{p}{z-z_0}+x_0+x_1(z-z_0)+\dots, \qquad v=\frac{q}{z-z_0}+y_0+y_1(z-z_0)+\dots. \tag{7}$$

- Recurrence relations:

$$-p^2+2pq+\alpha\,[p,q]+p=0, \qquad -q^2+2qp+\beta\,[q,p]+q=0, \tag{8}$$

$$(\mathcal{L}-k\,\mathbb{I})\begin{pmatrix}x_k\\ y_k\end{pmatrix}=\begin{pmatrix}f_\alpha\,(x_{k-1},y_{k-1})\\ f_\beta\,(y_{k-1},x_{k-1})\end{pmatrix}, \qquad k\in\mathbb{Z}_{\geq 0}, \tag{9}$$

where $f_\gamma\,(X_{-1},Y_{-1})\overset{def}{=}0$ and for any $k\in\mathbb{N}$ the function $f_\gamma\,(X_k,Y_k)$ is defined by

$$f_\gamma\,(X_k,Y_k)\overset{def}{=}\sum_{l=0}^{k}\left(\frac{1}{2}(X_lX_{k-l}+X_{k-l}X_l)-2X_lY_{k-l}-\gamma\,[X_l,Y_{k-l}]\right). \tag{10}$$

- The linear operator $\mathcal{L}:\mathrm{Mat}_n\,(\mathbb{C})\oplus\mathrm{Mat}_n\,(\mathbb{C})\to\mathrm{Mat}_n\,(\mathbb{C})\oplus\mathrm{Mat}_n\,(\mathbb{C})$ acts as

$$\mathcal{L}\begin{pmatrix}X\\ Y\end{pmatrix}=\begin{pmatrix}-pX-Xp+2\,(pY+Xq)+\alpha\,([p,Y]+[X,q])\\ -qY-Yq+2\,(qX+Yp)+\beta\,([q,X]+[Y,p])\end{pmatrix}. \tag{11}$$

Isomonodromic Lax pairs

The first approach: reductions

Idea:

- Find an integrable matrix PDE, whose reduction gives a matrix analog of a Painlevé equation.
- Rewrite the zero-curvature representation for the PDE as the compatibility condition for the isomonodromic problem for the matrix Painlevé generalization.

Remark.

- The reduction procedure is usually much more complicated in the matrix case.
- How to find such a PDE?

Example: ${}_{\mathrm{mat}}\mathrm{KdV} \to \mathrm{P}_1^0$

- The matrix KdV equation,

$$w_t + 6ww_x + 6w_xw + w_{xxx} = 0,$$

has the following zero-curvature representation $\partial_t U - \partial_x V = [V, U]$ with matrices

$$U(\mu, x, t) = \begin{pmatrix} 0 & \frac{1}{2}\mu\,\mathbb{I} + w \\ -2\,\mathbb{I} & 0 \end{pmatrix}, \quad V(\mu, x, t) = \begin{pmatrix} -2w_x & 2\mu^2\,\mathbb{I} + 2\mu w - 4w^2 - w_{xx} \\ -8\mu\,\mathbb{I} + 8w & 2w_x \end{pmatrix}.$$

- Using the transformation $w(x,t) = -y\,(z(x,t)) + t\,\mathbb{I}$, $z(x,t) = x - 6t^2$, $\lambda(t) = \mu + 2t$, this representation becomes

$$\partial_z A - \partial_\lambda B = [B, A], \qquad B(\lambda, z) = U\,(\lambda, z) = \begin{pmatrix} 0 & \frac{1}{2}\lambda\,\mathbb{I} - y \\ -2\,\mathbb{I} & 0 \end{pmatrix},$$

$$A(\lambda, z) = \tfrac{1}{2}V\,(\lambda, z) + 6tU\,(\lambda, z) = \begin{pmatrix} y' & \lambda^2\,\mathbb{I} - \lambda y + y^2 + \frac{1}{2}z\,\mathbb{I} + \frac{1}{2}a \\ -4\lambda\,\mathbb{I} - 4y & -y' \end{pmatrix}.$$

- The compatibility condition $\partial_z A - \partial_\lambda B = [B, A]$ is equivalent to the matrix P_1^0 equation.

The second approach: a non-abelinization

Idea:

- All scalar Painlevé equations have an isomonodromic Lax pair [Jimbo and Miwa, 1981]. Some of them has several Lax pairs.
- Try to write a non-abelian analog of a scalar pair, using the undetermined coefficients. Namely:
  - Form an ansatz for the matrix pair replacing parameters and variables by non-commutative ones such that in the case of 1×1-matrices this ansatz coincides with the scalar pair.
  - Rewrite all commutative terms in a non-abelian form, e.g.

$$u^2 v \mapsto k_1 u^2 v + k_2 uvu + k_3 vu^2, \qquad k_1 + k_2 + k_3 = 1, \qquad k_i \in \mathbb{C}.$$

  - Reconstruct all possible parameters that cannot be removed by the shifts or scalings in the matrix case.
- The compatibility condition gives restrictions on this coefficients.

Remark.

- The scalar pair should be written in the most general form, it cannot be simplified by the shifts, since such shifts may not exist in the matrix case.
- How to find a scalar pair that can be generalized to the matrix case?

Example: the P_1 equation

Scalar case

- The system

$$u' = v, \qquad v' = 6u^2 + z,$$

has the following isomonodromic representation

$$\partial_z A - \partial_\lambda B = [B, A].$$

- Matrices A and B are given by

$$A(\lambda, z) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \lambda^2 + \begin{pmatrix} 0 & u \\ 4 & 0 \end{pmatrix} \lambda + \begin{pmatrix} -v & u^2 + \frac{1}{2}z \\ -4u & v \end{pmatrix},$$

$$B(\lambda, z) = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix} \lambda + \begin{pmatrix} 0 & u \\ 4 & 0 \end{pmatrix}.$$

- Note that we can make the shift $z \mapsto z + a, \quad a \in \mathbb{C}$.

Matrix case

- When a becomes a matrix, no shift like the above can remove it.
- Reconstructing this shift in the pair, we obtain a Lax pair for the matrix P_1^0 equation:

$$A(\lambda, z) = \begin{pmatrix} 0 & \mathbb{I} \\ 0 & 0 \end{pmatrix} \lambda^2 + \begin{pmatrix} 0 & u \\ 4\,\mathbb{I} & 0 \end{pmatrix} \lambda + \begin{pmatrix} -v & u^2 + \frac{1}{2}z\,\mathbb{I} + \frac{1}{2}a \\ -4u & v \end{pmatrix},$$

$$B(\lambda, z) = \begin{pmatrix} 0 & \frac{1}{2}\mathbb{I} \\ 0 & 0 \end{pmatrix} \lambda + \begin{pmatrix} 0 & u \\ 4\,\mathbb{I} & 0 \end{pmatrix}.$$

Matrix generalizations of the Painlevé-4 equation

Homogeneous matrix P_4 type systems (1)

$$\begin{cases} u' = -u^2 + 2\,uv + \alpha[u,v], \\ v' = -v^2 + 2\,vu + \beta[v,u], \end{cases} \qquad u(z),\ v(z) \in \mathrm{Mat}_n(\mathbb{C}), \quad z,\ \alpha,\ \beta \in \mathbb{C}. \qquad (6)$$

Theorem. [Bobrova and Sokolov, 2021a]
System (6) has three different maximal solutions (7) iff $(\alpha, \beta) \in \Sigma$.

- For the points that are vertices of the star in the Figure, two of these series have commuting residues, and for the third one the residues do not commute.
- For the remaining seven points, the residues commute for all maximal series.

▶ When $[p, q] = 0$, the form of residues is

$$p = \mathrm{diag}(\,-\mathbb{I}_{k_1}, \mathbb{I}_{k_2}, 0_{k_3}, 0_{k_4}),$$
$$q = \mathrm{diag}(\,-\mathbb{I}_{k_1}, 0_{k_2}, \mathbb{I}_{k_3}, 0_{k_4});$$

1 : $k_1 = 1, \quad k_2 = k_3 = 0, \quad k_4 = n - 1;$

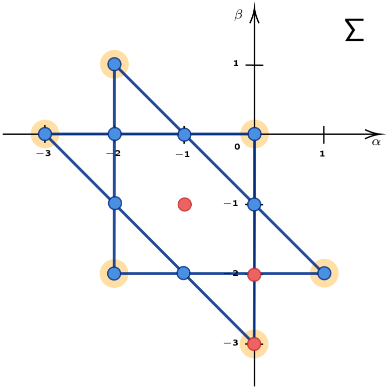
2 : $k_2 = 1, \quad k_1 = k_3 = 0, \quad k_4 = n - 1;$

3 : $k_3 = 1, \quad k_1 = k_2 = 0, \quad k_4 = n - 1.$

▶ When $[p, q] \neq 0$, we have

$$p = \mathrm{diag}(X_2, 0_{n-2}), \quad q = \mathrm{diag}(Y_2, 0_{n-2}),$$

where $X_2,\ Y_2$ are some 2×2 matrices defined by solutions of system (8).



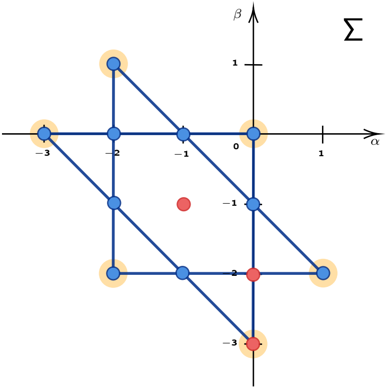
Homogeneous matrix P_4 type systems (2)

$$\begin{cases} u' = -u^2 + 2\,uv + \alpha[u,v], \\ v' = -v^2 + 2\,vu + \beta[v,u], \end{cases} \qquad u(z),\ v(z) \in \mathrm{Mat}_n(\mathbb{C}), \quad z,\ \alpha,\ \beta \in \mathbb{C}. \qquad (6)$$

- System (6) admits the following symmetries

$$\mathbf{1)}\quad u \leftrightarrow v, \quad \alpha \leftrightarrow \beta; \qquad \mathbf{2)}\quad u \mapsto u^T, \quad v \mapsto v^T, \quad \alpha \mapsto -\alpha - 2, \quad \beta \mapsto -\beta - 2;$$
$$\mathbf{3)}\quad u \mapsto -u, \quad v \mapsto v - u, \quad \alpha \mapsto \alpha, \quad \beta \mapsto -\alpha - \beta - 3. \qquad (12)$$

- The group generated by involutions (12) of the (α, β)-plane is isomorphic to D_{12}.
- The point $(-1, -1)$, the six point marked with the dots surrounded by an orange rim, and the remaining six points form three orbits of the group action.



Homogeneous matrix P_4 type systems (2)

$$\begin{cases} u' = -u^2 + 2\,uv + \alpha[u,v], \\ v' = -v^2 + 2\,vu + \beta[v,u], \end{cases} \qquad u(z),\ v(z) \in \mathrm{Mat}_n(\mathbb{C}), \quad z,\ \alpha,\ \beta \in \mathbb{C}. \qquad (6)$$

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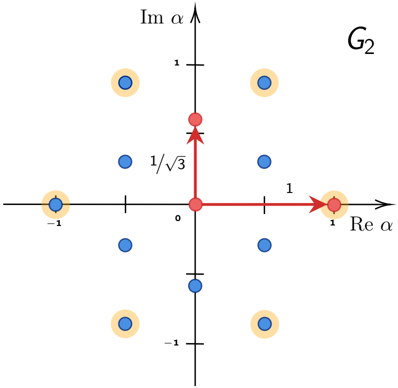
$$\textbf{1)}\quad u \leftrightarrow v, \quad \alpha \leftrightarrow \beta; \qquad \textbf{2)}\quad u \mapsto u^T, \quad v \mapsto v^T, \quad \alpha \mapsto -\alpha - 2, \quad \beta \mapsto -\beta - 2;$$
$$\textbf{3)}\quad u \mapsto -u, \quad v \mapsto v - u, \quad \alpha \mapsto \alpha, \quad \beta \mapsto -\alpha - \beta - 3. \qquad (12)$$

- The group generated by involutions (12) of the (α, β)-plane is isomorphic to D_{12}.
- The point $(-1, -1)$, the six point marked with the dots surrounded by an orange rim, and the remaining six points form three orbits of the group action.
- There is another equivalent version of system (6), where $\beta = \bar{\alpha}$ and α's are defined by the root system of G_2-type:

$$\begin{cases} u' = v^2 + \alpha[u,v], \\ v' = u^2 + \beta[v,u]. \end{cases}$$



Deformations of the matrix homogeneous P_4 type systems

Scalar case

- Consider the linear deformation of the homogeneous part of the P_4 system:

$$\begin{cases} u' = -u^2 + 2uv - 2zu + b_1 u + b_2 v + c_1, \\ v' = -v^2 + 2uv + 2zv + b_3 v + b_4 u + c_2, \end{cases} \qquad b_i, c_i \in \mathbb{C}. \tag{13}$$

- The PK test is passed for every type of the general solution of system (13), if the following conditions for the coefficients b_i hold

$$\mathbf{1}: \quad b_3 = -b_1, \quad b_4 = -b_2; \qquad \mathbf{2}: \quad b_3 = -b_1, \quad b_4 = 0; \qquad \mathbf{3}: \quad b_3 = -b_1, \quad b_2 = 0.$$

- We will reconstruct the P_4 system, if all three solutions are maximal.

All maximal solutions of a homogeneous system have to allow a prolongation to solutions of the inhomogeneous system while preserving the property of their maximality.

Matrix case

- We will consider the following linear deformation of (6)

$$\begin{cases} u' = -u^2 + 2\,uv + \alpha[u, v] - 2zu + b_1 u + ub_2 + b_3 v + vb_4 + b_5, \\ v' = -v^2 + 2\,vu + \beta[v, u] + 2zv + c_1 v + vc_2 + c_3 u + uc_4 + c_5, \end{cases} \qquad b_i, c_i \in \mathrm{Mat}_n(\mathbb{C}).$$

Matrix generalisations of Painlevé-4

$$\begin{cases} u' = -u^2 + 2\,uv + \alpha[u,v] - 2zu + b_1\,u + ub_2 + b_3\,v + vb_4 + b_5, \\ v' = -v^2 + 2\,vu + \beta[v,u] + 2zv + c_1\,v + vc_2 + c_3\,u + uc_4 + c_5. \end{cases} \tag{14}$$

1) $u \mapsto iv, \quad v \mapsto iu, \quad z \mapsto -iz;$
2) $u \mapsto u^T, \quad v \mapsto v^T;$
3) $u \mapsto -iu + R_1, \quad v \mapsto i(v - u - 2z) + R_2, \quad z \mapsto -iz, \quad R_i \in \mathrm{Mat}_n(\mathbb{C});$
4) $u \mapsto e^{zK}\,(u + Q_1)\,e^{-zK}, \quad v \mapsto e^{zK}\,(v + Q_2)\,e^{-zK}, \quad K,\ Q_i \quad \in \mathrm{Mat}_n(\mathbb{C}).$

Theorem. [Bobrova and Sokolov, 2021a]
Any system (14) that satisfies the matrix Painlevé-Kovalevskaya test can be reduced by transformations **1) – 4)** to one of the following:

$$\begin{cases} u' = -u^2 + uv + vu - 2zu + hu + \gamma_1\,\mathbb{I}, \\ v' = -v^2 + vu + uv + 2zv - vh + \gamma_2\,\mathbb{I}, \end{cases} \qquad \mathrm{P}_4^0$$

$$\begin{cases} u' = -u^2 + 2uv - 2zu + h, \\ v' = -v^2 + 2uv + 2zv + h + \gamma\,\mathbb{I}, \end{cases} \qquad \mathrm{P}_4^1$$

$$\begin{cases} u' = -u^2 + 2uv - 2zu + h_2, \\ v' = -v^2 + 3uv - vu + 2zv + h_1\,u + 2h_2 + \gamma\,\mathbb{I}. \end{cases} \qquad \mathrm{P}_4^2$$

Here γ, $\gamma_i \in \mathbb{C}$, h is an arbitrary matrix and two constant matrices h_1, h_2 are connected by the commutation relation $[h_2, h_1] = -2h_1$.

Isomonodromic Lax pairs

Theorem. [Bobrova and Sokolov, 2021b]
Each of systems P_4^0 – P_4^2 has an isomonodromic representation of the form

$$\partial_z A - \partial_\zeta B = [B, A], \tag{15}$$

where ζ is the spectral parameter and $A(\zeta, z)$, $B(\zeta, z)$ are some $2n \times 2n$-matrices.

Example: the P_4^0 system

- The system

$$\begin{cases} u' = -u^2 + uv + vu - 2zu + hu + \gamma_1 \mathbb{I}, \\ v' = -v^2 + vu + uv + 2zv - vh + \gamma_2 \mathbb{I}, \end{cases} \qquad \gamma_i \in \mathbb{C}. \qquad P_4^0$$

  is equivalent to the compatibility condition (15).
- The Lax pair is given by

$$\begin{aligned} B(\zeta, z) &= \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix} \zeta + \begin{pmatrix} v - z\,\mathbb{I} + h & uv + \gamma_1 \mathbb{I} \\ \mathbb{I} & z\,\mathbb{I} \end{pmatrix}, \\ A(\zeta, z) &= \begin{pmatrix} -\mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix} \zeta + \begin{pmatrix} z\,\mathbb{I} & -uv - \gamma_1 \mathbb{I} \\ -\mathbb{I} & -z\,\mathbb{I} + h \end{pmatrix} + \frac{1}{2} \begin{pmatrix} uv + \frac{1}{2}\gamma_2 \mathbb{I} & -uvu - \gamma_2 u \\ v & -vu - \frac{1}{2}\gamma_2 \mathbb{I} \end{pmatrix} \zeta^{-1}. \end{aligned} \tag{16}$$

Degeneracies: $P_4 \rightarrow P_2$

$\mathrm{P}_4 \to \mathrm{P}_2$: scalar case

$$\begin{cases} u' = -u^2 + 2uv - 2zu + \theta_1, \\ v' = -v^2 + 2uv + 2zv + \theta_2, \end{cases} \qquad \theta_i \in \mathbb{C}. \tag{17}$$

- Consider the following transformation for the scalar P_4 system (17)

$$z = \tfrac{1}{4}\varepsilon^{-3} - \varepsilon\, x, \qquad u(z) = -\tfrac{1}{4}\varepsilon^{-3} - \varepsilon^{-1} f(x), \qquad v(z) = -2\varepsilon\, g(x), \tag{18}$$

$$c_1 = -\,\tfrac{1}{16}\varepsilon^{-6}, \quad c_2 = 2\theta, \quad \theta \in \mathbb{C}.$$

- It leads to the ε-depended system:

$$\begin{cases} f' = 2\varepsilon^2 (2fg - xf) - f^2 + g - \tfrac{1}{2}x, \\ g' = 2\varepsilon^2 \left(-g^2 + xg\right) + 2fg + \theta. \end{cases} \tag{19}$$

- Taking $\varepsilon \to 0$, the resulting system is equivalent to the P_2 equation for $y(x) = f(x)$.
- Supplement (18) by $\zeta = 2\varepsilon\lambda$ to obtain from the scalar JM-pair for the P_4 system the Harnad–Tracy–Widom pair [Harnad et al., 1993] for the P_2 system:

$$\begin{aligned} B(\lambda, x) &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\lambda + \begin{pmatrix} -f & 0 \\ -1 & f \end{pmatrix}, \\ A(\lambda, x) &= \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\lambda + \begin{pmatrix} -2f & 2f^2 - g + x \\ -2 & 2f \end{pmatrix} + \begin{pmatrix} \tfrac{1}{2}\theta & 0 \\ -g & -\tfrac{1}{2}\theta \end{pmatrix}\lambda^{-1}. \end{aligned} \tag{20}$$

$P_4 \to P_2$: matrix case

- In the matrix case we have the same transformations as in the scalar one.
- The following degeneracies hold

$$P_4^0 \to P_2^0, \qquad P_4^1 \to P_2^1, \qquad P_4^2 \to P_2^2.$$

Example: $P_4^0 \to P_2^0$

- For the pair (16) let us apply transformations (18) and $\zeta = 2\varepsilon\lambda$ taking together with

$$h = 4\varepsilon b, \qquad \gamma_1 = -\frac{1}{16}\varepsilon^{-6}, \qquad \gamma_2 = 2\theta, \qquad b \in \mathrm{Mat}_n(\mathbb{C}), \qquad \theta \in \mathbb{C}.$$

- Then, passing to the limit $\varepsilon \to 0$, we obtain matrices

$$B(\lambda, x) = \begin{pmatrix} 0 & \mathbb{I} \\ 0 & 0 \end{pmatrix} \lambda + \begin{pmatrix} -f & 0 \\ -\mathbb{I} & f \end{pmatrix},$$
$$A(\lambda, x) = \begin{pmatrix} 0 & 2\,\mathbb{I} \\ 0 & 0 \end{pmatrix} \lambda + \begin{pmatrix} -2f & 2f^2 - g + x\,\mathbb{I} + 2b \\ -2\,\mathbb{I} & 2f \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \theta\,\mathbb{I} & 0 \\ -2g & -\theta\,\mathbb{I} \end{pmatrix} \lambda^{-1}. \tag{21}$$

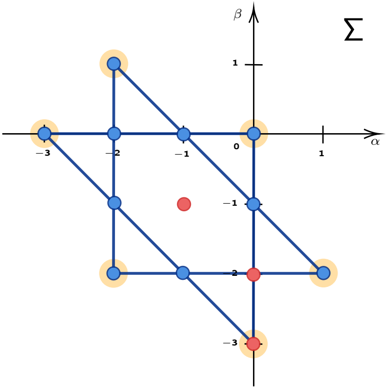
- The latter defines the Lax pair for a system equivalent to the P_2^0 equation:

$$\begin{cases} f' = -f^2 + g - \frac{1}{2}x\,\mathbb{I} - b, \\ g' = fg + gf + \theta\,\mathbb{I}. \end{cases} \tag{22}$$

Conclusion and further questions

Conclusion

- The matrix PK test allows us to find out such matrix Painlevé type systems that can admit an isomonodromic representation.
- Methods to construct isomonodormic Lax pairs:
  - reductions of integrable PDEs;
  - a non-abelinization of known scalar pairs.
- In the considered classes of matrix generalizations of the Painlevé type systems, there are three non-equivalent matrix P_4^i and P_2^i systems, $i = 0, 1, 2$.
- The degeneration procedure can be extended to the matrix case: $P_4^i \to P_2^i \to P_1^0$.



What else?

- Consider more general classes.
- Detect matrix analogs for other Painlevé equations.
- For a given matrix generalization of a Painlevé equation try to find its isomonodromic representation by a non-abelinization procedure of the well-known Lax pairs.
- Try to generalize known results related to the scalar Painlevé equations such as Bäcklund transformations, special solutions, hierarchies, etc.

Many thanks!

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