Bi-presymplectic separability theory

Maciej Błaszak

Poznań University, Poland

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Preliminaries

Given a manifold $\mathcal{M}$ of $\dim \mathcal{M} = m,$ a *Poisson operator* Π of co-rank r on $\mathcal{M}$ is a bivector $\Pi \in \Lambda^2(\mathcal{M})$ with vanishing Schouten bracket:

$$[\Pi, \Pi]_S = 0,$$

whose kernel is spanned by exact one-forms

$$\ker \Pi = Sp\{dc_i\}_{i=1,\ldots,r}.$$

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c_i functions are called Casimirs. In a local coordinate system $(x^1, \ldots, x^m)$ on $\mathcal{M}$ we have

$$\Pi = \sum_{i<j}^{m} \Pi^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j},$$

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while the Poisson property takes the form

$$\sum_l (\Pi^{jl}\partial_l \Pi^{ik} + \Pi^{il}\partial_l \Pi^{kj} + \Pi^{kl}\partial_l \Pi^{ji}) = 0, \quad \partial_i := \frac{\partial}{\partial x^i}.$$

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Let $C(\mathcal{M})$ denote the space of all smooth real-valued functions on $\mathcal{M}$.
Having a Poisson tensor we can define a Hamiltonian vector fields on $\mathcal{M}$. A vector field X_F related to a function $F \in C(\mathcal{M})$ by the relation

$$X_F = \Pi dF, \tag{3}$$

is called the *Hamiltonian vector field* with respect to the Poisson operator Π.

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A linear combination $\Pi_\lambda = \Pi_1 + \lambda \Pi_0$ ($\lambda \in \mathbb{R}$) of two Poisson operators Π_0 and Π_1 is called a *Poisson pencil* if the operator Π_λ is Poisson for any value of the parameter λ.

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When all Casimir functions of Π_λ are polynomials in parameter λ then we say that the pencil is of Gel'fand-Zakharevich (GZ) type.

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In local coordinate system $(x^1, \ldots, x^m)$ on $\mathcal{M}$ we can represent Ω as

$$\Omega = \sum_{i<j}^{m} \Omega_{ij} dx^i \wedge dx^j,$$

where the closeness condition takes the form

$$\partial_i \Omega_{jk} + \partial_k \Omega_{ij} + \partial_j \Omega_{ki} = 0.$$

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Any non-degenerate closed two form on $\mathcal{M}$ is called a *symplectic* form. The inverse of a symplectic form is an *implectic* operator, i.e. invertible Poisson tensor.

Preliminaries

A pair (Π, Ω) is called dual implectic-symplectic pair on $\mathcal{M}$ if Π is non-degenerate Poisson tensor, Ω is non-degenerate closed two-form and the following partition of unity holds on $T\mathcal{M}$, respectively on $T^*\mathcal{M}$:

$$I = \Pi\Omega \quad and \quad I = \Omega\Pi.$$

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So, in the non-degenerate case, a dual pair is a pair of mutually inverse operators on $\mathcal{M}$.

Moreover, the Hamiltonian and the inverse Hamiltonian representations are equivalent as for any implectic bivector Π there is a unique dual symplectic form $\Omega = \Pi^{-1}$ and hence a vector field Hamiltonian with respect to Π is an inverse Hamiltonian with respect to Ω.

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1. $\ker \Pi = Sp\{dc_i : i = 1, \ldots r\}$.
2. $\ker \Omega = Sp\{Z_i : i = 1, \ldots r\}$.
3. $Z_i(c_j) = \delta_{ij}$, $i = 1, 2 \ldots r$.

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3. $Z_i(c_j) = \delta_{ij}$, $i = 1, 2 \ldots r$.
4. The following partition of unity holds on $T\mathcal{M}$, respectively on $T^*\mathcal{M}$

$$I = \Pi\Omega + \sum_{i=1}^{r} Z_i \otimes dc_i, \qquad I = \Omega\Pi + \sum_{i=1}^{r} dc_i \otimes Z_i,$$

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A presymplectic form Ω plays the role of an 'inverse' of Poisson bivector Π in the sense that on any symplectic leaf of the foliation defined by $\ker \Pi$, the restrictions of Ω and Π are inverses of each other.

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The respective freedom exists for a new Π' dual to Ω.

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Then,

$$dF = \Omega(X_F) + \sum_{i=1}^{r} Z_i(F) dc_i, \qquad X_F = X^F - \sum_{i=1}^{r} X^F(c_i) Z_i.$$

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It means that an inverse Hamiltonian vector field X^F is simultaneously a Hamiltonian vector field X_F, i.e. $X^F = X_F$, if dF is annihilated by $\ker(\Omega)$ and X^F is annihilated by $\ker(\Pi)$.

Any dual pair (Π, Ω) defines a Poisson algebra on $C^\infty(M)$

$$\{F, G\}_\Pi := \Pi(dF, dG) = \Omega(\Pi dF, \Pi dG)$$
$$= \Omega(X_F, X_G) =: \{F, G\}^\Omega, \quad F, G \in C^\infty(M).$$

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Lemma Let (Π_0, Ω_0) be a dual implectic-symplectic pair.

(i) Let a Poisson tensor Π_1 be d-compatible with Π_0. Then, $\Omega_1 = \Omega_0 \Pi_1 \Omega_0$ is closed two-form d-compatible with Ω_0.

(ii) Let a closed two-form Ω_1 be d-compatible with Ω_0. Then, $\Pi_1 = \Pi_0 \Omega_1 \Pi_0$ is a Poisson tensor d-compatible with Π_0.

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Let us extend the notion of d-compatibility onto the degenerate case.

d-compatibility

A closed two-form Ω_1 is d-compatible with a closed two-form Ω_0 if there exists a Poisson tensor Π_0, dual to Ω_0, such that $\Pi_0\Omega_1\Pi_0$ is Poisson. Then we say that the pair (Ω_0, Ω_1) is d-compatible with respect to Π_0.

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d-compatibility $\Rightarrow$ compatibility

The inverse relation is true provided that

$$\Omega_0(L_{Z_i}\Pi_1)\Omega_0 = 0, \qquad\qquad i = 1, ..., r.$$

Bi-presymplectic chains

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Assume further, that they form bi-presymplectic chains of one-forms

$$\beta_i^{(k)} = \Omega_0 Y_i^{(k)} = \Omega_1 Y_{i-1}^{(k)}, \quad i = 1, 2, \ldots, n_k$$

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$$\beta_i^{(k)} = \Omega_0 Y_i^{(k)} = \Omega_1 Y_{i-1}^{(k)}, \quad i = 1, 2, \ldots, n_k$$

where $k = 1, ..., r$, $n_1 + ... + n_r = n$ and each chain starts with a kernel vector field $Y_0^{(k)}$ of Ω_0 and terminates with a kernel vector field $Y_{n_k}^{(k)}$ of Ω_1.

Bi-presymplectic chains

Then

(i)

$$\Omega_0(Y_i^{(k)}, Y_j^{(m)}) = \Omega_1(Y_i^{(k)}, Y_j^{(m)}) = 0,$$

for $k, m = 1, ..., r$, $i = 1, 2, \ldots, n_k$, $j = 1, 2, \ldots, n_m$.

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for $k, m = 1, ..., r$, $i = 1, 2, \ldots, n_k$, $j = 1, 2, \ldots, n_m$.

Moreover, let us assume that

$$X_i^{(k)} = \Pi_0 \beta_i^{(k)} = \Pi_0 dH_i^{(k)},$$

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for $k = 1, ..., r,\ i = 1, 2, \ldots, n_k.$ Then,

(ii)

$$\Pi_0(dH_i^{(k)}, dH_j^{(m)}) = 0, \quad [X_i^{(k)}, X_j^{(m)}] = 0$$

and bi-presymplectic chain defines a Liouville integrable system.

Bi-presymplectic chains

Additionally, if

$$Y_0^{(k)}(H_1^{(m)}) = Y_0^{(m)}(H_1^{(k)})$$

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(iii) vector fields $X_i^{(k)}$ form bi-Hamiltonian chains

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Algorithmic procedure of separability.

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Algorithmic procedure of separability.
ω_0, ω_1- restrictions of Ω_0 and Ω_1 to any symplectic leaf of Π_0. Separation coordinates are eigenvalues of the recursion operator

$$N = \omega_0^{-1}\omega_1.$$

Stäckel system

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Separation coordinates (λ, μ) and separation relations:

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$$\sum_{k=1}^{n} S_i^k(\lambda_i, \mu_i) a_k = \psi_i(\lambda_i, \mu_i), \qquad i = 1, \ldots, n,$$

where $a_k = h_k(\lambda, \mu)$ and matrix $S = (S_i^k)$ is called a *generalized Stäckel matrix*.

Stäckel system

For further convenience, let us collect the terms from the l.h.s. as follows:

$$\sum_{k=1}^{r} \varphi_i^k(\lambda^i, \mu_i) h^{(k)}(\lambda^i) = \psi_i(\lambda^i, \mu_i), \qquad i = 1, \ldots, n,$$

where

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On the extended phase space $M \to \mathcal{M}$: $(\lambda, \mu) \to (\lambda, \mu, c)$, where $\dim \mathcal{M} = 2n + r$, differentials $dh_i^{(k)}$ form bi-inverse-Hamiltonian chains

Stäckel system

$$\Omega_0 Y_{i+1}^{(k)} = dh_{i+1}^{(k)} = \Omega_1 Y_i^{(k)}, \quad i = 1, 2, \ldots, n_k, \quad k = 1, ..., r,$$

which starts with a kernel vector field $Y_0^{(k)} = \frac{\partial}{\partial c_k}$ of Ω_0 and terminates with a kernel vector field $Y_{n_k}^{(k)}$ of Ω_1,

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$$\Omega_0 = -\sum_i d\lambda^i \wedge d\mu_i, \quad \Omega_1 = -\sum_i \lambda^i d\lambda^i \wedge d\mu_i + \sum_{k=1}^r dh_1^{(k)} \wedge dc_k,$$

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$$\Omega_0 Y_{i+1}^{(k)} = dh_{i+1}^{(k)} = \Omega_1 Y_i^{(k)}, \quad i = 1, 2, \ldots, n_k, \quad k = 1, ..., r,$$

which starts with a kernel vector field $Y_0^{(k)} = \frac{\partial}{\partial c_k}$ of Ω_0 and terminates with a kernel vector field $Y_{n_k}^{(k)}$ of Ω_1,

$$\Omega_0 = -\sum_i d\lambda^i \wedge d\mu_i, \quad \Omega_1 = -\sum_i \lambda^i d\lambda^i \wedge d\mu_i + \sum_{k=1}^r dh_1^{(k)} \wedge dc_k,$$

$$Y_i^{(k)} = \Pi_0 dh_i^{(k)} - \sum_{l=1}^r F_i^{(k,l)} Y_0^{(l)}, \quad \Pi_0 = \sum_i \frac{\partial}{\partial \lambda_i} \wedge \frac{\partial}{\partial \mu_i}$$

and $F_i^{(k,l)}$ are an appropriate separable potentials.

Stäckel system

Moreover, Ω_0 and Ω_1 are d-compatible with respect to Π_0 but vector fields $X_i^{(k)} = \Pi_0 dh_i^{(k)}$ are not bi-Hamiltonian as $Y_i^{(k)}(h_0^{(l)}) = -F_i^{(k,l)} \neq 0$ while $Y_0^{(k)}(H_i^{(m)}) = 0$.

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In order to construct on $\mathcal{M}$ related bi-Hamiltonian chains of vector fields, one has to extend the original Hamiltonians

$$h_i^{(k)} \to H_i^{(k)} = h_i^{(k)} - \sum_{l=1}^{r} F_i^{(k,l)} c_l, \quad i = 1, ..., n.$$

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Then, on $\mathcal{M}$, vector fields $K_i^{(k)} = \Pi_0 dH_i^{(k)}$ form a bi-Hamiltonian chains

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$$\Pi_0 dH_{i+1}^{(k)} = K_{i+1}^{(k)} = \Pi_1 dH_i^{(k)}, \quad i = 1, 2, \ldots, n_k, \quad k = 1, ..., r,$$

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$$\Pi_0 dH_{i+1}^{(k)} = K_{i+1}^{(k)} = \Pi_1 dH_i^{(k)}, \quad i = 1, 2, \ldots, n_k, \quad k = 1, ..., r,$$

where

$$\Pi_1 = \Pi_0 \Omega_1 \Pi_0 + \sum_{m=1}^{r} K_1^{(m)} \wedge Y_0^{(m)}.$$

Each chain starts with the Casimir of Π_0: $H_0^{(k)} = c_k$, and terminates with the Casimir of Π_1: $H_{n_k}^{(k)}$. Poisson tensors Π_0 and Π_1 are d-compatible with respect to Ω_0.

Stäckel system

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Differentials $H^{(k)}_i$ do not form bi-inverse-Hamoltonian chains as

$$Y^{(k)}_0(H^{(m)}_1) = -F^{(m,k)}_1 \neq -F^{(k,m)}_1 = Y^{(m)}_0(H^{(k)}_1).$$

Example

On $M = \mathbb{R}^4$ consider separation relations

$$h_1\lambda_i + h_2 = \frac{1}{2}\lambda_i\mu_i^2 + \lambda_i^4, \quad i = 1, 2$$

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The canonical point transformation

$$q_1 = \lambda_1 + \lambda_2, \quad \frac{1}{4}q_2^2 = -\lambda_1\lambda_2,$$

transforms the system to flat coordinates (q, p) with

$$h_1 = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + q_1^3 + \frac{1}{2}q_1q_2^2,$$

$$h_2 = \frac{1}{2}q_2p_1p_2 - \frac{1}{2}q_1p_2^2 + \frac{1}{16}q_2^4 + \frac{1}{4}q_1^2q_2^2.$$

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We recognize the Henon-Heiles system.

Example

On $M=\mathbb{R}^5$ differentials dh_1 and dh_2 form bi-inverse-Hamiltonian chain

$$\begin{aligned}\Omega_0 Y_0 &= 0\\ \Omega_0 Y_1 &= dh_1 = \Omega_1 Y_0\\ \Omega_0 Y_2 &= dh_2 = \Omega_1 Y_1\\ 0 &= \Omega_1 Y_2\end{aligned}$$

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\end{aligned}
$$

with vector fields

$$
\begin{aligned}
Y_0 &= (0,0,0,0,1)^T\\
Y_1 &= (p_1, p_2, -3q_1^2 - \frac{1}{2}q_2^2, -q_1 q_2, -q_1)^T\\
Y_2 &= (\frac{1}{2}q_2 p_2, \frac{1}{2}q_2 p_1 - q_1 p_1, \frac{1}{2}p_2^2 - \frac{1}{2}q_1 q_2^2,\\
&\quad -\frac{1}{2}p_1 p_2 - \frac{1}{4}q_2^3 - \frac{1}{2}q_1^2 q_2, -\frac{1}{4}q_2^2)^T
\end{aligned}
$$

Example

and presymplectic forms

$$\Omega_0 = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

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$$\Omega_1 = \begin{pmatrix} 0 & -\frac{1}{2}p_2 & -q_1 & -\frac{1}{2}q_2 & 3q_1^2 + \frac{1}{2}q_2^2 \\ \frac{1}{2}p_2 & 0 & -\frac{1}{2}q_2 & 0 & q_1 q_2 \\ q_1 & \frac{1}{2}q_2 & 0 & 0 & p_1 \\ \frac{1}{2}q_2 & 0 & 0 & 0 & p_2 \\ -3q_1^2 - \frac{1}{2}q_2^2 & -q_1 q_2 & -p_1 & -p_2 & 0 \end{pmatrix}.$$

Example

which are d-compatible with respect to the canonical Poisson tensor Π_0 dual to Ω_0 one.

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$$H_1 = h_1 - cq_1, \quad H_2 = h_2 - \frac{1}{4}cq_2^2.$$

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$$H_1 = h_1 - cq_1, \qquad H_2 = h_2 - \frac{1}{4}cq_2^2.$$

Then

$$\begin{aligned} \pi_0 dH_0 &= 0 \\ \pi_0 dH_1 &= K_1 = \pi_1 dH_0 \\ \pi_0 dH_2 &= K_2 = \pi_1 dH_1 \\ 0 &= \pi_1 dH_2 \end{aligned}$$

Example

where

$$\Pi_0 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Pi_1 = \begin{pmatrix} 0 & 0 & q^1 & \frac{1}{2}q^2 & p_1 \\ 0 & 0 & \frac{1}{2}q^2 & 0 & p_2 \\ -q^1 & -\frac{1}{2}q^2 & 0 & \frac{1}{2}p_2 & -3(q^1)^2 - \frac{1}{2}(q^2)^2 + c \\ -\frac{1}{2}q^2 & 0 & -\frac{1}{2}p_2 & 0 & -q^1q^2 \\ -p_1 & -p_2 & 3(q^1)^2 + \frac{1}{2}(q^2)^2 - c & q^1q^2 & 0 \end{pmatrix}$$

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Π_1 is d-compatible to Π_0 with respect to Ω_0.

Example

Notice that

$$Y_1 = X_1 - q_1 \frac{\partial}{\partial c}, \quad Y_2 = X_2 - \frac{1}{4} q_2^2 \frac{\partial}{\partial c}.$$

$$H_1 = h_1 - q_1 c, \quad H_2 = h_2 - \frac{1}{4} q_2^2 c,$$

$$X_i = \Pi_0 d h_i, \qquad K_i = \Pi_0 d H_i.$$

THE END

Preliminaries

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