Symplectization of 1-jet space $J^1\mathbb{R}^n$
and point classification of PDE's

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Let $\mathbb{R}^n$ be real space with coordinates $x := (x_1, \ldots, x_n)$, $J^1\mathbb{R}^n$ be 1-jet space of smooth functions $f: \mathbb{R}^n \to \mathbb{R}$ with canonic coordinates $(x, y, y')$, where $y' = (y_1, \ldots, y_n)$ and

$$x([f]_a^1) = a, \quad y([f]_a^1) = f(a), \quad y_k([f]_a^1) = \frac{\partial f}{\partial x_k}(a).$$

Abel partial differential equation of degree $d$ is a differential equation, which is polynomial in derivatives $y_k$:

$$\sum_{i_1 + \ldots + i_n \leq d} A_{i_1 \ldots i_n}(x, y) \cdot y_1^{i_1} \ldots y_n^{i_n} = 0.$$
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*Point pseudogroup* is pseudogroup $G := \text{Diff}(J^0 \mathbb{R}^n)$ of diffeomorphisms of space $J^0 \mathbb{R}^n \cong \mathbb{R}^{n+1}$. It acts on Abel differential equations:

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x \mapsto X = X(x, y), \quad y \mapsto Y = Y(x, y), \quad y' \mapsto Y' = \frac{Y_y + Y_x \cdot y'}{X_y + X_x \cdot y'}.
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Problem

Classify Abel partial differential equations with respect to the action of point pseudogroup.
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Why is this problem complicated? The action of point pseudogroup on the fiber $\mathbb{RP}^n$ of the canonic projection $\pi_{1,0}: J^1 \mathbb{R}^n \to J^0 \mathbb{R}^n$ is projective

$\Rightarrow$ it is difficult to calculate invariants of such action

(for example, affine curvature of the plane curve has order 3, whereas projective curvature of the plane curve has order 7).

Main idea. We replace projection $\pi_{1,0}: J^1 \mathbb{R}^n \to J^0 \mathbb{R}^n$ by some bundle with the base $J^0 \mathbb{R}^n$ and fiber $\mathbb{R}^{n+1}$, with the linear (not projective) action of the point pseudogroup on the fibers.

So, we just “homogenize” the fiber $\mathbb{RP}^n$ and obtain the fiber $\mathbb{R}^{n+1}$, whose projectivization is $\mathbb{RP}^n$. 
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Symplectization of contact space $J^1\mathbb{R}$

Further we will assume that $n = 1$; all constructions and theorems can be easily generated on arbitrary $n$.

Cartan distribution $\mathcal{C} : \theta \mapsto \mathcal{C}_\theta$ on $J^1\mathbb{R}$ is defined by Cartan differential 1-form $\kappa := dy - y_1\,dx$, i.e.

$$\mathcal{C}_\theta = \ker \kappa_\theta,$$

where $\theta \in J^1\mathbb{R}$.

$\Rightarrow$ contact structure on $J^1\mathbb{R}$.

Linear 1-form $\alpha_\theta \in T^*_\theta(J^1\mathbb{R})$ is said to be contact, if $\ker \alpha_\theta = \mathcal{C}_\theta$.

It is clear that $\alpha_\theta = \lambda \cdot \kappa_\theta$, where $\lambda \in \mathbb{R}^*$.

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Symplectization $\text{Symp}(J^1\mathbb{R})$ of contact space $J^1\mathbb{R}$ is the set of all contact 1-forms $\alpha_\theta$.

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Symplectic structure

Theorem

The following diffeomorphism holds:

\[ \text{Symp}(J^1\mathbb{R}) \simeq T^*(J^0\mathbb{R}) \setminus \{s_0\}, \]

where \( s_0 \) is the image of zero section of cotangent bundle \( T^*(J^0\mathbb{R}) \).

Proof.

We construct diffeomorphism \( T^*(J^0\mathbb{R}) \setminus \{s_0\} \cong \text{Symp}(J^1\mathbb{R}) \).

Let \( \beta_a \in T_a^*(J^0\mathbb{R}) \Rightarrow L_a := \ker \beta_a \subset T_a(J^0\mathbb{R}) \Rightarrow (a, L_a) =: \theta \in J^1\mathbb{R} \Rightarrow \alpha_\theta := \pi_{1,0}^*(\beta_a). \)

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where $s_0$ is the image of zero section of cotangent bundle $T^*(J^0\mathbb{R})$.

Proof.

We construct diffeomorphism $T^*(J^0\mathbb{R}) \setminus \{s_0\} \cong \text{Symp}(J^1\mathbb{R})$.

Let $\beta_a \in T_a^*(J^0\mathbb{R}) \Rightarrow L_a := \ker \beta_a \subset T_a(J^0\mathbb{R})$

$\Rightarrow (a, L_a) \models \theta \in J^1\mathbb{R} \Rightarrow \alpha_\theta := \pi_{1,0}^*(\beta_a)$.

$$\alpha_\theta |_{C_\theta} = \beta_a |_{L_a} = 0$$

$\Rightarrow \alpha_\theta$ is contact.
Put

\[ q := (x, y) \quad \text{and} \quad p := (-\lambda y_1, \lambda). \]

Then

\[ \omega := \lambda \cdot \pi^*(\kappa) = p \, dq \]

is canonic 1-form

and

\[ \Omega := d\omega = dp \wedge dq \]

is symplectic structure on symplectization \( \text{Symp}(J^1\mathbb{R}) \).
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Symplectic coordinates

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Symplectization of point transformations and PDE’s

Point transformation $\varphi: J^1\mathbb{R} \to J^1\mathbb{R}$ acts on contact forms $\alpha_\theta \Rightarrow$$\varphi$ prolongs to symplectomorphism of cotangent bundle $\tau^*: T^*(J^0\mathbb{R}) \to J^0\mathbb{R}: q \mapsto Q(q), \quad p \mapsto Q_*^{-1}p$ (here $Q_*$ is Jackobi matrix of diffeomorphism $Q: J^0\mathbb{R} \to J^0\mathbb{R}$).

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— smooth function on $T^*(J^0\mathbb{R})$, which is homogeneous in fiber coordinates $p = (p_1, p_2)$.

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Classify smooth functions on $T^*(J^0\mathbb{R})$, which are homogeneous in fiber coordinates $p$, with respect to the action of symplectomorphisms $q \mapsto Q(q), \quad p \mapsto Q_*^{-1}p$. 

Pavel Bibikov  
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Pavel Bibikov
Teplice nad Bečvou–2013
Function $F = F(q, p)$ is homogeneous with respect to $p \iff p \cdot F_p = d \cdot F$,

i.e. $F$ is solution of the Euler equation.

Let us consider function $F$ as function only on $p = (p_1, p_2)$

$\Rightarrow$ linear action of group $\text{GL}_2(\mathbb{R})$ on the solutions of the Euler equation

$\Rightarrow$ the same problem as in case of binary forms!
Euler equation, algebra

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Differential invariants

Let $F \in C^\infty(T^*(J^0\mathbb{R}))$.

Let $J^k$ be $k$-jet space of such functions with canonic coordinates $(q, p, u, u_\sigma)$.

Point pseudogroup $G$ acts on $J^k$.

- **Differential invariant of group $G$** is rational function $J \in C^\infty(J^\infty)$ such that $g \circ J = J$ for all $g \in G$.

- **Invariant derivative** is derivative $\nabla$ of algebra $C^\infty(J^\infty)$ such that $[\nabla, \xi] = 0$ for all $\xi \in g$. 
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Some differential invariants

Theorem

1. **Differential forms**

\[ \omega := p dq = p_1 dq_1 + p_2 dq_2 \quad \text{and} \quad \psi := \frac{u p_1 dp_1 + u p_2 dp_2}{u} \]

are \( G \)-invariant.

2. **Put**

\[ H := \frac{u p_1 p_1 u p_2 p_2 - u^2}{u^2} \quad \text{and} \quad \nabla := \frac{u p_2}{u} \cdot \frac{d}{dp_1} - \frac{u p_1}{u} \cdot \frac{d}{dp_2} \]

Then

\[ J := \frac{(\nabla H)^2}{H^3}, \quad r := p_1 \cdot \frac{d}{dp_1} + p_2 \cdot \frac{d}{dp_2} \quad \text{and} \quad \delta := \frac{\nabla}{\nabla H} \]

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Invariant coordinate system

Consider invariants $J_1 := J, J_2 := \delta J, J_3 := \delta^2 J, J_4 := \delta^3 J$. We assume, that they are independent $\Rightarrow$ they define «coordinate system». Let us rewrite vector field $\delta$, canonic 1-form $\omega$ and invariant 1-form $\psi$ in this coordinate system:

$$
\delta = \sum_{i=1}^{4} \delta(J_i) \frac{D}{DJ_i},
$$

$$
\omega = \sum_{i=1}^{4} \omega \left( \frac{D}{DJ_i} \right) \hat{d}J_i,
$$

$$
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Classification theorem

Let us consider functions $\mathcal{P} : T^*(J^0 \mathbb{R}) \to \mathbb{R}$, $\mathcal{Q}$, $\mathcal{R} : T^*(J^0 \mathbb{R}) \to \mathbb{R}^4$, where $\mathcal{Q} := (Q_i)$, $\mathcal{R} := (R_i)$, $i = 1, \ldots, 4$, and

$$
\delta^4 J = \mathcal{P}(J_1, \ldots, J_4), \quad \omega\left(\frac{D}{D J_i}\right) = Q_i(J_1, \ldots, J_4),
$$

$$
\psi\left(\frac{D}{D J_i}\right) = R_i(J_1, \ldots, J_4).
$$

Theorem

Two Abel PDE’s, which correspond to the homogeneous functions $F$ and $\tilde{F}$ are point–equivalent, iff

$$(\mathcal{P}, \mathcal{Q}, \mathcal{R}) = (\tilde{\mathcal{P}}, \tilde{\mathcal{Q}}, \tilde{\mathcal{R}}).$$
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\]

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\psi\left(\frac{D}{D J_i}\right) = R_i(J_1, \ldots, J_4).
\]

Theorem

Two Abel PDE’s, which correspond to the homogeneous functions $F$ and $\tilde{F}$ are point–equivalent, iff

\[ (\mathcal{P}, \mathcal{Q}, \mathcal{R}) = (\tilde{\mathcal{P}}, \tilde{\mathcal{Q}}, \tilde{\mathcal{R}}). \]
Classification theorem

Let us consider functions $\mathcal{P} : T^*(J^0\mathbb{R}) \to \mathbb{R}$, $\mathcal{Q}$, $\mathcal{R} : T^*(J^0\mathbb{R}) \to \mathbb{R}^4$, where $\mathcal{Q} := (Q_i)$, $\mathcal{R} := (R_i)$, $i = 1, \ldots, 4$, and

$$
\delta^4 J = \mathcal{P}(J_1, \ldots, J_4), \quad \omega \left( \frac{D}{DJ_i} \right) = Q_i(J_1, \ldots, J_4),
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\[(\mathcal{P}, \mathcal{Q}, \mathcal{R}) = (\tilde{\mathcal{P}}, \tilde{\mathcal{Q}}, \tilde{\mathcal{R}}).\]
Proof

“⇒” — obvious.

“⇐” Let

\[(\mathcal{P}, \mathcal{Q}, \mathcal{R}) = (\tilde{\mathcal{P}}, \tilde{\mathcal{Q}}, \tilde{\mathcal{R}})\]

for two homogeneous functions \(F\) and \(\tilde{F}\).

Consider “invariant coordinate systems”

\[S := (J_1(F), J_2(F), J_3(F), J_4(F)), \quad \tilde{S} := (J_1(\tilde{F}), J_2(\tilde{F}), J_3(\tilde{F}), J_4(\tilde{F})).\]

Let

\[
\Phi: T^*(J^0\mathbb{R}) \to T^*(J^0\mathbb{R}), \quad \Phi(S) = \tilde{S}.
\]

Then

- \(\Phi \in G\), because \(\Phi\) preserves \(J^0\mathbb{R}\) (\(\mathcal{P}\)) and \(\omega\) (\(\mathcal{Q}\));
- \(\Phi \circ F = \mu \cdot \tilde{F}\), because \(\Phi\) preserves \(\psi\) (\(\mathcal{R}\)).

Hence, \(F\) and \(\tilde{F}\) are point–equivalent.
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Proof

“⇒” — obvious.

“⇐” Let

\[(P, Q, R) = (\tilde{P}, \tilde{Q}, \tilde{R})\]

for two homogeneous functions \(F\) and \(\tilde{F}\).

Consider “invariant coordinate systems”

\[S := (J_1(F), J_2(F), J_3(F), J_4(F)), \quad \tilde{S} := (J_1(\tilde{F}), J_2(\tilde{F}), J_3(\tilde{F}), J_4(\tilde{F})).\]

Let

\[\Phi: T^* (J^0 R) \to T^* (J^0 R), \quad \Phi(S) = \tilde{S}.\]

Then

- \(\Phi \in G\), because \(\Phi\) preserves \(J^0 R (P)\) and \(\omega (Q)\);
- \(\Phi \circ F = \mu \cdot \tilde{F}\), because \(\Phi\) preserves \(\psi (R)\).

Hence, \(F\) and \(\tilde{F}\) are point–equivalent.
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\[\Phi: T^*(J^0\mathbb{R}) \to T^*(J^0\mathbb{R}), \quad \Phi(S) = \tilde{S}.\]

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- \(\Phi \in G\), because \(\Phi\) preserves \(J^0\mathbb{R} (\mathcal{P})\) and \(\omega (\mathcal{Q})\);
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for two homogeneous functions \(F\) and \(\tilde{F}\).

Consider “invariant coordinate systems”

\[S := (J_1(F), J_2(F), J_3(F), J_4(F)), \quad \tilde{S} := (J_1(\tilde{F}), J_2(\tilde{F}), J_3(\tilde{F}), J_4(\tilde{F})).\]

Let

\[\Phi: T^*(J^0\mathbb{R}) \to T^*(J^0\mathbb{R}), \quad \Phi(S) = \tilde{S}.\]

Then

- \(\Phi \in G\), because \(\Phi\) preserves \(J^0\mathbb{R} (\mathcal{P})\) and \(\omega (\mathcal{Q})\);
- \(\Phi \circ F = \mu \cdot \tilde{F}\), because \(\Phi\) preserves \(\psi (\mathcal{R})\).

Hence, \(F\) and \(\tilde{F}\) are point–equivalent.
Proof

“⇒” — obvious.

“⇐” Let

\[(\mathcal{P}, \mathcal{Q}, \mathcal{R}) = (\mathcal{\tilde{P}}, \mathcal{\tilde{Q}}, \mathcal{\tilde{R}})\]

for two homogeneous functions \(F\) and \(\tilde{F}\).

Consider “invariant coordinate systems”

\[S := (J_1(F), J_2(F), J_3(F), J_4(F)), \quad \tilde{S} := (J_1(\tilde{F}), J_2(\tilde{F}), J_3(\tilde{F}), J_4(\tilde{F})).\]

Let

\[\Phi : T^*(J^0\mathbb{R}) \rightarrow T^*(J^0\mathbb{R}), \quad \Phi(S) = \tilde{S}.\]

Then

- \(\Phi \in G\), because \(\Phi\) preserves \(J^0\mathbb{R} (\mathcal{P})\) and \(\omega (\mathcal{Q})\);
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Hence, \(F\) and \(\tilde{F}\) are point-equivalent.
Proof

“⇒” — obvious.

“⇐” Let

$$( P, Q, R ) = ( \tilde{P}, \tilde{Q}, \tilde{R} )$$

for two homogeneous functions $F$ and $\tilde{F}$.

Consider “invariant coordinate systems”

$$S := ( J_1(F), J_2(F), J_3(F), J_4(F) ), \quad \tilde{S} := ( J_1(\tilde{F}), J_2(\tilde{F}), J_3(\tilde{F}), J_4(\tilde{F}) ).$$

Let

$$\Phi : T^*(J^0\mathbb{R}) \rightarrow T^*(J^0\mathbb{R}), \quad \Phi(S) = \tilde{S}.$$ 

Then

- $\Phi \in G$, because $\Phi$ preserves $J^0\mathbb{R} ( P )$ and $\omega ( Q )$;
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Hence, $F$ and $\tilde{F}$ are point–equivalent.
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for two homogeneous functions $F$ and $\tilde{F}$.

Consider “invariant coordinate systems”

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Let

$$\Phi: T^* ( J^0 \mathbb{R} ) \rightarrow T^* ( J^0 \mathbb{R} ), \quad \Phi(S) = \tilde{S}.$$ 

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- $\Phi \in G$, because $\Phi$ preserves $J^0 \mathbb{R} ( \mathcal{P} )$ and $\omega ( \mathcal{Q} )$;
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Hence, $F$ and $\tilde{F}$ are point–equivalent.
Proof

“⇒” — obvious.

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for two homogeneous functions \(F\) and \(\tilde{F}\).

Consider “invariant coordinate systems”

\[S := (J_1(F), J_2(F), J_3(F), J_4(F)), \quad \tilde{S} := (J_1(\tilde{F}), J_2(\tilde{F}), J_3(\tilde{F}), J_4(\tilde{F})).\]

Let

\[\Phi: T^*(J^0\mathbb{R}) \rightarrow T^*(J^0\mathbb{R}), \quad \Phi(S) = \tilde{S}.\]

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- \(\Phi \in G\), because \(\Phi\) preserves \(J^0\mathbb{R} (P)\) and \(\omega (Q)\);
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Hence, \(F\) and \(\tilde{F}\) are point–equivalent.