Symplectization of 1-jet space $J^1\mathbb{R}^n$ and point classification of PDE's

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Problem

Let $\mathbb{R}^n$ be real space with coordinates $\mathbf{x} := (x_1, \ldots, x_n)$,

$J^1\mathbb{R}^n$ be 1-jet space of smooth functions $f\colon \mathbb{R}^n \to \mathbb{R}$

with canonic coordinates $(\mathbf{x}, y, \mathbf{y}')$, where $\mathbf{y}' = (y_1, \ldots, y_n)$ and

$$\mathbf{x}([f]^1_{\mathbf{a}}) = \mathbf{a}, \quad y([f]^1_{\mathbf{a}}) = f(\mathbf{a}), \quad y_k([f]^1_{\mathbf{a}}) = \frac{\partial f}{\partial x_k}(\mathbf{a}).$$

Abel partial differential equation of degree d is a differential equation, which is polynomial in derivatives y_k:

$$\sum_{i_1+\ldots+i_n \leqslant d} A_{i_1\ldots i_n}(\mathbf{x}, y) \cdot y_1^{i_1} \ldots y_n^{i_n} = 0.$$

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Point pseudogroup is pseudogroup $G := \mathrm{Diff}(J^0\mathbb{R}^n)$ of diffeomorphisms of space $J^0\mathbb{R}^n \simeq \mathbb{R}^{n+1}$. It acts on Abel differential equations:

$$\mathbf{x} \mapsto \mathbf{X} = \mathbf{X}(\mathbf{x}, y), \quad y \mapsto Y = Y(\mathbf{x}, y), \quad \mathbf{y}' \mapsto \mathbf{Y}' = \frac{Y_y + Y_{\mathbf{x}} \cdot \mathbf{y}'}{\mathbf{X}_y + \mathbf{X}_{\mathbf{x}} \cdot \mathbf{y}'}.$$

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Classify Abel partial differential equations with respect to the action of point pseudogroup.

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Why is this problem complicated? The action of point pseudogroup on the fiber $\mathbb{R}P^n$ of the canonic projection $\pi_{1,0}\colon J^1\mathbb{R}^n \to J^0\mathbb{R}^n$ is *projective*

$\Rightarrow$ it is difficult to calculate invariants of such action

(for example, affine curvature of the plane curve has order 3, whereas projective curvature of the plane curve has order 7).

Main idea. We replace projection $\pi_{1,0}\colon J^1\mathbb{R}^n \to J^0\mathbb{R}^n$ by some bundle with the base $J^0\mathbb{R}^n$ and fiber $\mathbb{R}^{n+1}$, with the *linear* (not projective) action of the point pseudogroup on the fibers.

So, we just "*homogenize*" the fiber $\mathbb{R}P^n$ and obtain the fiber $\mathbb{R}^{n+1}$, whose projectivization is $\mathbb{R}P^n$.

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Symplectization of contact space $J^1\mathbb{R}$

Further we will assume that $n = 1$; all constructions and theorems can be easily generated on arbitrary n.

Cartan distribution $\mathcal{C}\colon \theta \mapsto \mathcal{C}_\theta$ on $J^1\mathbb{R}$ is defined by Cartan differential 1-form $\varkappa := dy - y_1\,dx$, i.e.

$$\mathcal{C}_\theta = \ker \varkappa_\theta, \quad \text{where } \theta \in J^1\mathbb{R}.$$

$\Rightarrow$ *contact structure* on $J^1\mathbb{R}$.

Linear 1-form $\alpha_\theta \in T^*_\theta(J^1\mathbb{R})$ is said to be *contact*, if $\ker \alpha_\theta = \mathcal{C}_\theta$.

It is clear that $\alpha_\theta = \lambda \cdot \varkappa_\theta$, where $\lambda \in \mathbb{R}^*$.

Definition

Symplectization $\mathrm{Symp}(J^1\mathbb{R})$ *of contact space* $J^1\mathbb{R}$ *is the set of all contact 1-forms* α_θ.

Natural bundle $\pi\colon \mathrm{Symp}(J^1\mathbb{R}) \to J^1\mathbb{R}$ with fiber $\mathbb{R}^*$ and coordinate λ.

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Symplectic structure

Theorem

The following diffeomorphism holds:

$$\mathrm{Symp}(J^1\mathbb{R}) \simeq T^*(J^0\mathbb{R}) \setminus \{s_0\},$$

where s_0 is the image of zero section of cotangent bundle $T^*(J^0\mathbb{R})$.

Proof.

We construct diffeomorphism $T^*(J^0\mathbb{R}) \setminus \{s_0\} \xrightarrow{\sim} \mathrm{Symp}(J^1\mathbb{R})$.
Let $\beta_a \in T_a^*(J^0\mathbb{R}) \Rightarrow L_a := \ker \beta_a \subset T_a(J^0\mathbb{R})$
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$\Rightarrow \alpha_\theta$ is contact. □

Symplectic structure

Theorem

The following diffeomorphism holds:

$$\mathrm{Symp}(J^1\mathbb{R}) \simeq T^*(J^0\mathbb{R}) \setminus \{s_0\},$$

where s_0 is the image of zero section of cotangent bundle $T^*(J^0\mathbb{R})$.

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Put

$$\mathbf{q} := (x, y) \quad \text{and} \quad \mathbf{p} := (-\lambda y_1, \lambda).$$

Then

$$\omega := \lambda \cdot \pi^*(\varkappa) = \mathbf{p}\, d\mathbf{q}$$

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Point transformation $\varphi\colon J^1\mathbb{R} \to J^1\mathbb{R}$ acts on contact forms $\alpha_\theta \Rightarrow$ φ prolongs to symplectomorphism of cotangent bundle $\tau^*\colon T^*(J^0\mathbb{R}) \to J^0\mathbb{R}$: $\mathbf{q} \mapsto \mathbf{Q}(\mathbf{q}), \quad \mathbf{p} \mapsto \mathbf{Q}_*^{-1}\mathbf{p}$ (here $\mathbf{Q}_*$ is Jackobi matrix of diffeomorphism $\mathbf{Q}\colon J^0\mathbb{R} \to J^0\mathbb{R}$).

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Euler equation, algebra

Function $F = F(\mathbf{q}, \mathbf{p})$ is homogeneous with respect to $\mathbf{p} \Leftrightarrow$

$$\mathbf{p} \cdot F_{\mathbf{p}} = d \cdot F,$$

i.e. F is solution of the Euler equation.

Let us consider function F as function only on $\mathbf{p} = (p_1, p_2)$

$\Rightarrow$ linear action of group $\mathrm{GL}_2(\mathbb{R})$ on the solutions of the Euler equation

$\Rightarrow$ the same problem as in case of binary forms!

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Differential invariants

Let $F \in C^\infty(T^*(J^0\mathbb{R}))$.

Let $\mathbf{J}^k$ be k-jet space of such functions with canonic coordinates $(\mathbf{q}, \mathbf{p}, u, u_\sigma)$.

Point pseudogroup G acts on $\mathbf{J}^k$.

- *Differential invariant of group G* is rational function $J \in C^\infty(\mathbf{J}^\infty)$ such that $g \circ J = J$ for all $g \in G$.
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are G-invariant.

2. *Put* $H := \dfrac{u_{p_1p_1}u_{p_2p_2} - u_{p_1p_2}^2}{u^2}$ *and* $\nabla := \dfrac{u_{p_2}}{u} \cdot \dfrac{d}{dp_1} - \dfrac{u_{p_1}}{u} \cdot \dfrac{d}{dp_2}$. *Then*

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Invariant coordinate system

Consider invariants $J_1 := J$, $J_2 := \delta J$, $J_3 := \delta^2 J$, $J_4 := \delta^3 J$. We assume, that they are independent $\Rightarrow$ they define «coordinate system». Let us rewrite vector field δ, canonic 1-form ω and invariant 1-form ψ in this coordinate system:

$$\delta = \sum_{i=1}^{4} \delta(J_i) \frac{D}{DJ_i},$$

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Invariant coordinate system

Consider invariants $J_1 := J$, $J_2 := \delta J$, $J_3 := \delta^2 J$, $J_4 := \delta^3 J$. We assume, that they are independent $\Rightarrow$ they define «coordinate system». Let us rewrite vector field δ, canonic 1-form ω and invariant 1-form ψ in this coordinate system:

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Classification theorem

Let us consider functions $\mathscr{P}\colon T^*(J^0\mathbb{R}) \to \mathbb{R}$, $\mathscr{Q}$, $\mathscr{R}\colon T^*(J^0\mathbb{R}) \to \mathbb{R}^4$, where $\mathscr{Q} := (Q_i)$, $\mathscr{R} := (R_i)$, $i = 1, \ldots, 4$, and

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Theorem

Two Abel PDE's, which correspond to the homogeneous functions F and $\widetilde{F}$ are point–equivalent, iff

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Proof

"$\Rightarrow$" — obvious.

"$\Leftarrow$" Let

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for two homogeneous functions F and $\widetilde{F}$.

Consider "invariant coordinate systems"

$$S := (J_1(F), J_2(F), J_3(F), J_4(F)), \quad \widetilde{S} := (J_1(\widetilde{F}), J_2(\widetilde{F}), J_3(\widetilde{F}), J_4(\widetilde{F})).$$

Let

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- $\Phi \in G$, because Φ preserves $J^0\mathbb{R}$ ($\mathscr{P}$) and ω ($\mathscr{Q}$);
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