Differential geometry and representations of semi-simple algebraic groups

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Introduction

The problem of studying orbit spaces Ω/G for actions $G : \Omega$ of groups G on spaces Ω is one the most important problems, which has a lot of different applications in many areas (representation theory, geometry, differential equations, etc.).

Most of the cases of this problem fall into the following groups:

- Ω is a smooth manifold and G is a Lie group (*geometric situation*);
- Ω is an algebraic manifold and G is an algebraic Lie group, acting algebraically on Ω (*algebraic situation*).

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- Ω is an algebraic manifold and G is an algebraic Lie group, acting algebraically on Ω (*algebraic situation*).

In the first case it was proved by J. L. Koszul and R. Palais, that if the action $G : \Omega$ is proper and free, then the orbit space Ω/G is a smooth manifold and G-orbits are separated by smooth invariants.

The algebraic case has a very long and interesting history.

- G is reductive, Ω is an algebraic manifold $\Rightarrow$ algebra of polynomial invariants $\mathbb{C}[\Omega]^G$ is finite-generated (D. Hilbert, 1899).
- G is not reductive (14-th Hilbert problem) $\Rightarrow$ counterexample (Nagata, Steinberg, 1954).
- G is semi-simple $\Rightarrow$ field of rational invariants $\mathbb{C}(\Omega)^G$ is finite-generated (Rosenlicht, 1956).
- Projective action $G : \Omega \Rightarrow$ Geometrical Invariant Theory (D. Mumford, 1960-th).

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Example: classification of binary forms

A binary form is a homogeneous polynomial on $\mathbb{C}^2$:

$$f(x,y) = \sum_{k=0}^{n} p_k x^k y^{n-k},$$

where $p_k \in \mathbb{C}$.
The space of all binary forms of degree n is denoted by V_n.
The action of the group $\mathbf{GL}_2(\mathbb{C}) = \mathbf{SL}_2(\mathbb{C}) \rtimes \mathbb{C}^*$:

$$\mathbf{SL}_2(\mathbb{C}) \ni A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} : f(x,y) \longmapsto f(a_{22}x - a_{12}y, a_{11}y - a_{21}x),$$

$$\mathbb{C}^* \ni \lambda \colon f \longmapsto \lambda f.$$

Problem

When are two binary forms $\mathbf{GL}_2(\mathbb{C})$-equivalent?

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Classical approach: invariant theory

In algebra we usually consider the action of the group $\mathbf{SL}_2(\mathbb{C})$. To describe the orbits one can calculate the algebra *of polynomial invariants*, i.e. $\mathbf{SL}_2(\mathbb{C})$-invariant polynomials $I(p_0, \ldots, p_n)$. Let $\mathbf{A}_n := \mathbb{C}[V_n]^{\mathbf{SL}_2(\mathbb{C})}$ be the invariant algebra.

$n = 1$: Trivial case:

$$\mathbf{A}_1 = \mathbb{C}.$$

$n = 2$: $V_2 = \{ax^2 + 2bxy + cy^2\}$ — quadrics;

$$\mathbf{A}_2 = \mathbb{C}[b^2 - ac].$$

Generator — discriminant (= Hessian).

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- $n = 3$: Bool, 1841 (debut of the classical invariant theory)
- $n = 4$: Bool, Cayle, Eisinstine, 1840–1850 (cross-ratio, j-invariant)
- $n = 5$: Hermite (1954): invariant I_{18} of degree 18, which contains more than ***800*** terms + syzygy

$$I_4 I_8^4 + 8 I_8^3 I_{12} - 2 I_4^2 I_8^2 I_{12} - 72 I_4 I_8 I_{12}^2 - 432 I_{12}^3 + I_4^3 I_{12}^2 - 16 I_{18}^2 = 0!$$

- $6 \leqslant n \leqslant 10$, $n = 12$: Gordan, Shioda, Dixmier, Bedratuke, Brauer, Popovich (1860–2016).

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New approach

Consider binary forms as solutions of the Euler equation
$xf_x + yf_y = nf$.

Consider the action of group $\mathbf{GL}_2(\mathbb{C})$ *on this differential equation*!

Let us find the invariants of this action.

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Necessary definitions

- (Jet space.) *k-jet of function f in point* $a \in \mathbb{C}^2$:

$$[f]^k_a := \Big(a, f(a), f_x(a), f_y(a), f_{xx}(a), f_{xy}(a), f_{yy}(a), \dots\Big).$$

k-jet space $\mathbf{J}^k := \mathbf{J}^k\mathbb{C}^2$ with the canonical coordinates

$$(x, y, u, u_{10}, u_{01}, u_{20}, u_{11}, u_{02}, \dots, u_\sigma),$$

where $u_{ij}([f]^k_a) = \frac{\partial^{i+j} f}{\partial^i x \partial^j y}(a)$.

- (Euler equation.) Euler differential equation $\rightsquigarrow$ algebraic manifold

$$\mathcal{E} := \{x \cdot u_{10} + y \cdot u_{01} = n \cdot u\} \subset \mathbf{J}^1.$$

$\mathcal{E}^{(k-1)} \subset \mathbf{J}^k$ — prolongations.

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Problem

We want to describe the orbits of the action $\mathbf{GL}_2(\mathbb{C}) : \mathcal{E}^{(k-1)}$ *for all* k.

- *Differential invariant of order* k is a function

$$I \in \mathbb{C}(\mathcal{E}^{(k-1)})^{\mathbf{GL}_2(\mathbb{C})}.$$

- *Invariant derivation* is derivation

$$\nabla = A\frac{d}{dx} + B\frac{d}{dy},$$

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Algebra of differential invariants

- Function ("Hessian")

$$H = \frac{u_{20}u_{02} - u_{11}^2}{u^2}$$

is differential invariant of order 2.

- Derivative

$$\nabla = \frac{u_{01}}{u}\frac{d}{dx} - \frac{u_{10}}{u}\frac{d}{dy}$$

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Classification

Let f be a binary form. Consider the restrictions

$$H(f), \quad \nabla H(f), \quad \nabla^2 H(f).$$

They are homogeneous rational functions in variables x, y $\Rightarrow$ there is an *algebraic dependence* between them:

$$F(H(f), \nabla H(f), \nabla^2 H(f)) = 0.$$

Theorem

Binary forms f_1, f_2 of the same degree are $\mathbf{GL}_2(\mathbb{C})$-equivalent iff

$$F_1 \equiv F_2.$$

Dependence F is a resultant of two polynomials $\Rightarrow$ this theorem provides an effective criterion for the equivalence of binary forms.

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Generalization

Let G be a connected semi-simple complex Lie group, and let

$$\rho_\lambda \colon G \to \mathrm{GL}(V)$$

be its irreducible representation with highest weight λ.

We want to apply theory of differential invariants to study the action $G : V$.

But where are *the functions* in this problem?

Main idea: Borel–Weil–Bott theorem.

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Main idea: Borel–Weil–Bott theorem.

1. Let $B \subset G$ be Borel group and $M := G/B$ be homogeneous complex flag manifold.
2. Consider the action $B : G$ by the right shifts: $g \mapsto gb^{-1}$, where $g \in G$, $b \in B$.
3. Consider the bundle product $E := G \times_B \mathbb{C} = G \times \mathbb{C}/\sim$, where

$$(g, c) \sim (gb^{-1}, \chi_\lambda(b)c),$$

and where $\chi_\lambda \in \mathfrak{X}(T)$ is the character corresponding to the highest weight λ of the maximal torus $T \subset B$.

Theorem (BWB)

Consider bundle $\pi^\lambda \colon E \to M$, $\pi^\lambda(g, c) = gB$ and the module $\Gamma(\pi^\lambda)$ of its holomorphic sections. Then representation ρ_λ is isomorphic to the action $G : \Gamma(\pi^\lambda)$ by left shifts.

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Example: binary forms, $G = \mathbf{SL}_2(\mathbb{C})$

- Weight $\lambda = n\alpha/2$, where α is the positive root of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ and $n \in \mathbb{Z}_+$.
- Borel group $B = \mathbf{B}_2(\mathbb{C})$ consists of upper triangular matrices.
- Flag manifold $M = \mathbf{SL}_2/\mathbf{B}_2 \simeq \mathbb{C}P^1$.
- The character χ_λ acts on B in the following way:

$$\chi_\lambda \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = a^n.$$

If we denote the homogeneous coordinates on M by $(x : y)$, then the holomorphic sections of bundle π^λ are just the homogeneous polynomials of degree n in variables x and y. Thus, the study of invariants of representations of group $\mathbf{SL}_2(\mathbb{C})$ is reduced to the classification $\mathbf{SL}_2(\mathbb{C})$-orbits of binary forms.

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Using BWB-theorem, one can describe the algebra of differential invariants for the action $G : \Gamma(\pi^\lambda)$ and obtain the equivalence criterion.

Finally, for an arbitrary algebraic action $G : \Omega$ on the algebraic manifold Ω there exists a linearization in the following sense.

According to Sumihiro's linearization theorem, each algebraic G-manifold Ω can be embedded into a G-invariant sub-manifold in an irreducible finite-dimensional G-module V.

Computing the invariants for the linear action $G : V$ and restricting them on Ω, we get the invariant on Ω, which separate G-orbits.

This result "closes" the classical invariant theory.

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THANK YOU FOR YOUR ATTENTION!