Five-Dimensional Lax-Integrable Equation, Its Reductions and Recursion Operator

Hynek Baran
(Silesian University in Opava)

in collaboration with
Iosif S. Krasil′shchik, Oleg I. Morozov and Petr Vojčák

WINE 2015, Mikulov, Czech Republic

Consider an 5D equation

$$u_{yz} = u_{ts} + u_s\, u_{xz} - u_z\, u_{xs} \tag{1}$$

This equation is related to the five-dimensional equation considered by [Martínez Alonso Shabat 2004]. It is a particular case of [Manakov-Santini 2006, 2014] equation.

We show that (1) has differential coverings with non-removable parameters (*Lax-integrability*). Two known 4-dimensional Lax-integrable equations

$$u_{yz} = u_{tx} + u_x\, u_{xy} - u_y\, u_{xx} \tag{2}$$

and

$$u_{ty} = u_z\, u_{xy} - u_y\, u_{xz} \tag{3}$$

introduced by [Ferapontov Khusnutdinova 2003] and [Martínez Alonso Shabat 2004], respectively, and their coverings with non-removable parameters are simple reductions of (1) and one of its coverings.

3-dimensional Lax-integrable reductions of equation (3) were considered in [Morozov Sergyeyev 2014]. We also find two reductions of equation (2) to known 3-dimensional Lax-integrable equations. One of them turns out to be also a reduction of (3).

Consider an 5D equation

$$u_{yz} = u_{ts} + u_s\, u_{xz} - u_z\, u_{xs} \tag{1}$$

This equation is related to the five-dimensional equation considered by [Martínez Alonso Shabat 2004]. It is a particular case of [Manakov-Santini 2006, 2014] equation.

We show that (1) has differential coverings with non-removable parameters (*Lax-integrability*). Two known 4-dimensional Lax-integrable equations

$$u_{yz} = u_{tx} + u_x\, u_{xy} - u_y\, u_{xx} \tag{2}$$

and

$$u_{ty} = u_z\, u_{xy} - u_y\, u_{xz} \tag{3}$$

introduced by [Ferapontov Khusnutdinova 2003] and [Martínez Alonso Shabat 2004], respectively, and their coverings with non-removable parameters are simple reductions of (1) and one of its coverings.

3-dimensional Lax-integrable reductions of equation (3) were considered in [Morozov Sergyeyev 2014]. We also find two reductions of equation (2) to known 3-dimensional Lax-integrable equations. One of them turns out to be also a reduction of (3).

Consider an 5D equation

$$u_{yz} = u_{ts} + u_s\, u_{xz} - u_z\, u_{xs} \tag{1}$$

This equation is related to the five-dimensional equation considered by [Martínez Alonso Shabat 2004]. It is a particular case of [Manakov-Santini 2006, 2014] equation.
We show that (1) has differential coverings with non-removable parameters (*Lax-integrability*). Two known 4-dimensional Lax-integrable equations

$$u_{yz} = u_{tx} + u_x\, u_{xy} - u_y\, u_{xx} \tag{2}$$

and

$$u_{ty} = u_z\, u_{xy} - u_y\, u_{xz} \tag{3}$$

introduced by [Ferapontov Khusnutdinova 2003] and [Martínez Alonso Shabat 2004], respectively, and their coverings with non-removable parameters are simple reductions of (1) and one of its coverings.
3-dimensional Lax-integrable reductions of equation (3) were considered in [Morozov Sergyeyev 2014]. We also find two reductions of equation (2) to known 3-dimensional Lax-integrable equations. One of them turns out to be also a reduction of (3).

Two coverings

Using the method described by [Morozov 2009, 2014] we find a contact integrable extension for the structure equations of the symmetry pseudo-group of our equation (1). We obtain two coverings of (1):
The first covering is defined by system

$$\left\{ \begin{array}{lcl} q_t & = & H\,q_z + u_z\,q_x, \\ q_y & = & H\,q_s + u_s\,q_x, \end{array} \right. \tag{4}$$

where H is a solution to

$$\left\{ \begin{array}{lcl} H_t & = & H\,H_z, \\ H_y & = & H\,H_s, \\ H_x & = & 0. \end{array} \right. \tag{5}$$

The second covering is defined by

$$\left\{ \begin{array}{lcl} q_t & = & q\,q_z + u_z\,q_x, \\ q_y & = & q\,q_s + u_s\,q_x. \end{array} \right. \tag{6}$$

The first covering

The first covering system (5) has constant solutions $H(t, y, z, s) \equiv \lambda = \text{const}$. The parameter λ in the corresponding system

$$\left\{ \begin{array}{lcl} q_t & = & \lambda\, q_z + u_z\, q_x, \\ q_y & = & \lambda\, q_s + u_s\, q_x \end{array} \right. \tag{7}$$

is non-removable, that is, it can not be eliminated from (7) by a gauge transformation (a transformation of the covering that is identical on the base equation).

Moreover, one can input *hidden spectral parameters* into the covering (4) with $H \not\equiv \text{const}$. Such a covering of (1) may be defined by the system

$$\left\{ \begin{array}{lcl} q_t & = & -\dfrac{\alpha_1\, z + \alpha_2\, s + \alpha_3}{\alpha_1\, t + \alpha_2\, y + \alpha_4}\, q_z + u_z\, q_x, \\ q_y & = & -\dfrac{\alpha_1\, z + \alpha_2\, s + \alpha_3}{\alpha_1\, t + \alpha_2\, y + \alpha_4}\, q_s + u_s\, q_x \end{array} \right. \tag{8}$$

with four non-removable parameters $\alpha_1, \ldots, \alpha_4 \in \mathbb{R}$.

The second covering

Upon excluding u from (7) we arrive to the equation

$$\left(\frac{q_y - \lambda\, q_s}{q_x}\right)_z = \left(\frac{q_t - \lambda\, q_z}{q_x}\right)_s \tag{9}$$

or

$$q_x\,(q_{yz} - q_{ts}) + (q_y - \lambda\, q_s)\, q_{xz} - (q_t - \lambda\, q_z)\, q_{xs} = 0, \tag{10}$$

which can be considered as a 5-dimensional generalization for the so-called ABC equation

$$A\, q_x\, q_{ty} + B\, q_y\, q_{tx} + C\, q_t\, q_{xy} = 0, \qquad A + B + C = 0, \tag{11}$$

which describes three-dimensional Veronese webs [Zakharevich 2000], [Dunajski Krynski 2013], [Ferapontov Kruglikov 2012], [Ferapontov Moss 2012], and for the 4-dimensional equation

$$q_y\, q_{xz} + \lambda\, q_x\, q_{ty} - (q_z + \lambda\, q_t)\, q_{xy} = 0, \tag{12}$$

from [Morozov Sergyeyev 2014].

In the case of $\lambda = 0$ equation (9) coincides with equation (16) from [Martínez Alonso Shabat 2004].

The second covering – cont.

Renaming $q \mapsto r$, $\lambda \mapsto \mu$ in the covering (10) gives equation

$$r_x\,(r_{yz} - r_{ts}) + (r_y - \mu\,r_s)\,r_{xz} - (r_t - \mu\,r_z)\,r_{xs} = 0.$$

This equation is related to equation (10) by the Bäcklund transformation

$$\left\{\begin{array}{lcl} r_t & = & \dfrac{q_t - \lambda\,q_z}{q_x}\,r_x + \mu\,r_z, \\ r_y & = & \dfrac{q_y - \lambda\,q_s}{q_x}\,r_x + \mu\,r_s, \end{array}\right. \tag{13}$$

which is the generalization of the corresponding Bäcklund transformations for ABC (11) and (12) from [Zakharevich 2000] and [Morozov Sergyeyev 2014], respectively. Unlike those transformations, the case of $\mu = \lambda$ is not excluded in (13). In this case we obtain the Bäcklund auto-transformation for equation (10).

First reduction (5D → **4D Pavlov** → **3D Pavlov**)

Put $s = x$ and rename $y = z$, $z = y$. This reduces equation (1) to the 4D Pavlov equation

$$u_{yz} = u_{tx} + u_x\, u_{xy} - u_y\, u_{xx} \tag{2}$$

The corresponding reduction of (7) provides a covering with non-removable parameter for (2) defined by system

$$\begin{cases} q_t &= \lambda\, q_y + u_y\, q_x, \\ q_z &= (u_x + \lambda)\, q_x. \end{cases} \tag{14}$$

In its turn, equation (2) admits two reductions to three-dimensional Lax-integrable equations.

When $z = y$, (2) reduces to equation

$$u_{yy} = u_{tx} + u_x\, u_{xy} - u_y\, u_{xx} \tag{15}$$

while (7) transforms into its covering

$$\begin{cases} q_t &= (\lambda^2 + \lambda\, u_x + u_y)\, q_x, \\ q_y &= (u_x + \lambda)\, q_x \end{cases} \tag{16}$$

found in [Pavlov 2003, Dunajski 2004].

First reduction (5D $\to$ 4D Pavlov $\to$ **3D rdDym**)

Otherwise, if we suppose that u and q in (1) and (7) are independent of t, so $u_t = 0$, $q_t = 0$, and then rename $z = -t$, we obtain equation, [Blaszak 2002, Pavlov 2006, Ovsienko 2010, Morozov 2009],

$$u_{ty} = u_y\, u_{xx} - u_x\, u_{xy} \tag{17}$$

and its covering, [Pavlov 2006, Morozov 2009],

$$\begin{cases} q_t &=& -(u_x + \lambda)\, q_x, \\ q_y &=& -\lambda^{-1}\, u_y\, q_x. \end{cases} \tag{18}$$

Second reduction (5D $\to$ **4D rdDym** $\to$ **3D rdDym**)

If we put $s = y$ in (1) and (7), then change the variables and the parameter $t = \tilde{t}$, $x = \tilde{x}$, $y = \tilde{y}$, $z = -\tilde{t} - \tilde{z}$, $u = \tilde{u}$, $q = \tilde{q}$, $\lambda = 1 - \tilde{\lambda}^{-1}$, and finally drop tildes, we get equation

$$u_{ty} = u_z\, u_{xy} - u_y\, u_{xz}. \tag{3}$$

The covering (7) transfroms into the covering

$$\begin{cases} q_y &= u_z\, q_x - \lambda^{-1}\, q_z, \\ q_z &= \lambda\, u_y\, q_x \end{cases} \tag{19}$$

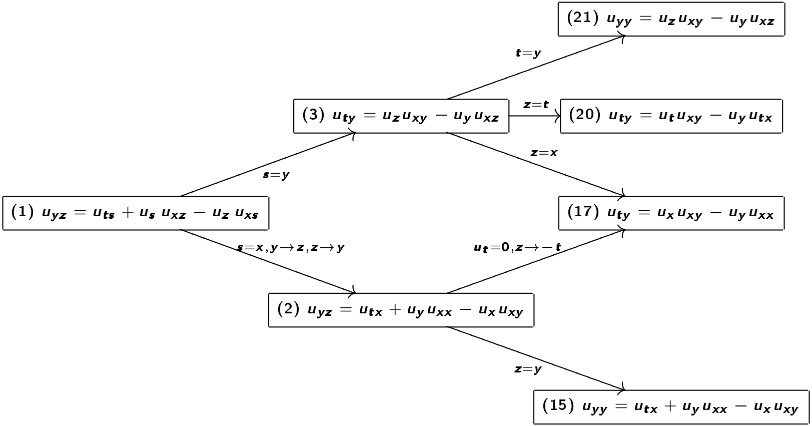
found in [Morozov 2014].
Reductions of (3) to (17) together with *basic Veronese web equation*

$$u_{ty} = u_t\, u_{xy} - u_y\, u_{tx} \tag{20}$$

and *universal hierarchy equation*

$$u_{yy} = u_z\, u_{xy} - u_y\, u_{xz} \tag{21}$$

were found in [Morozov Sergyeyev 2014].



(3) 4D rdDym equation (4D Alonso-Shabat equation);
(2) 4D Pavlov equation;
(21) universal hierarchy equation;
(20) basic Veronese web equation;
(17) 3D rdDym equation;
(15) 3D Pavlov's equation.

Symmetries, shadows

Local symmetries of $u_{yz} = u_{ts} + u_s\, u_{xz} - u_z\, u_{xs}$ (1) are solutions to its *linearization*

$$D_{yz}(\varphi) = D_{ts}(\varphi) + u_s\, D_{xz}(\varphi) + u_{xz}\, D_s(\varphi) - u_z\, D_{xs}(\varphi) - u_{xs}\, D_z(\varphi), \quad (22)$$

such that φ is a fuction of t, x, y, z, s, u and derivatives of u w.r.t. t, ... , s up to some order.

The recursion operator is a Bäcklund auto-transformation for system (1), (22). To construct it we consider *shadows of nonlocal symmetries* (or merely shadows) of equation (1) in covering (7). Informally speaking, the shadows are solutions to (22) such that φ is a function of $t, \ldots, s, u, q$ and their derivatives up to some order.

For simplicity of calculations we rewrite (7) in the form

$$q_z = \lambda^{-1}\,(q_t - u_z\,q_x), \quad q_s = \lambda^{-1}\,(q_y - u_s\,q_x). \tag{23}$$

Direct computations [Jets software, HB & MM]: Every shadow of (1) in covering (23) that depend on derivatives of u and q up to the first order is the sum of a local symmetry and the function

$$r = (c_1\,q_t + c_2\,q_y + Q)\,q_x^{-1}, \tag{24}$$

where $c_1, c_2 \in \mathbb{R}$ and $Q = Q(t, y, z, s, q)$ is a solution to system

$$Q_y = \lambda\,Q_s, \qquad Q_t = \lambda\,Q_z. \tag{25}$$

We observe that, first, function (24) is a solution to system

$$\left\{ \begin{array}{lcl} D_z(r) & = & \lambda^{-1}\,(D_t(r) - u_z\,D_x(r) + u_{xz}\,r - c_1\,u_{tz} - c_2\,u_{yz}), \\ D_s(r) & = & \lambda^{-1}\,(D_y(r) - u_s\,D_x(r) + u_{xs}\,r - c_1\,u_{ts} - c_2\,u_{ys}), \end{array} \right. \tag{26}$$

and, second, shadows of the first order for (1) in covering (26) up to adding a local symmetry are either of form r when $c_1^2 + c_2^2 \neq 0$ or of form $R \cdot r$ when $c_1 = c_2 = 0$, here $R = R(t, y, z, s)$ is a solution to system (25). In any case we conclude that solutions to (26) are also solutions to linearized eq. (22).

Now suppose that a formal power series

$$r = \sum_{n=-\infty}^{+\infty} \lambda^n \, r_n \tag{27}$$

is a solution to (26). Then r is a solution to linearized eq. (22), and, since (22) is independent of λ, each r_n is also a solution to (22). Substituting for (27) into (26) yields the infinite series of systems

$$\left\{ \begin{array}{lcl} D_z(r_n) & = & D_t(r_{n+1}) - u_z \, D_x(r_{n+1}) + u_{xz} \, r_{n+1} - c_1 \, u_{tz} - c_2 \, u_{yz}, \\ D_s(r_n) & = & D_y(r_{n+1}) - u_s \, D_x(r_{n+1}) + u_{xs} \, r_{n+1} - c_1 \, u_{ts} - c_2 \, u_{ys}. \end{array} \right. \tag{28}$$

Now for a fixed $n \in \mathbb{Z}$ in the corresponding system (28) we rename $r_n = \psi$, $r_{n+1} = \varphi + c_1 \, u_z + c_2 \, u_s$ and obtain

The recursion operator and its inverse

$$\begin{cases} D_z(\psi) = D_t(\varphi) - u_z\, D_x(\varphi) + u_{xz}\,\varphi, \\ D_s(\psi) = D_y(\varphi) - u_s\, D_x(\varphi) + u_{xs}\,\varphi. \end{cases} \qquad (29)$$

Since r_n, r_{n+1}, u_z and u_s are solutions to linearized eq. (22), φ and ψ are also its solutions. Direct computations show that by virtue of (1) system (29) is compatible whenever φ is a solution to (22). Expressing $D_t(\varphi)$ and $D_y(\varphi)$ from (29) gives

$$\begin{cases} D_t(\varphi) = D_z(\psi) + u_z\, D_x(\varphi) - u_{xz}\,\varphi, \\ D_y(\varphi) = D_s(\psi) + u_s\, D_x(\varphi) - u_{xs}\,\varphi. \end{cases} \qquad (30)$$

This system is compatible by virtue of (1) whenever ψ is a solution to (22). Thus (29) and (30) define a Bäcklund auto-transformation for (22). In other words, (29) and (30) define a recursion operator $\psi = \mathcal{R}(\varphi)$ and the inverse recursion operator $\varphi = \mathcal{R}^{-1}(\psi)$, respectively.

Reduced covering

Consider again the covering

$$\left\{ \begin{array}{lcl} q_z & = & \lambda^{-1}\,(q_t - u_z\, q_x), \\ q_s & = & \lambda^{-1}\,(q_y - u_s\, q_x) \end{array} \right. \tag{23}$$

and expand it in powers of λ (summation over repeating indices):

$$\lambda^i q_{i,z} = \lambda^{i-1}(\lambda^i q_{i,t} - u_z \lambda^i q_{i,x}) = \lambda^{i-1} q_{i,t} - u_z \lambda^{i-1} q_{i,x},$$
$$\lambda^i q_{i,s} = \lambda^{i-1}(\lambda^i q_{i,y} - u_s \lambda^i q_{i,x}) = \lambda^{i-1} q_{i,y} - u_s \lambda^{i-1} q_{i,x},$$

which gives

$$q_{i,z} = q_{i+1,t} - u_z q_{i+1,x}, \qquad q_{i,s} = q_{i+1,y} - u_s q_{i+1,x}. \tag{31}$$

Consider the two following reductions of (31):

- *positive*, $\tau_+ = \{q_{i<0} = 0\}$;
- *negative*, $\tau_- = \{q_{i>0} = 0\}$.

Positive covering

The positive one, ie.

$$\tau_+ = \{q_{i<0} = 0\},$$

is an infinite tower of *non-Abelian* coverings

$$\tau^+: \quad \begin{aligned} q_{0,t} &= u_z q_{0,x}, & q_{0,y} &= u_s q_{0,x}, \\ q_{1,t} &= u_z q_{1,x} + q_{0,z}, & q_{1,y} &= u_s q_{1,x} + q_{0,s}, \\ &\dots & &\dots \\ q_{i,t} &= u_z q_{i,x} + q_{0,z}, & q_{i,y} &= u_s q_{i,x} + q_{0,s}, \\ &\dots & &\dots \end{aligned}$$

Negative covering

The first two pairs of equations in the negative reductions are

$$q_{0,z} = 0, \qquad q_{0,s} = 0;$$

$$q_{-1,z} = q_{0,t} - u_z q_{0,x}, \qquad q_{-1,s} = q_{0,y} - u_s q_{0,x}.$$

The first pair implies $q_0 = q_0(x, y, t)$. Set $q_0 = x$. Then from the second one we obtain $q_{-1} = u$. Now, relabelling $w_i = q_{-i-2}$, $i = 0, 1, \ldots$, we obtain the infinite series of 2-component conservation laws

$$\tau^- : \quad \begin{aligned} & w_{0,z} = u_t - u_z u_x, && w_{0,s} = u_y - u_s u_x, \\ & w_{1,z} = w_{0,t} - u_z w_{0,x}, && w_{1,s} = w_{0,y} - u_s w_{0,x}; \\ & \ldots && \ldots \\ & w_{i+1,z} = w_{i,t} - u_z w_{i,x}, && w_{i+1,s} = w_{i,y} - u_s w_{i,x}, \\ & \ldots && \ldots \end{aligned}$$

The first one is local, the others are nonlocal.

Some explicit results – local symmetries

The basis of local symmetries of (1) is

$$\phi_1 = ysu_s + tsu_z + yu + tyu_t + y^2 u_y, \qquad \phi_8 = u_z,$$

$$\phi_2 = su_s + yu_y, \qquad \phi_9 = su_z + yu_t,$$

$$\phi_3 = tu + t^2 u_t + tyu_y + tzu_z + yzu_s, \qquad \phi_{10} = u + tu_t + yu_y$$

$$\phi_4 = tu_y + zu_s, \qquad \phi_{11} = u_t,$$

$$\phi_5 = tu_z + yu_s, \qquad \phi_{12} = u_y,$$

$$\phi_{13}(U(t,x,y)) = sU_y - uU_x + zU_t + Uu_x, \qquad \phi_6 = u_s$$

$$\phi_{14}(U(t,x,y)) = U, \qquad \phi_7 = tu_t + zu_z,$$

Lie algebra structure

On the next slide we will see commutators of above symmetries.
All U_i, V_i, V_{ji}, $i = 1, 2$, $j = 1, \ldots, 13$ are functions of t, x, y, and

$$V_{1i} = -(i-1)yV_i - tyV_{i,t} - y^2 V_{i,y}, \quad V_{2i} = -yV_{i,y},$$

$$V_{3i} = -(i-1)tV_i - t^2 V_{i,t} - tyV_{i,y}, \quad V_{4i} = -tV_{i,y},$$

$$V_{51} = -tV_{1,t} - yV_{1,y}, \quad V_{61} = -V_{1,y},$$

$$V_{7i} = -tV_{i,t}, \quad V_{81} = -V_{1,t},$$

$$V_{9i} = -yV_{i,t}, \quad V_{10i} = -(i-1)V_i - tV_{i,t} - yV_{i,y},$$

$$V_{11i} = -V_{i,t}, \quad V_{12i} = -V_{i,y},$$

$$V_{13i} = U_{1,x}V_i - U_1 V_{i,x}.$$

	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	ϕ_8	ϕ_9	ϕ_{10}	ϕ_{11}	ϕ_{12}	ϕ_{13}	ϕ_{14}
ϕ_1	ϕ_1	0	ϕ_3	0	ϕ_5	0	0	0	ϕ_1	ϕ_9	$\phi_2 + \phi_{10}$	$\phi_{13}(V_{11})$	$\phi_{14}(V_{12})$
ϕ_2		0	ϕ_4	0	ϕ_6	0	0	$-\phi_9$	0	0	ϕ_{12}	$\phi_{13}(V_{21})$	$\phi_{14}(V_{22})$
ϕ_3			0	0	0	ϕ_3	ϕ_5	ϕ_1	ϕ_3	$\phi_{10} + \phi_7$	ϕ_4	$\phi_{13}(V_{31})$	$\phi_{14}(V_{32})$
ϕ_4				0	0	ϕ_4	ϕ_6	$\phi_2 - \phi_7$	0	ϕ_{12}	0	$\phi_{13}(V_{41})$	$\phi_{14}(V_{41})$
ϕ_5					0	0	0	0	ϕ_5	ϕ_8	ϕ_6	$\phi_{14}(V_{51})$	0
ϕ_6						0	0	$-\phi_8$	0	0	0	$\phi_{14}(V_{61})$	0
ϕ_7							ϕ_8	ϕ_9	0	ϕ_{11}	0	$\phi_{13}(V_{71})$	$\phi_{14}(V_{72})$
ϕ_8								0	0	0	0	$\phi_{14}(V_{81})$	0
ϕ_9									0	0	ϕ_{11}	$\phi_{13}(V_{91})$	$\phi_{14}(V_{92})$
ϕ_{10}										ϕ_{11}	ϕ_{12}	$\phi_{13}(V_{101})$	$\phi_{14}(V_{102})$
ϕ_{11}											0	$\phi_{13}(V_{111})$	$\phi_{14}(V_{112})$
ϕ_{12}												$\phi_{13}(V_{121})$	$\phi_{14}(V_{122})$
ϕ_{13}												$\psi_{13}(V_{131})$	$\phi_{14}(V_{132})$
ϕ_{14}													0

Action of the recursion operator on the basis symmetries

Denoting $\psi_i = \mathcal{R}(\phi_i)$, $i = 1 \dots 14$ and using nonlocal variable w_1 defined by $w_{1,z} = u_t + uu_{xz}, w_{1,s} = u_y + uu_{xs}$ we obtain the following results:

$$\psi_1 = s\phi_{10} + y\psi_{10}, \quad \psi_2 = s\phi_{12} + y\psi_{12},$$
$$\psi_3 = z\phi_{10} + t\psi_{10}, \quad \psi_4 = z\phi_{12} + y\psi_{12},$$
$$\psi_7 = z\phi_{11} + t\psi_{11}, \quad \psi_9 = s\phi_{11} + y\psi_{11},$$

$$\psi_5 = \phi_{10}, \quad \psi_6 = \phi_{12}, \quad \psi_8 = \phi_{11}, \quad \psi_{14} = \phi_{13},$$

$$\psi_{10} = -uu_x + 2w_1 + t\psi_{11} + y\psi_{12},$$

$$\psi_{11} = -uu_{tx} + w_{1,t},$$
$$\psi_{12} = -uu_{xy} + w_{1,y},$$

$$\psi_{13} = \frac{1}{2}(z^2 U_{tt} + s^2 U_{yy} + u^2 U_{xx}) + z(u_x U_t - uU_{tx} + sU_{ty}) + s(u_x U_y - uU_{xy}) - uu_{xx} U - w_1 U_x + w_{1,x} U.$$

Action of the recursion operator

The action of the direct recursion operator $\psi = \mathcal{R}(\varphi)$ generates nonlocal symmetries in τ_-.

Moreover, the action of its inverse $\varphi = \mathcal{R}^{-1}(\psi)$ generates those in τ_+.

Thank you

for your attention

http://jets.math.slu.cz/

Baran H., Marvan M. Jets. A software for differential calculus on jet spaces and diffieties. `http://jets.math.slu.cz/`

Błaszak M. Classical R-matrices on Poisson algebras and related dispersionless systems, Phys. Lett. A 2002, **297**, 191–195

Bocharov A.V., Chetverikov V.N., Duzhin S.V., Khor′kova N.G., Krasil′shchik I.S., Samokhin A.V., Torkhov Yu.N., Verbovetsky A.M., Vinogradov A.M., Symmetries and Conservation Laws for Differential Equations of Mathematical Physics, Transl. Math. Monogr., 182, American Mathematical Society, Providence, 1999

Burtsev S. P., Zakharov V. E., Mikhailov A. V. Inverse scattering method with variable spectral parameter. *Theor. Math. Phys.*, 1987, **70**, no. 3, 227–240

Cieśliński J. Non-local symmetries and a working algorithm to isolate integrable geometries. *J. Phys. A.*, 1993, **26**, L267–L271

Cieśliński J. Group interpretation of the spectral parameter in the case of nonhomogeneous, nonlinear Schrödinger system. *J. Math. Phys.*, 1993, **34**, 2372–2384

- Dunajski M. A class of Einstein – Weil spaces associated to an integrable system of hydrodynamic type. *J. Geom. Phys.*, 2004, **51**, 126–137
- M. Dunajski, W. Kryński, Einstein–Weyl geometry, dispersionless Hirota equation and Veronese webs, arXiv:1301.0621.
- Ferapontov E.V., Khusnutdinova K.R. Hydrodynamic reductions of multi-dimensional dispersionless PDEs: the test for integrability. arXiv:nlin/0312015
- E. Ferapontov, B. Kruglikov, Dispersionless integrable systems in 3D and Einstein–Weyl geometry, arXiv:1208.2728.
- E. Ferapontov, J. Moss, Linearly degenerate PDEs and quadratic line complexes, arXiv:1204.2777.
- Igonin, S., Krasil′shchik, J. On one-parametric families of Bäcklund transformations. Preprint arXiv:nlin/0010040 (2000)
- Krasil′shchik, I.S.: On one-parametric families of Bäcklund transformations. Preprint DIPS-1/2000, The Diffiety Institute, Pereslavl-Zalessky (2000)

- Krasil′shchik I.S., Kersten P.H.M., Deformations and recursion operators for evolution equations, In: Geometry in Partial Differential Equations, World Scientific, River Edge, 1994, 114âĂŞ154
- Krasil′shchik I.S., Kersten P.H.M., Graded differential equations and their deformations: a computational theory for recursion operators, Acta Appl. Math., 1995, 41(1-3), 167âĂŞ191
- Krasil′shchik I.S., Lychagin V.V., Vinogradov A.M., Geometry of Jet Spaces and Nonlinear Differential Equations, Gordon and Breach, New York, 1986
- Krasil′shchik I.S., Vinogradov A.M. Nonlocal symmetries and the theory of coverings: and addendum to Vinogradov's "Local symmetries and conservation laws". Acta Appl. Math., 1984, Vol. 2, 79–96
- Krasil′shchik, I.S., Vinogradov, A.M. Nonlocal trends in the geometry of differential equations: symmetries, conservation laws, and Bäcklund transformations. Acta Appl. Math., 1989, Vol. 15, 161–209

Malykh A.A., Nutku Y., Sheftel M.B. Partner symmetries and non-invariant solutions of four-dimensional heavenly equations. J. Phys. A, 2004, Vol. 37 7527–7546

Martínez Alonso L., Shabat A.B. Hydrodynamic reductions and solutions of a universal hierarchy. Theor. Math. Phys., 2004, **104**, 1073–1085

Marvan M. Another look on recursion operators, In: Differential Geometry and Applications, Brno, August 28âĂŞ September 1, 1995, Masaryk University, Brno, 1996, 393–402

Marvan M. On the horizontal gauge cohomology and nonremovability of the spectral parameter. Acta Appl. Math., 2002, Vol. 72, 51–65

Marvan M. Sufficient set of integrability conditions of an orthonomic system. *Foundations of Computational Mathematics*, 2009, **9** 651–674

Marvan M., Sergyeyev A., Recursion operators for dispersionless integrable systems in any dimension, Inverse Problems 2012 28 (2), 025011

Morozov O.I. Contact integrable extensions of symmetry pseudo-groups and coverings of (2+1) dispersionless integrable equations, J. Geom. Phys., 2009, 59, 1461–1475

Morozov O.I. The four-dimensional Martínez Alonso–Shabat equation: differential coverings and recursion operators. arXin:1309.4993 (2013)

Morozov O.I., Sergyeyev A. The four-dimensional Martínez Alonso–Shabat equation: reductions and nonlocal symmetries. arXiv:1401.7942 (2014)

Ovsienko V. Bi-Hamiltionian nature of the equation $u_{tx} = u_{xy}\, u_y - u_{yy}\, u_x$ *Adv. Pure Appl. Math.* **1** 7–17 (2010)

Pavlov M.V. Integrable hydrodynamic chains. J. Math. Phys., **44**, 4134–4156 (2003)

Pavlov M.V. The Kupershmidt hydrodynamics chains and lattices. Intern. Math. Research Notes, **2006** article ID 46987, 1 âĂŞ 43 (2006)

Vinogradov A.M., Local symmetries and conservation laws, Acta Appl. Math., 1984, 2, 21–78

Zakharevich I. Nonlinear wave equation, nonlinear Riemann problem, and the twistor transform of Veronese webs, arXiv:math-ph/0006001.