Workshop on Integrable Nonlinear Equations (WINE-2015)

Coverings and nonlocal symmetries of the 3DrdDym equation

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References

⋆ HB, JK, OM, PV:
Coverings over Lax integrable equations and their nonlocal symmetries, arXiv:1507.00897v2 [nlin.SI]

The equation

The 3DrdDym equation $\mathscr{E}$

$$u_{ty} = u_x u_{xy} - u_y u_{xx}$$

is one of a few linearly degenerate integrable PDEs (see Ferapontov, Moss, *Communications in Analysis and Geometry*, **23** (2015) no. 1, 91–127; `arXiv:1204.2777` [math.DG])

It is homogeneous with respect to the weights

$$|x| = 1, \qquad |u| = 2, \qquad |y| = |t| = 0;$$

then $|\mathscr{E}| = 2$.

Local symmetries

The algebra $\mathrm{sym}(\mathscr{E})$ of local symmetries is generated by

$$\begin{aligned} \psi_0 &= xu_x - 2x, \\ \upsilon_0(B) &= Bu_y, && B = B(y), \\ \theta_0(A) &= Au_t + A'(xu_x - u) + \frac{1}{2}A''x^2, \\ \theta_{-1}(A) &= Au_x + A'x, \\ \theta_{-2}(A) &= A, && A = A(t). \end{aligned}$$

The subscripts coincide with the weights of the corresponding vector fields.

Local symmetries

The commutators are

	ψ_0	$\upsilon_0(\bar{B})$	$\theta_0(\bar{A})$	$\theta_{-1}(\bar{A})$	$\theta_{-2}(\bar{A})$
ψ_0	0	0	0	$-\theta_{-1}(\bar{A})$	$-2\theta_{-2}(\bar{A})$
$\upsilon_0(B)$	...	$\upsilon_0(B\bar{B}' - B'\bar{B})$	0	0	0
$\theta_0(A)$	...	...	$\theta_0(\bar{A}A' - \bar{A}'A)$	$\theta_{-1}(\bar{A}A' - \bar{A}'A)$	$\theta_{-2}(\bar{A}A' - \bar{A}'A)$
$\theta_{-1}(A)$	...	...	...	$\theta_{-2}(\bar{A}A' - \bar{A}'A)$	0
$\theta_{-2}(A)$	...	...	...	...	0

One has

$$\{\upsilon_0(B)\} = \langle \text{vector fields on } \mathbb{R}^1(y)\rangle$$

and

$$\{\theta_i(A)\} = \langle \text{vector fields on } \mathbb{R}^1(t)\rangle \otimes \mathbb{R}_{-3}[z],$$

where $\mathbb{R}_{-3}[z] = \mathbb{R}[z^{-1}]/(z^{-3} = 0)$ is the ring of truncated polynomials.

Integrability features

Recursion operators (Morozov, arXiv:1202.2308[nlin.SI])

$$\mathscr{R}^+ : \begin{aligned} u_y D_t(\hat{\varphi}) &= u_y D_x(\varphi) - u_x D_y(\varphi) + (u_x u_{xy} - u_y u_{xx})\hat{\varphi}, \\ u_y D_x(\hat{\varphi}) &= u_y \hat{\varphi} - D_y(\varphi); \end{aligned}$$

$$\mathscr{R}^- : \begin{aligned} D_x(\varphi) &= D_t(\hat{\varphi}) - u_x D_x(\hat{\varphi}) + u_{xx}\hat{\varphi}, \\ D_y(\varphi) &= u_{xy}\hat{\varphi} - u_y D_x(\hat{\varphi}). \end{aligned}$$

They are "mutually inverse".
Lax pair

$$w_t = (u_x - \lambda) w_x,$$
$$w_y = \frac{u_y w_x}{\lambda},$$

where $\lambda \in \mathbb{R} \setminus \{0\}$.

Coverings

Expanding $w = \sum_{i=-\infty}^{+\infty} w_i \lambda^i$ we get

$$w_{i,t} = u_x w_{i,x} - w_{i-1,x},$$
$$w_{i,y} = u_y w_{i+1,x}, \quad i \in \mathbb{Z},$$

(cf. Pavlov, J. Math. Phys., **44** (2003), 4134–4156) and cut in into two parts:

1. The positive hierarchy, $w_i = 0$ for $i < 0$.
2. The negative hierarchy, $w_i = 0$ for $i > 0$.

We study symmetries in the resulting coverings.

The positive case, τ_+

Here we have $w_{0,t} = w_{0,x} = 0$; hence $w_0 = G(y)$ and

$$
\left.\begin{aligned} w_{1,t} &= \frac{u_x}{u_y} G', \\ w_{1,x} &= \frac{G'}{u_y}; \end{aligned}\ \right| \ \begin{aligned} w_{i,t} &= \frac{u_x}{u_y} w_{i-1,y} - w_{i-1,x}, \\ w_{i,x} &= \frac{w_{i-1,y}}{u_y}, \quad i > 1. \end{aligned}
$$

Changing $y \mapsto G(y)$ (a symmetry of $\mathscr{E}$) and relabeling, we arrive to

$$
\tau_+ : \quad \left.\begin{aligned} q_{1,t} &= \frac{u_x}{u_y}, \\ q_{1,x} &= \frac{1}{u_y}; \end{aligned}\ \right| \ \begin{aligned} q_{i,t} &= \frac{u_x}{u_y} q_{i-1,y} - q_{i-1,x}, \\ q_{i,x} &= \frac{q_{i-1,y}}{u_y}, \quad i > 1; \end{aligned}
$$

plus additional nonlocal variables $q_i^{(j)}$: $q_i^{(0)} = q_i$, $q_i^{(j+1)} = \left(q_i^{(j)}\right)_y$.

The negative case, τ_-

In this case,

$$w_{0,x} = 0, \quad w_{0,y} = 0; \quad \Big| \quad w_{-1,x} = u_x w_{0,x} - w_{0,t}, \quad w_{-1,y} = u_y w_{0,x}; \quad \Big| \quad w_{-2,x} = u_x w_{-1,x} - w_{-1,t}, \quad w_{-2,y} = u_y w_{-1,x};$$

Consequently,

$$w_0 = \bar{F}, \quad w_{-1} = -x\bar{F}' + G, \quad w_{-2} = -\bar{F}\frac{1}{2}x^2\bar{F}'' - G'x + H,$$

where $\bar{F}$, G, H are functions in t.
Skipping G and H and relabeling, we get

$$\bar{r}_{1,x} = F(u_x^2 - u_t) - F'(u + xu_x) + \tfrac{1}{2}x^2F'', \quad \bar{r}_{1,y} = u_y(Fu_x - xF'); \quad \Big| \quad \bar{r}_{i,x} = u_x\bar{r}_{i-1,x} - \bar{r}_{i-1,t}, \quad \bar{r}_{i,y} = u_y\bar{r}_{i-1,x},$$

where $F = -\bar{F}'$.

The negative case, $\boldsymbol{\tau}_-$

All these coverings are equivalent to the case $F = 1$: set

$$r_1 = -F\bar{r}_1 - F'xu + \frac{1}{6}F''x^3,$$
$$r_i = \frac{1}{i+2}\mathscr{Y}_-(r_{i-1}), \quad i > 1,$$

where

$$\mathscr{Y}_- = -x\frac{\partial}{\partial t} + 2u\frac{\partial}{\partial x} - 3\bar{r}_1\frac{\partial}{\partial u} + \sum_{i\geq 1}(i+2)\bar{r}_{i+1}\frac{\partial}{\partial \bar{r}_i}.$$

Then

$$\begin{array}{l|l} r_{1,x} = u_x^2 - u_t, & r_{i,x} = u_x r_{i-1,x} - r_{i-1,t}, \\ r_{1,y} = u_y u_x; & r_{i,y} = u_y r_{i-1,x} \end{array}$$

plus additional nonlocal variables $r_i^{(j)}$: $r_i^{(0)} = r_i$, $r_i^{(j+1)} = \left(r_i^{(j)}\right)_t$.

Nonlocal symmetries in the positive case, $\mathrm{sym}\,\tau_+$

Any symmetry $X \in \mathrm{sym}\,\tau_+$ is of the form

$$X = \mathbf{E}_\varphi + \sum_i \left(\varphi^i \frac{\partial}{\partial q_i} + \sum_j D_y^j(\varphi^i) \frac{\partial}{\partial q_i^{(j)}} \right),$$

where the vector-function $\Phi = \langle \varphi, \varphi^1, \dots, \varphi^i, \dots \rangle$ enjoys

$$D_t D_y(\varphi) = u_{xy} D_x(\varphi) + u_x D_x D_y(\varphi) - u_{xx} D_y(\varphi) - u_y D_x^2(\varphi),$$
$$q_{1,t} D_y(\varphi) + u_y D_t(\varphi^1) = D_x(\varphi),$$
$$q_{1,x} D_y(\varphi) + u_y D_x(\varphi^1) = 0,$$

and

$$u_y(D_t(\varphi^i) + D_x(\varphi^{i-1})) + (q_{i,t} + q_{i-1,x}) D_y(\varphi) = q_{i-1,y} D_x(\varphi) + u_x D_y(\varphi^{i-1}),$$
$$q_{i,x} D_y(\varphi) + u_y D_x(\varphi^i) = D_y(\varphi^{i-1})$$

for $i > 1$.

Nonlocal symmetries in the positive case, lifts of ψ_0 and $\theta_j(A)$

By direct check, one can see that $\Psi_0 = \langle \psi_0, \psi_0^1, \dots, \psi_0^i, \dots \rangle$, where

$$\psi_0^i = iq_i + xq_{i,x}.$$

Also $\Theta_j(A) = \langle \theta_j(A), \theta_j^1(A), \dots, \theta_j^i(A), \dots \rangle$, $j = 0, -1, -2$, with

$$\begin{aligned}
\theta_{-2}^i(A) &= 0, \\
\theta_{-1}^i(A) &= Aq_{i,x}, \\
\theta_0^1(A) &= \theta_{-1}(A)q_{1,x}, \\
\theta_0^i(A) &= \theta_{-1}(A)q_{i,x} - Aq_{i-1,x}, \quad i > 1.
\end{aligned}$$

Nonlocal symmetries in the positive case, invisible symmetries

An invisible symmetry of depth k is

$$\Upsilon_k(B) = \langle \underbrace{0, \dots, 0}_{k \text{ times}}, \varphi^1_{\text{inv}}(B), \dots, \varphi^i_{\text{inv}}(B), \dots \rangle.$$

They do exist:

$$\varphi^1_{\text{inv}}(B) = B(y), \qquad \varphi^{i+1}_{\text{inv}}(B) = \frac{1}{i} \mathscr{X}(\varphi^i_{\text{inv}}(B)), \quad i > 1,$$

where

$$\mathscr{X} = q_1 \frac{\partial}{\partial y} + \sum_{i \geq 1} (i+1) q_{i+1} \frac{\partial}{\partial q_i}.$$

One has $|\Upsilon_k(B)| = k$.

Nonlocal symmetries in the positive case, two hierarchies

There also exist two nonlocal symmetries

$$\Psi_{-1} = \langle q_1 u_y, \psi^1_{-1}, \dots, \psi^i_{-1}, \dots \rangle,$$
$$\Psi_{-2} = \langle (2q_2 - q_1 q_1^{(1)}) u_y, \psi^1_{-2}, \dots, \psi^i_{-2}, \dots \rangle$$

with

$$\psi^i_{-1} = -(i+1) q_{i+1} + q_i^{(1)} q_1,$$
$$\psi^i_{-2} = -(i+2) q_{i+2} + q_1 q_{i+1}^{(1)} + (2q_2 - q_1 q_1^{(1)}) q_i^{(1)}.$$

We set

$$\Psi_{-k} = \{\{\dots\{\Psi_{-2}, \underbrace{\Psi_{-1}\}, \dots\}, \Psi_{-1}\}}_{k-2 \text{ times}}, \quad k \geq 3,$$
$$\Upsilon_{-k}(B) = \{\Psi_{-k-1}, \Upsilon_1(B)\}, \quad k \geq 0.$$

In particular, $\Upsilon_0(B)$ is the lift of $\upsilon_0(B)$. One has $|\Psi_{-k}| = |\Upsilon_{-k}(B)| = -k$.

Nonlocal symmetries in the positive case, an intermediate summary

weight	$\dots$	$-l$	$\dots$	-2	-1	0	1	$\dots$	k	$\dots$
	$\dots$	Ψ_{-l}	$\dots$	Ψ_{-2}	Ψ_{-1}	Ψ_{0}	■	■	■	■
	■	■	■	$\Theta_{-2}(A)$	$\Theta_{-1}(A)$	$\Theta_{0}(A)$	■	■	■	■
	$\dots$	$\Upsilon_{-l}(B)$	$\dots$	$\Upsilon_{-2}(B)$	$\Upsilon_{-1}(B)$	$\Upsilon_{0}(B)$	$\Upsilon_{1}(B)$	$\dots$	$\Upsilon_{k}(B)$	$\dots$

■ denotes unoccupied slots

Nonlocal symmetries in the positive case, commutators

Computing the generalized Jacobi brackets, one obtains

$$\{\Psi_{-k}, \Psi_{-l}\} = \frac{(k-2)!(l-2)!}{(k+l-2)!}(k-l)\Psi_{-k-l},$$

$$\{\Psi_{-k}, \Upsilon_l(B)\} = \frac{l(-l-1)!(k-2)!}{(l-k-1)!}\Upsilon_{l-k}(B),$$

$$\{\Upsilon_k(B), \Upsilon_l(\bar{B})\} = \frac{(-k-1)!(-l-1)!}{(-k-l-1)!}\Upsilon_{k+l}(B\bar{B}' - B'\bar{B})$$

(we set $n! = 1$ for $n < 0$). After gauging the basis

$$\Psi_{-k} \mapsto \frac{1}{(k-2)1}\Psi_{-k}, \qquad \Upsilon_l(B) \mapsto \frac{1}{(-l-1)!}\Upsilon_l(B),$$

we arrive to the final description of $\operatorname{sym}\tau_+$.

Nonlocal symmetries in the positive case, Lie algebra structure

Notation:

$$\mathfrak{V}[s] = \langle \text{vector fields on } \mathbb{R}^1(s) \rangle,$$

$$\mathfrak{W}^- = \langle z^{-k+1}\frac{\partial}{\partial z},\ k = 0, 1, \dots \rangle,\ \text{non-positive Witt algebra},$$

$$\mathfrak{L}[s] = \langle z^m B(s)\frac{\partial}{\partial z},\ m \in \mathbb{Z} \rangle = \mathfrak{V}[s] \otimes \mathbb{R}[z, z^{-1}],\ \text{loop algebra},$$

$$\mathfrak{V}_{-3}[s] = \mathfrak{V}[s] \otimes \mathbb{R}[z^{-1}]/(z^{-3} = 0).$$

Theorem

One has the isomorphism

$$\text{sym}\, \tau_+ \sim \mathfrak{W}^- \ltimes \left(\mathfrak{L}[y] \oplus \mathfrak{V}_{-3}[t]\right)$$

with the natural action of $\mathfrak{W}^-$ *on* $\mathfrak{L}[y]$ *and* $\mathfrak{V}_{-3}[t]$.

Nonlocal symmetries in the negative case, sym τ_-

Any symmetry $X \in \mathrm{sym}\,\tau_+$ is of the form

$$X = \mathbf{E}_\varphi + \sum_i \left(\varphi^i \frac{\partial}{\partial r_i} + \sum_j D_t^j(\varphi^i) \frac{\partial}{\partial r_i^{(j)}} \right),$$

where the vector-function $\Phi = \langle \varphi, \varphi^1, \dots, \varphi^i, \dots \rangle$ enjoys

$$\begin{aligned}
D_t D_y(\varphi) &= u_{xy} D_x(\varphi) + u_x D_x D_y(\varphi) - u_{xx} D_y(\varphi) - u_y D_x^2(\varphi),\\
D_x(\varphi^1) &= 2u_x D_x(\varphi) - D_t(\varphi),\\
D_y(\varphi^1) &= u_x D_y(\varphi) + u_y D_x(\varphi),
\end{aligned}$$

and

$$\begin{aligned}
D_x(\varphi^i) &= r_{i-1,x} D_x(\varphi) + u_x D_x(\varphi^{i-1}) - D_t(\varphi^{i-1}),\\
D_y(\varphi^i) &= r_{i-1,x} D_y(\varphi) + u_y D_x(\varphi^{i-1})
\end{aligned}$$

for $i > 1$.

Nonlocal symmetries in the negative case, an overview

All computations follow the same lines as in the positive case, but are more complicated technically. The results for τ_- and τ_- are compared here:

sym τ_-	$\dots$	$-l$	$\dots$	-2	-1	0	1	$\dots$	k	$\dots$
	■	■	■	■	■	$\Psi_{\mathbf{0}}$	$\Psi_{\mathbf{1}}$	$\dots$	Ψ_k	$\dots$
	$\dots$	$\Theta_{-l}(A)$	$\dots$	$\Theta_{-\mathbf{2}}(A)$	$\Theta_{-\mathbf{1}}(A)$	$\Theta_{\mathbf{0}}(A)$	$\Theta_{\mathbf{1}}(A)$	$\dots$	$\Theta_k(A)$	$\dots$
	■	■	■	■	■	$\Upsilon_{\mathbf{0}}(B)$	■	■	■	■
sym τ_+	$\dots$	Ψ_{-l}	$\dots$	$\Psi_{-\mathbf{2}}$	$\Psi_{-\mathbf{1}}$	$\Psi_{\mathbf{0}}$	■	■	■	■
	■	■	■	$\Theta_{-\mathbf{2}}(A)$	$\Theta_{-\mathbf{1}}(A)$	$\Theta_{\mathbf{0}}(A)$	■	■	■	■
	$\dots$	$\Upsilon_{-l}(B)$	$\dots$	$\Upsilon_{-\mathbf{2}}(B)$	$\Upsilon_{-\mathbf{1}}(B)$	$\Upsilon_{\mathbf{0}}(B)$	$\Upsilon_{\mathbf{1}}(B)$	$\dots$	$\Upsilon_k(B)$	$\dots$

In particular, $\Theta_{-l}(A)$, $l \geq 3$, are invisible symmetries in τ_- that are obtained by the operator

$$\mathscr{Y}_+ = -x\frac{\partial}{\partial t} + 2u\frac{\partial}{\partial x} + 3r_1\frac{\partial}{\partial u} + sum_{\geq 1}(i+3)r_{i+1}\frac{\partial}{\partial r_i}.$$

Nonlocal symmetries in the negative case, Lie algebra structure

Theorem

One has the isomorphism

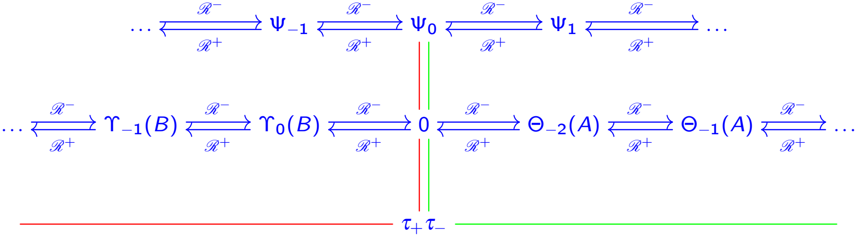
$$\operatorname{sym}\tau_{-} \sim \left(\mathfrak{W}^{+} \ltimes \mathfrak{L}[t]\right) \oplus \mathfrak{V}[y]$$

with the natural action of $\mathfrak{W}^{+}$ *on* $\mathfrak{L}[t]$, *where*

$$\mathfrak{M}^{+} = \langle z^{k+1}\frac{\partial}{\partial z},\ k = 0, 1, \dots\rangle$$

is the nonnegative Witt algebra.

Action of recursion operators



THANK YOU

FOR YOUR ATTENTION