Dispersionless integrable hierarchies and GL(2) geometry

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(based on joint work with E. Ferapontov)

Paraconformal structures

Paraconformal or $GL(2)$ geometry on an n-dimensional manifold M is defined by a field of rational normal curves of degree $n-1$ in the projectivised cotangent bundle $\mathbb{P}T^*M$. Equivalently, for a coframe $\{\omega_i\}$ on M it can be viewed as a field of 1-forms

$$\omega(\lambda) = \omega_0 + \lambda\omega_1 + \cdots + \lambda^{n-1}\omega_{n-1}.$$

This field and the parameter λ are defined up to transformations

$$\lambda \mapsto \frac{a\lambda + b}{c\lambda + d}, \quad \omega(\lambda) \mapsto r(c\lambda + d)^{n-1}\omega(\lambda),$$

where $a, b, c, d, r \in C^\infty(M)$, $ad - bc = 1$, $r \neq 0$.

Conventionally, a $GL(2)$ geometry is defined by a field of rational normal curves in the projectivised *tangent* bundle $\mathbb{P}TM$.

Both pictures are projectively dual: the equation $\omega(\lambda) = 0$ defines a one-parameter family of hyperplanes that osculate a dual rational normal curve $\tilde{\omega}(\lambda) \subset \mathbb{P}TM$.

$GL(2)$ geometry arises in:

- **Poisson geometry:** Compatible bivectors of Kronecker type on an odd-dimensional manifold give rise to $GL(2)$ structures, identified by Gelfand and Zakharevich as Veronese webs in the theory of bi-Hamiltonian integrable systems.
- **Exotic holonomy:** In 4D there exist torsion-free affine connections whose holonomy group is the irreducible representation of $GL(2)$, yielding parallel $GL(2)$ structures. This was observed by Bryant, giving the first example of a holonomy not appearing on the Berger list.
- **Algebraic geometry:** By the deformation theory of Kodaira, the moduli space of rational curves in a compact complex surface $C \subset X$ with the normal bundle $\mathcal{O}(n)$ has dimension $(n+1)$. It also carries a canonical $GL(2)$ structure, studied by Cartan and Hitchin for $n = 2$, by Bryant for $n = 3$ and by Dunajski-Tod and Krynski for general n.

$GL(2)$ geometry arises in:

- **Ordinary differential equations:** For every scalar ODE of order n with vanishing Wünschmann (Doubrov-Wilczynski) invariants, its local solution space M^n is canonically endowed with a $GL(2)$ structure. This class of ODEs, by a result of Cap-Doubrov-The, satisfies the C-property of Cartan: every differential invariant is an integral.
- **Grassmann geometry:** The Segre structure induces a generalised conformal structure on submanifolds M in Grassmannians. Thus, a four-fold $\mathcal{E}^4 \subset Gr(3,5)$ carries a canonical $GL(2)$ structure. In Doubrov-Ferapontov-Kruglikov-Novikov this encoded integrablity of the resp 1st order system of PDEs with 2 independent and 3 dependent variables. Similarly, a canonical $GL(2)$ structure on a hypersurface in the Lagrangian Grassmannian $\mathcal{E}^5 \subset LG(3,6)$ encodes integrability of the resp 2nd order PDE.

$GL(2)$ geometry on integrable hierarchies

Jointly with E. Ferapontov we established that dispersionless integrable hierarchies of PDEs in 3D provide $GL(2)$ geometry on the solutions. In fact, the $GL(2)$ structure is given by the characteristic variety of the truncated hierarchy.

Moreover, the structures arising in such hierarchies provide explicit examples of particularly interesting classes of involutive $GL(2)$ structures studied in the literature.

In this way we obtain torsion-free $GL(2)$ structures of Bryant as well as totally geodesic $GL(2)$ structures of Krynski. The latter possess a compatible affine connection (with torsion) and a two-parameter family of totally geodesic α-manifolds, making them a natural generalisation of the Einstein-Weyl geometry.

Ex 1. Veronese web hierarchy:

compatible system for $1 \leq i < j < k \leq n$ starting from $n = 3$:

$$(c_i - c_j)u_k u_{ij} + (c_j - c_k)u_i u_{jk} + (c_k - c_i)u_j u_{ik} = 0.$$

The characteristic variety is defined by a system of quadrics

$$(c_i - c_j)u_k p_i p_j + (c_j - c_k)u_i p_j p_k + (c_k - c_i)u_j p_i p_k = 0,$$

which specify a rational normal curve in $\mathbb{P}T^*M$ parametrised as $p_i = \frac{u_i}{\lambda - c_i}$ (the ideal of a rational normal curve is generated by quadrics)

$$\omega(\lambda) = p_i dx^i = \sum \frac{u_i}{\lambda - c_i}\, dx^i;$$

Veronese system is equivalent to the Frobenius intergability conditions of the isospectral Lax representation

$$X_i = u_1(\lambda - c_i)\partial_{x^i} - u_i(\lambda - c_1)\partial_{x^1}, \quad 1 < i \leq n.$$

Ex 2. The dKP hierarchy:

$$u_{i,j+1}-u_{j,i+1}+\sum_{k=1}^{i}u_{i-k}u_{jk}-\sum_{k=1}^{j}u_{j-k}u_{ik}=0,\quad 1\leq i<j.$$

For $i=1,j=2$, $x^1=x$, $x^2=y$, $x^3=t$ we get the dispersionless Kadomtsev-Petviashvili equation $u_{xt}-u_{yy}-u_xu_{xx}=0$.
The characteristic variety is the intersection of quadrics

$$p_ip_{j+1}-p_jp_{i+1}+\sum_{k=1}^{i}u_{i-k}p_jp_k-\sum_{k=1}^{j}u_{j-k}p_ip_k=0.$$

It defines a rational normal curve, parametrised as $p_1=1$ and

$$p_{i+1}=\lambda p_i+\sum_{k=1}^{i-1}u_{i-k}p_k,\quad i\geq 1.$$

The dispersionless Lax representation is nonisospectral

$$X_i=\partial_{x^i}-\lambda\partial_{x^{i-1}}-\sum_{k=1}^{i-2}u_{i-k-1}\partial_{x^k}+u_{1,i-1}\partial_\lambda,\quad 1<i.$$

Ex 3. The universal hierarchy:

$$u_{i,j+1} - u_{i+1,j} + u_j u_{1,i} - u_i u_{1,j} = 0, \quad 0 < i < j < n,$$

starting from $n = 3$, in which case the equation is

$$u_{xt} - u_{yy} + u_y u_{xx} - u_x u_{xy} = 0.$$

The characteristic variety of this system is a rational normal curve in $\mathbb{P}T^*M$ parametrised as

$$p_x = 1,\ p_y = \lambda - u_x,\ p_t = \lambda^2 - u_x\lambda - u_y,\ p_z = \lambda^3 - u_x\lambda^2 - u_y\lambda - u_t,\ \dots$$

and so the $GL(2)$ structure is given by

$$\omega(\lambda) = \sum_{i=1}^{n} (\lambda^{i-1} - u_1\lambda^{i-2} - \cdots - u_{i-1})\, dx^i;$$

The dispersionless Lax representation is

$$X_i = \partial_{x^i} - (\lambda^{i-1} - u_1\lambda^{i-2} - \cdots - u_{i-1})\partial_{x^1}, \quad 1 < i \le n.$$

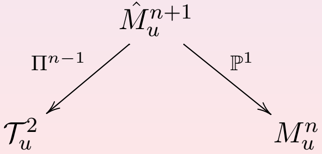
α-integrability and twistors

For $x \in M$, α-hyperplanes in T_xM are given by the equation $\omega(\lambda) = 0$; parametrised by $\lambda \in \mathbb{P}^1$. A hypersurface $S \subset M$ is an *α-manifold* if T_xS is α-hyperplane for each $x \in S$.

Definition

A $GL(2)$ structure is *involutive* $\equiv$ *α-integrable* if every α-hyperplane is tangential to some α-manifold.

In other words, the correspondence space $\hat{M}^{n+1}$, which is a $\mathbb{P}^1$ bundle over M^n, is locally foliated by α-manifolds Π^{n-1}. The following generalizes twistor correspondence for every solution u:



Adapted affine connections

An affine connection ∇ is $GL(2)$ compatible if $\nabla g = g$, where $g = \langle g_s \rangle$ is the ideal of quadratic forms defining the rational normal curve. In this form α-integrability generalizes the Weyl condition.

Types of canonical connections associated with $GL(2)$ structure:

- Torsion-free: $\exists$ only in 4D, equiv to involutivity
- Totally geodesic: shown by Krynski to satisfy the generalized Einstein-Weyl property $\mathrm{Ric}_{\nabla}^{\text{sym}} \in \tilde{g}$;
  $\exists$ for linearly degenerate (isospectral) hierarchies
- Normal: T_{∇} is trace-free and preserves α-hyperlances; totally geodesic connections are normal but not vise versa
- Projective: torsion-free connections possessing 2-parameter family of totally geodesic α-manifolds, not $GL(2)$-compatible

Normal and projective connections $\exists$ for all hierarchies.

Theorem 1: Paremetrization

Every involutive $GL(2)$ structure is locally of the form

$$\omega(\lambda) = \sum_{i=1}^{n} \Big[\prod_{j\neq i} \Big(\lambda - \frac{u_j}{v_j} \Big) \Big] u_i dx^i.$$

Here u and v are functions of $(x^1, \dots, x^n)$, and subscripts denote partial derivatives: $u_i = u_{x^i}$, $v_i = v_{x^i}$. These functions satisfy a system of second-order PDEs, 2 equations for each quadruple of indices $1 \leq i < j < k < l \leq n$: $E_{ijkl} = 0$, $F_{ijkl} = 0$ with

$$E_{ijkl} = \underset{(jkl)}{\mathfrak{S}} (a_i - a_j)(a_k - a_l) \left(\frac{2u_{ij} - (a_i + a_j)v_{ij}}{u_i u_j} + \frac{2u_{kl} - (a_k + a_l)v_{kl}}{u_k u_l} \right)$$

$$F_{ijkl} = \underset{(jkl)}{\mathfrak{S}} (b_i - b_j)(b_k - b_l) \left(\frac{2v_{ij} - (b_i + b_j)u_{ij}}{v_i v_j} + \frac{2v_{kl} - (b_k + b_l)u_{kl}}{v_k v_l} \right)$$

where $a_i = \frac{u_i}{v_i}$, $b_i = \frac{v_i}{u_i}$, and $\mathfrak{S}$ denotes cyclic summation.

Proof of Thm 1 (sketch)

The space of α-manifolds is parametrised by 1 arbitrary function of 1 variable. Choose n 1-parameter family of α-manifolds $\equiv$ (local) foliations of M given by $\lambda = a_i(x)$ and rectify them: $\omega(a_i) = f_i dx^i$ (no summation). In this coordinate system:

$$\omega(\lambda) = \sum_{i=1}^{n} \Big[\prod_{j\neq i} \frac{\lambda - a_j}{a_i - a_j} \Big] f_i dx^i.$$

Choose two extra 1-parameter families of α-manifolds: $\omega(a_{n+1}) = f_{n+1} du$ and $\omega(a_{n+2}) = f_{n+2} dv$, i.e.

$$f_i \prod_{j\neq i} \frac{a_{n+1} - a_j}{a_i - a_j} = f_{n+1} u_i, \quad f_i \prod_{j\neq i} \frac{a_{n+2} - a_j}{a_i - a_j} = f_{n+2} v_i.$$

Using the coordinate freedom send $a_{n+1} \to \infty$ and $a_{n+2} \to 0$ and use conformal freedom of $\omega(\lambda)$ to get its required formula.

The above overdetermined PDE system (EF) is obtained from the integrability condition $d\omega(\lambda) \wedge \omega(\lambda) = 0$.

Proof of Thm 1 - end of the argument

Indeed, collecting coefficients at $dx^i \wedge dx^j \wedge dx^k$ we obtain

$$\mathop{\mathfrak{S}}_{(jkl)} \frac{\lambda - a_i}{u_i} \left(\frac{1}{\lambda - a_k} - \frac{1}{\lambda - a_j} \right) \lambda_i + S_{ijk} = 0, \qquad (\dagger)$$

where $\lambda_i = \lambda_{x^i}$ and S_{ijk} is given by

$$u_{ij} \frac{a_j - a_i}{u_i u_j} \left(\frac{\lambda}{\lambda - a_i} + \frac{\lambda}{\lambda - a_j} \right) + u_{ik} \frac{a_i - a_k}{u_i u_k} \left(\frac{\lambda}{\lambda - a_i} + \frac{\lambda}{\lambda - a_k} \right) + u_{jk} \frac{a_k - a_j}{u_j u_k} \left(\frac{\lambda}{\lambda - a_j} + \frac{\lambda}{\lambda - a_k} \right)$$
$$- v_{ij} \frac{a_j - a_i}{u_i u_j} \left(\frac{\lambda a_i}{\lambda - a_i} + \frac{\lambda a_j}{\lambda - a_j} \right) - v_{ik} \frac{a_i - a_k}{u_i u_k} \left(\frac{\lambda a_i}{\lambda - a_i} + \frac{\lambda a_k}{\lambda - a_k} \right) - v_{jk} \frac{a_k - a_j}{u_j u_k} \left(\frac{\lambda a_j}{\lambda - a_j} + \frac{\lambda a_k}{\lambda - a_k} \right).$$

Denote by T_{ijk} the left-hand side of ($\dagger$). For four distinct indices $i \neq j \neq k \neq l$ there are only two non-trivial linear combinations that do not contain derivatives of λ:

$$T_{ikj} + T_{ijl} + T_{ilk} + T_{jkl} \quad \text{and} \quad \frac{1}{\lambda - a_l} T_{ikj} + \frac{1}{\lambda - a_k} T_{ijl} + \frac{1}{\lambda - a_j} T_{ilk} + \frac{1}{\lambda - a_i} T_{jkl}.$$

The first linear combination is equal to zero identically, while the second combination vanishes iff relations (EF) are satisfied.

Thus system (EF) governing general involutive $GL(2)$ structures results on elimination of the derivatives of λ from equations ($\dagger$).

Theorem 2: Involutivity

For every value of n, the following holds:

- The characteristic variety of system (EF) is the tangential variety of the rational normal curve $\mathbb{P}^1 \ni \lambda \mapsto \omega(\lambda) \in \mathbb{P}^{n-1}$.
- The characteristic variety has degree $2n-4$, and the rational normal curve can be recovered as its singular locus.
- System (EF) is in involution and its general solution depends on $2n-4$ functions of 3 variables (for analytic/formal case).

Note that although the PDE system (EF) formally consists of $2\binom{n}{4}$ equations, only $2\binom{n-2}{2}$ of them are linearly independent: we can restrict to equations $E_{12kl}=0$ and $F_{12kl}=0$ for $3 \leq k < l \leq n$ since all other equations are their linear combinations.

For $n=4$ system (EF) is determined: it consists of 2 second-order PDEs for 2 functions u and v of 4 independent variables, so the claim is instant, and this implies the count of Bryant.

Proof of Thm 2 - part 1

We parametrize the rational normal curve as

$$\lambda \mapsto [p_1 : \cdots : p_n] \in \mathbb{P}T^*M, \quad p_i = \frac{u_i}{\lambda - a_i}, \quad a_i = \frac{u_i}{v_i}, \qquad (\dagger)$$

so that its tangential variety is given by

$$(\lambda, \mu) \mapsto [p_1 : \cdots : p_n] \in \mathbb{P}T^*M, \quad p_i = \frac{u_i}{\lambda - a_i} + \frac{u_i \mu}{(\lambda - a_i)^2}. \quad (\ddagger)$$

The symbol of $\mathcal{E} = \{E = 0,\ F = 0\}$ is given by the matrix

$$\ell_{\mathcal{E}}(p) = \begin{bmatrix} \ell^u_E(p) & \ell^v_E(p) \\ \ell^u_F(p) & \ell^v_F(p) \end{bmatrix},$$

where $\ell^u_E(p) =$ is the symbol of u-linearization of E is given by

$$\ell^u_{E_{ijkl}}(p) = \sum_{a \le b} \frac{\partial E_{ijkl}}{\partial u_{ab}} p_a p_b = 2 \underset{(jkl)}{\mathfrak{S}} (a_i - a_j)(a_k - a_l)\left(\frac{p_i p_j}{u_i u_j} + \frac{p_k p_l}{u_k u_l}\right)$$

and similarly for other entries. Substitution of $(\dagger)$ yields $\ell_{\mathcal{E}}(p) = 0$, while substitution of $(\ddagger)$ outside $(\dagger)$ yields $\operatorname{rank}\big(\ell_{\mathcal{E}}(p)\big) = 1$.

Proof of involution: A. Projecitve modules

If an involutive PDE system $\mathcal{E}$ is linear, its symbolic module $\mathcal{M}_{\mathcal{E}} = g^*$ over the ring $\mathcal{R} = ST$ is projective (locally free).

Let $\mathcal{E}$ be defined by a k-th order differential operator $\Delta : \Gamma(\pi) \to \Gamma(\nu)$, corresponding to morphisms $\psi^{\Delta}_{k+i} : J^{k+i}\pi \to J^i\nu$, i.e. $\mathcal{E}_{k+i} = \mathrm{Ker}(\psi^{\Delta}_{k+i})$ for $i \geq 0$.

We construct a minimal free resolution of the symbolic module:

$$\cdots \to \mathcal{R} \otimes \varpi^* \xrightarrow{\psi^*} \mathcal{R} \otimes \nu^* \xrightarrow{\sigma^*_\Delta} \mathcal{R} \otimes \pi^* \longrightarrow \mathcal{M}_{\mathcal{E}} \to 0$$

and applying the functor $* = \mathrm{Hom}_{\mathbb{R}}(\cdot, \mathbb{R})$ get the exact sequence

$$0 \to g \hookrightarrow ST^* \otimes \pi \xrightarrow{\sigma_\Delta} ST^* \otimes \nu \xrightarrow{\psi} ST^* \otimes \varpi \to \dots$$

from which the compatibility conditions of $\mathcal{E} = \{\Delta = 0\}$ are $\Psi \circ \Delta|_{\mathcal{E}} = 0$ for $\Psi \in \mathrm{Diff}(\nu, \varpi)$ with the symbol ψ at x.

For nonlinear equations, apply the linearisation operator on a solution instead of Δ. Its symbol yields a syzygy, and hence compatibility operators, and Ψ is an operator in total derivatives.

Proof of involution: B. Explicit differential syzygies

The symbol $\ell_{\mathcal{E}}$ of the nonlinear vector-operator defining $\mathcal{E}$ in new coordinates $\xi_i = \frac{p_i}{u_i}$ on T_x^*M has components

$$\ell^u_{E_{ijkl}}(\xi) = 2 \underset{(jkl)}{\mathfrak{S}} (a_i - a_j)(a_k - a_l)\big(\xi_i\xi_j + \xi_k\xi_l\big),$$

$$\ell^v_{E_{ijkl}}(\xi) = - \underset{(jkl)}{\mathfrak{S}} (a_i - a_j)(a_k - a_l)\big((a_i + a_j)\xi_i\xi_j + (a_k + a_l)\xi_k\xi_l\big),$$

$$\ell^u_{F_{ijkl}}(\xi) = - \underset{(jkl)}{\mathfrak{S}} \frac{(a_i - a_j)(a_k - a_l)}{a_i a_j a_k a_l}\big((a_i + a_j)\xi_i\xi_j + (a_k + a_l)\xi_k\xi_l\big),$$

$$\ell^v_{F_{ijkl}}(\xi) = 2 \underset{(jkl)}{\mathfrak{S}} (a_i - a_j)(a_k - a_l)\left(\frac{\xi_i\xi_j}{a_k a_l} + \frac{\xi_k\xi_l}{a_i a_j}\right),$$

in the basis e_u, e_v of $\mathcal{R}^2$ and basis $e_{E_{ijkl}}, e_{F_{ijkl}}$ of $\mathcal{R}^{2\binom{n-2}{2}}$, where we restrict to indices $i = 1, j = 2$, $2 < k < l \leq n$.

This means that the homomorphism $\ell_{\mathcal{E}}$ maps $f(\xi)e_u$ to $f(\xi)\sum_{k<l}(\ell^u_{E_{12kl}}(\xi)e_{E_{12kl}} + \ell^u_{F_{12kl}}(\xi)e_{F_{12kl}})$ and similarly for $h(\xi)e_v$.

Proof of involution: B — cont'd

Now we resolve $\ell_{\mathcal{E}}$ by a homomorphism $\mathcal{C} = \mathcal{C}_{\mathcal{E}}$. The image $\mathcal{C}(\xi)(w)$ for $w = \sum_{i<j}(w_{E_{12ij}} e_{E_{12ij}} + w_{F_{12ij}} e_{F_{12ij}})$ has the following components $(2 < i < j < k \le n)$:

$$\mathcal{C}^{I}_{ijk} = \underset{(ijk)}{\mathfrak{S}} \big((a_2 - a_k)\xi_1 + (a_k - a_1)\xi_2 + (a_1 - a_2)\xi_k\big) w_{E_{12ij}}$$

$$\mathcal{C}^{II}_{ijk} = \underset{(ijk)}{\mathfrak{S}} \Big[\big((a_1 - a_2)(a_2 - a_k)a_1\xi_1 + (a_2 - a_1)(a_1 - a_k)a_2\xi_2 + \big((a_2 - a_k)^2 a_1 + (a_1 - a_k)^2 a_2\big)\xi_k\big) w_{E_{12ij}} + 2a_1a_2a_ia_j(a_1 - a_k)(a_2 - a_k)\xi_k w_{F_{12ij}}\Big]$$

$$\mathcal{C}^{III}_{ijk} = \underset{(ijk)}{\mathfrak{S}} \Big[2(a_1 - a_k)(a_2 - a_k)\xi_k w_{E_{12ij}} + \big((a_1 - a_2)(a_2 - a_k)a_1\xi_1 + (a_2 - a_1)(a_1 - a_k)a_2\xi_2 + \big((a_2 - a_k)^2 a_1 + (a_1 - a_k)^2 a_2\big)\xi_k\big) a_ia_j w_{F_{12ij}}\Big]$$

$$\mathcal{C}^{IV}_{ijk} = \underset{(ijk)}{\mathfrak{S}} \big((a_2 - a_k)a_1^2\xi_1 + (a_k - a_1)a_2^2\xi_2 + (a_1 - a_2)a_k^2\xi_k\big) a_ia_j w_{F_{12ij}}$$

Proof of involution: B — finish'd

One verifies that with these homomorphisms the following sequence is exact:

$$\mathcal{R}^2 \xrightarrow{\ell_{\mathcal{E}}} \mathcal{R}^{2\binom{n-2}{2}} \xrightarrow{\mathcal{C}_{\mathcal{E}}} \mathcal{R}^{4\binom{n-2}{3}}.$$

In other words, $\mathcal{C}_{\mathcal{E}}$ is the first syzygy for the module

$$\mathcal{M}^{\star}_{\mathcal{E}} = \mathrm{Ker}(\ell_{\mathcal{E}}) = \mathrm{Hom}_{\mathcal{R}}(\mathcal{M}_{\mathcal{E}}, \mathcal{R}).$$

Therefore, the differential syzygies for $\mathcal{E}$ are enumerated by 5 different indices $(12ijk)$, $2 < i < j < k \leq n$.

Consequently to verify compatibility conditions for each of these 5-tuples one can work in the corresponding 5-dimensional space, which is verified straightforwardly. And this yields involutivity.

Compatibility via free resolution

For a homomorphism $\varphi : \mathcal{R}^{n-2} \to \mathcal{R}^2$ the following sequence is known as the Eagon-Northcott complex ($\star = \mathrm{Hom}_{\mathcal{R}}(\cdot, \mathcal{R})$)

$$\cdots \to S^3\mathcal{R}^{\star 2} \otimes \Lambda^5 \mathcal{R}^{n-2} \xrightarrow{\partial} S^2\mathcal{R}^{\star 2} \otimes \Lambda^4 \mathcal{R}^{n-2} \xrightarrow{\partial} \mathcal{R}^{\star 2} \otimes \Lambda^3 \mathcal{R}^{n-2} \xrightarrow{\partial} \Lambda^2 \mathcal{R}^{n-2} \xrightarrow{\epsilon} \mathcal{R}.$$

It is exact when the Fitting ideal $I(\varphi)$, generated by 2×2 determinants of φ, contains a regular sequence of length $(n-3)$.

For the system $\mathcal{E}$ the map $\ell_{\mathcal{E}}$ split: $\ell_{\mathcal{E}}(e_u)$ and $\ell_{\mathcal{E}}(e_v)$ generate two complementary submodules $\Lambda^2\mathcal{R}^{n-2} \subset \mathcal{R}^{2\binom{n-2}{2}}$. Therefore two copies of the $\star$-dual Eagon-Northcott complex yield the following resolution of the $\star$-dual symbolic module:

$$0 \to \mathcal{M}_{\mathcal{E}}^{\star} \to \mathcal{R}^2 \xrightarrow{\ell_{\mathcal{E}}} \mathcal{R}^2 \otimes \Lambda^2 \mathcal{R}^{n-2} \xrightarrow{\mathcal{C}_{\mathcal{E}}} \mathcal{R}^{\star 2} \otimes \mathcal{R}^2 \otimes \Lambda^3 \mathcal{R}^{n-2} \xrightarrow{\partial^\star} S^2\mathcal{R}^{\star 2} \otimes \mathcal{R}^2 \otimes \Lambda^4 \mathcal{R}^{n-2} \to \dots$$

The Fitting condition corresponds to codimension $n-3$ of the zero set of $I(\ell_{\mathcal{E}})$ is the tangential variety to the rational normal curve.

That only 5-tuples of distinct indices enter the compatibility conditions we read off $\Lambda^3\mathcal{R}^{n-2}$: triples (ijk) yield 5-tuples $(12ijk)$.

Theorem 3: Intergability

For every n system (EF) is integrable via a dispersionless Lax representation. Thus letting $n \to \infty$ we obtain a dispersionless integrable hierarchy.

Indeed, on $\hat{M}_u^{n+1}$ the vector fields

$$V_{ijk} = \frac{\lambda - a_i}{u_i}\left(\frac{1}{\lambda - a_k} - \frac{1}{\lambda - a_j}\right)\partial_{x^i} + \frac{\lambda - a_j}{u_j}\left(\frac{1}{\lambda - a_i} - \frac{1}{\lambda - a_k}\right)\partial_{x^j} + \frac{\lambda - a_k}{u_k}\left(\frac{1}{\lambda - a_j} - \frac{1}{\lambda - a_i}\right)\partial_{x^k} - S_{ijk}\partial_\lambda,$$

generate distribution $V = span\langle V_{ijk}\rangle$ in $T\hat{M}_u$ of dimension $n-2$: due to identities for T_{ijk} a basis is given by V_{12l} for $3 \le l \le n$. Direct calculation based on the Frobenius theorem shows that on equations (EF) the distribution V is involutive.

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