

Shortest and straightest geodesics of an invariant sub-Riemannian metric on a homogeneous manifold

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Introduction

The efficiency of Riemannian geometry and its important role in applications based on the fact many important equations, arising in mechanics, mathematical physics, biology, economy, information theory etc can be reduced to the geodesic equation associated with a Riemannian metric. Moreover, Riemannian geometry give an effective tool for investigation of geodesic equations and other equations associated with the metric (Laplace, wave, heat and Schrödinger equation, Einstein equation etc).

There are many equivalent definitions of Riemannian geodesics . They are naturally generalised to sub-Riemannian manifold, but become non-equivalent.

We give a review of different definitions of geodesics of a sub-Riemannian manifold and interrelation between them.

Herz indicated two characterisations of geodesics:

geodesics as **shortest curves** based on Mopertrui's principle of least action (variational approach)) and

geodesics as **straightest curves** based on d'Alembert's principle of virtual work.

We recall three variational definitions of geodesics as (locally) shortest curves (Euler-Lagrange, Pontryagin and Hamilton) and three definitions of geodesics as straightest curves (d'Alembert , Levi-Civita-Schouten and Cartan-Tanaka), used in nonholonomic mechanics and discuss their interrelations.

A. Vershik and L. Faddeev showed that shortest geodesics for a generic sub-Riemannian manifolds (Q, D, g) are different from straightest geodesics on a open dense submanifold of Q . They gave first example (compact Lie group with the bi-invariant metric) when shortest geodesics coincides with straightest geodesics.

We generalised this example and consider a big class of sub-Riemannian manifolds associated with principal bundle over a Riemannian manifolds, for which shortest geodesics coincides with straightest geodesics. Using the geometry of flag manifolds, we describe some classes of compact homogeneous sub-Riemannian manifolds (including contact sub-Riemannian manifolds and symmetric sub-Riemannian manifolds) where straightest geodesics coincides with shortest geodesics. Construction of geodesics in these cases reduces to description of Riemannian geodesics of the Riemannian homogeneous manifold or left-invariant metric on a Lie group.

Geodesics as shortest curves. Euler-Lagrange variational definition of geodesics (EL-geodesics) on a Riemannian manifold (Q, g^Q)

Any Lagrangian $L(q, \dot{q})$ (homogeneous degree 1 in velocity \dot{q} function of TQ) defines a variational problem

$$\delta A_L = \delta \int_a^b L(q(t), \dot{q}(t)) dt = 0$$

in the space $C(q_0, q_1) = \{q(t), t \in [a, b], q_0 = \gamma(a), q_1 = \gamma(b)\}$. Solutions are critical points of A_L , i.e. solutions of the Euler-Lagrange equation

$$(EL) \quad \delta L_i := \frac{d}{dt} L_{\dot{q}_i} - L_{q_i} = 0.$$

which is a vector field $\Gamma^L \in \mathcal{X}(TQ)$.

EL-geodesics are critical point of the length functional $(L = \sqrt{g(\dot{q}, \dot{q})})$ or energy functional $(L = \frac{1}{2}g(\dot{q}, \dot{q}))$.

EL-geodesic on bracket generated sub-Riemannian manifold (Q, D, g^D)

$D \subset TM$ is a submanifold. **EL-geodesics** are solution of the variational problem

$$\delta A_L = \delta \int_a^b L(q(t), \dot{q}(t)) dt = 0$$

for length functional or energy functional in the space $C^h(q_0, q_1) = \{q(t), t \in [a, b], \dot{q} \in D, q_0 = \gamma(a), q_1 = \gamma(b)\}$.
i.e. a solution of EL-equation

$$\delta L_i := \frac{d}{dt} L_{\dot{q}_i} - L_{q_i} - \dot{\lambda}_a \omega_a - \lambda_a (\dot{q}) \omega_a$$

where $D = \{\omega_a = 0, a = 1, \dots, k\}$ This equation is a vector field $\Gamma^L \in \mathcal{X}(D \times D^0)$ where $D^0 = \text{Ann}(D) \subset T^*Q$.

Pontryagin optimal control definition of geodesics (P-geodesics) for bracket generated sub-Riemannian manifold (Q, D, g^D)

Let $L(q, \dot{q}) \in C^\infty(Q)$ be a Lagrangian on D , which is either the velocity $L = g^Q(\dot{q}, \dot{q})$ or the energy $\frac{1}{2}g^Q(\dot{q}, \dot{q})$. and X_i, \dots, X_m be an orthonormal (local) frame.

Any horizontal curve $q(t) \in C^h(q_0, q_1)$ is a solution of the ODE

$$(*) \quad \dot{q}^i(t) = \sum_{i=1}^n u^i(t) X_i(q^i(t)), \quad q(a) = q_0, q(b) = q_1.$$

where vector function $u(t)$ is an admissible control parameter and $A^h(q(t)) = \int_a^b L(q, \dot{q}) dt$ is the cost functional.

Pontryagin definition of geodesics

P-geodesic (resp., **minimal P-geodesic**) is a curves $q(t) \in C^h(q_0, q_1)$ which delivers a critical point (respectively, a minimum) for the cost functional

$$A_L = \int_a^b L(q, \dot{q}) dt$$

in the space of admissible control.

P-geodesics coincides with EL-geodesics.

For $D = TQ$, P-geodesics are geodesics of the Riemannian manifold (Q, g^Q) .

Hamiltonian sub-Riemannian geodesics (H-geodesics) in (Q, D, g^D)

Let p_D be the restriction of a linear form $p \in T_q^*Q$ to D and $g^*(p, p) = g^{-1}(p_D, p_D) \in C^\infty(T^*Q)$ the cometric.

H-geodesics are projection to Q . of orbits of Hamiltonian vector field $\vec{H} = \Omega^{-1}dH \in \mathcal{X}(T^*Q)$ with quadratic (degenerate) Hamiltonian $H = \frac{1}{2}g^*(p, p)$, defined by cometric.

Pontryagin Maximum Principle for bracket generated sub-Riemannian manifold

P-geodesics are exhausted by **normal geodesics**, which are exactly H-geodesics, and **abnormal geodesics**, which depend only on D .

By variational (or shortest) geodesics we will understand H-geodesics.

Geodesics as straightest curves. d'Alembert definition of sub-Riemannian geodesics (dA geodesics) in $Q, D.g^D$

To define D -geodesics, we extend g^D to g^Q . Let $D^0 = \text{Ann}(D) \subset T^*Q$ the codistribution. The **dA-geodesics** are solutions of the equation

$$\frac{d}{dt}L_{\dot{q}_i} - L_{q_i} \equiv 0 \pmod{D^0}$$

where $L(q, \dot{q}) = L = \sqrt{g(\dot{q}, \dot{q})}$ or $L = \frac{1}{2}g(\dot{q}, \dot{q})$, which is the appropriate projection of the Euler Lagrange geodesic vector field Γ^{g^Q} to D .

In general, the equation for M-geodesic is neither Lagrangian nor Hamiltonian

Levi-Civita-Schouten definition of Riemannian geodesics (L-geodesics) and Schouten partial connection

According to Levi-Civita, Riemannian geodesics (**L-geodesics**) are autoparallel curves, i.e. solution of the equation

$$\nabla_{\dot{\gamma}} \dot{\gamma} = \ddot{q}^i + \Gamma_{jk}^i \dot{q}_j \dot{q}^k = 0.$$

Schouten gave the following generalisation of this definition to (Q, D, g^D) .

A **rigging** of on sub-Riemannian manifold (Q, D, g^D) is a complementary to D (vertical) distribution V s.t. $TQ = D + V$. It defines (by Koszul formula) a **partial connection** $\nabla : \Gamma D \times \Gamma D \rightarrow \Gamma D$ (called **Schouten connection**) in D which preserves g and has zero "torsion"

$$T(X, Y) = \nabla_X Y - \nabla_Y X - \pi_D[X, Y], \quad X, Y \in \Gamma D$$

where π_D is the parallel to V projection to D .

Schouten definition of sub-Riemannian geodesic (L-geodesics)

If $V = D^\perp$ w.r.t. a Riemannian extension g^Q of g^D , then the Schouten connection is extended to a connection

$$\nabla_X^D = \pi_D \nabla^{g^Q} |_D$$

in D .

L-geodesics are geodesics of the Schouten connection, i.e. horizontal curves $\gamma(t)$ which satisfies

$$\nabla_{\dot{\gamma}} \dot{\gamma} = \pi_D \nabla_{\dot{\gamma}}^{g^Q} \dot{\gamma} = 0.$$

Theorem (Schouten, Synge, Vershik-Faddeev) dA-geodesics coincides with L-geodesics and they describe evolution of the free mechanical system with kinetic energy g^Q in configuration space Q with nonholonomic constraints D .

Cartan frame bundle definition of geodesics (C-geodesics)

Let $\pi : P \rightarrow Q = P/SO_n$ be the SO_n -principal bundle of orthonormal coframes (isometries $f : T_x M \rightarrow \mathbb{R}^n = V$) with the tautological soldering form

$$\theta : TP \rightarrow \mathfrak{so}_n, \theta_f(X) := f(\pi_* X).$$

The total space P admits a canonical SO_n -equivariant absolute parallelism

$$\kappa = \theta + \omega : TP \rightarrow V + \mathfrak{so}(V),$$

which is an extension of the vertical parallelism (defined by the free action of SO_n on P).

C-geodesics are projection to Q of constant horizontal vector fields from $\kappa^{-1}(V)$.

Non-holonomic C-geodesics on regular sub-Riemannian manifold

Regular distributions A distribution $D \subset TQ$ on a manifold defines an increasing filtration

$$0 \subset \mathcal{F}^1 := \mathcal{D}^{-1} = \Gamma D \subset \mathcal{D}^2 := \mathcal{D}^{-2} = \mathcal{D}^{-1} + [\mathcal{D}^{-1}, \mathcal{D}^{-1}] \subset \dots$$

in $\mathcal{X}(Q)$ and for $x \in Q$ a flag $0 \subset \mathcal{D}_x^{-1} \subset \mathcal{D}_x^{-2} \subset \dots \subset T_x Q$. The Lie bracket induces in the associated graded space

$$\mathfrak{m}_x = \mathfrak{m}_x^{-1} + \mathfrak{m}_x^{-2} + \dots + \mathfrak{m}_x^{-k} := \mathcal{D}_x^{-1} + \mathcal{D}_x^{-2} / \mathcal{D}_x^{-1} + \dots + \mathcal{D}_x^{-k} / \mathcal{D}_x^{-(k-1)}$$

the structure of a graded Lie algebra, called the **symbol algebra at a point x** .

The distribution $D \subset TQ$ is called **bracket generated or totally nonholonomic of depth k** if $\mathcal{D}_x^{-k} = T_x Q$ for all $x \in Q$. If, moreover $\mathcal{D}^i = \Gamma D^i$, $\forall i < 0$ is the space of sections of a vector bundle D^i and all symbol algebras \mathfrak{m}_x are isomorphic to a fixed graded algebra $\mathfrak{m} = \mathfrak{m}^{-1} + \dots + \mathfrak{m}^{-k}$ the distribution D is called a **regular distribution of type \mathfrak{m}** .

Sub-Riemannian metric as Tanaka structure

Let $\mathfrak{g}^0 \subset \mathfrak{so}(\mathfrak{m}) = \text{der}(\mathfrak{m}, \mathfrak{g}^{\mathfrak{m}})$ and $\mathfrak{g} = \mathfrak{m} + \mathfrak{g}^0$ the extended symbol of the sub-Riemannian structure (D, \mathfrak{g}^D) of type \mathfrak{m} . The group $G^0 = \text{Aut}(\mathfrak{m}, \mathfrak{g}^{\mathfrak{m}})$ acts simply transitively on $\text{Fr}(\mathfrak{m}^{-1})$.

Let $\text{gr}TQ = \cup_x \text{gr}T_q Q = \cup_x \mathfrak{m}_x$ be the graded tangent bundle of a regular sub-Riemannian manifold, where $\text{gr}T_q Q = \mathfrak{m}_x$ is the symbol algebra at a point x .

The group G^0 acts simply transitively on the set $\text{Fr}(D_x)$ of orthonormal frames in $D_x = \mathfrak{m}_x^{-1}$. Hence, the bundle

$$\pi : P = \text{Fr}(Q) = \cup_{q \in Q} \text{Fr}(D_x) \rightarrow P/G^0 = Q.$$

of orthonormal frames in D is a G^0 -principal bundle, called the **Tanaka G^0 -structure** (or relative G^0 -structure).

Tanaka and Morimoto theorems

N. Tanaka generalised the theory of \mathbf{G} -structures to Tanaka structures. In particular, he defined the full prolongation

$$\mathfrak{g}^\infty = \mathfrak{g} + \mathfrak{g}^{(1)} + \mathfrak{g}^{(2)} + \dots$$

Theorem (Tanaka) Assume that $\dim \mathfrak{g}^\infty < \infty$. Then there is a canonical bundle $P^\infty \rightarrow Q$ with an absolute parallelism.

Hence, one can define generalised geodesics as projection to Q of the orbits of constant vector fields.

Theorem (T. Morimoto) The first prolongation of the graded Lie algebra $\mathfrak{g} = \mathfrak{m} + \mathfrak{g}^0$, which is the extended symbol algebra of a regular sub-Riemannian manifold is trivial. Moreover, any regular sub-Riemannian manifold (Q, D, g^D) admits unique normal Cartan connection

$$\kappa : T(\text{Fr}(Q)) \rightarrow \mathfrak{g} = \mathfrak{m} + \mathfrak{g}^0.$$

Nonholonomic C-geodesics as projection of constant vector fields

Let (Q, D, g^D) be a regular sub-Riemannian manifold and

$$\kappa : T(\text{Fr}(Q)) \rightarrow \mathfrak{g} = \mathfrak{m} + \mathfrak{g}^0$$

associated [normal](#) Cartan connection.

[C-geodesics](#) are the projection to Q of the orbit of constant vector fields from $\kappa^{-1}(\mathfrak{m}^{-1})$.

It is not easy to construct a normal Cartan connection.

In [A- Medvedev- Slovak], it is shown that any admissible rigging

$V = V_2 + V_3 + \dots + \dots V_k$ s.t. $D^i = D + V_2 + \dots + V_i$

defines an isomorphism $grTQ \simeq TQ$ and isomorphism of the Tanaka bundle to a G^0 -structure with trivial first prolongation, Moreover, the Cartan connection of the Tanaka structure is described in terms of the Cartan connection of this G^0 -structure

$$\kappa = \theta + \omega : TP \rightarrow V + \mathfrak{g}^0$$

which is the direct sum of the soldering form and the connection form). Hence Any admissible ridding defines a Cartan connection and associated notion of C-geodesics as projection to Q of constant vector fields.

Sub-Riemannian geometry on principal bundle

Let $\pi^M : Q \rightarrow M = Q/G$ be a G -principal bundle with a connection $\omega : TQ \rightarrow \mathfrak{g}$ (a horizontal G -equivariant 1-form)) over a Riemannian manifold (M, g^M) .

The metric g^M induces a G -invariant sub-Riemannian metric g^D in D . Using vertical parallelism

$$\mathfrak{g} \simeq T_q M^\vee$$

we consider the **natural** extension of g^D to a Riemannian metric g^Q s.t. $TQ = T^\vee Q + D$ is orthogonal and the metric on $T^\vee Q \simeq \mathfrak{g}$ corresponds to a bi-invariant metric on \mathfrak{g} .

Montgomery theorem

Let $\pi^M : Q \rightarrow M = Q/G$ be a principal bundle with a connection ω over (M, g^M) , $D = \ker \omega, g^D$ the associated G -invariant sub-Riemannian structure and g^Q a Riemannian metric, which is the natural extension of g^D .

The following theorem characterises sub-Riemannian H-geodesics of (D, g^D) as horizontal geodesics of (Q, g^Q) and as horizontal lift of geodesics of the base manifold (M, g) . In other words, it establishes a bijection between (minimal) H-geodesics consider up to a G -equivalence and (minimal) geodesics in M .

Theorem

- i) A geodesic $\gamma(t)$ of the metric g^Q with horizontal data $\dot{\gamma}(a) \in D$ is a sub-Riemannian H-geodesic of (D, g^D) .
- ii) The projection $\bar{\gamma}(t) = \pi^M(\gamma(t))$ of a (minimal) H-geodesic $\gamma(t)$ is a (minimal) geodesic of (M, g^M) .
- iii) The horizontal lift of a (minimal) geodesic in (M, g) is a (minimal) H-geodesic of (D, g^D) .

Theorem Let (Q, D, g^D) be the sub-Riemannian manifold associated with a principal G -bundle $\pi_M : Q \rightarrow M = Q/G$ with a connection $D \subset TQ$. L-geodesics of the sub-Riemannian manifold $(Q, D = \ker \omega, g^D)$ coincide with H-geodesics. In particular, they do not depend on the extension of g^D to a Riemannian metric on Q .

In particular, L-geodesics of sub-Riemannian manifold, associated with a principal bundle, can be written in Hamiltonian form and we may speak about Liouville–Arnold integrability of the associated geodesic flow.

Sub-Riemannian structures associated to a Riemannian homogeneous manifold $M = G/H$

Let $(M = G/H, g^M)$ be a homogeneous Riemannian manifold with reductive decomposition $\mathfrak{g} = \mathfrak{gh} + \mathfrak{m}$ with an Ad_H -invariant metric $g^{\mathfrak{m}}$.

Assume that $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{a}$. Then

$\pi_M : Q = G/K \rightarrow M = G/H = G/K \cdot A$ is a principal A -bundle.

The manifold $Q = G/K$ has the reductive decomposition

$\mathfrak{g} = \mathfrak{k} + \mathfrak{n} = \mathfrak{k} + (\mathfrak{a} + \mathfrak{m})$ and $(\mathfrak{m}, g^{\mathfrak{m}})$ defines an invariant sub-Riemannian structure (D, g^D) of codimension $\dim \mathfrak{a}$ in $Q = G/K$. As above, we extend g^D to a Riemannian metric g^Q .

Montgomery theorem establishes an equivalence of three problems:

i) Description of geodesics of the Riemannian manifold

$$MG/H = G./K \cdot A ;$$

ii) Description of H-geodesics on the associated left invariant sub-Riemannian metric (D, g^D) in $Q = G/K$;

iii) Description of geodesics of the left invariant metric g^Q on $Q = G/K$, which is the bi-invariant extension of the metric g^D .

Theorem

Let $\pi_M : Q = G/K \rightarrow M = G/K \cdot A$ be the homogeneous A -principal connection and g^D is the invariant sub-Riemannian metric on D and g^Q its bi-invariant extension to a Riemannian metric, described above. Then H-geodesics of the sub-Riemannian metric g^D coincides with L-geodesics and are horizontal geodesics of the Riemannian metric g^Q and horizontal lifts of geodesics of the base manifold $(M = G/H, g^M)$.

Application to compact homogeneous sub-Riemannian structures of contact type

This result can be applied to regular homogeneous compact contact sub-Riemannian manifolds $(Q = G/K, D = \ker \theta, g^D)$ which described as follows.

Let $(F = G/H, g^F)$ be a flag manifold with an invariant Riemannian metric and reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ and $Z \in \mathfrak{h}$ is s.t. $C_{\mathfrak{g}}(Z) = \mathfrak{h}$ and $\mathfrak{h} = \mathfrak{k} \oplus \mathbb{R}Z$. Then $\pi_F = Q = G/K \rightarrow F = G/H$ is a principal T^1 -bundle and g^F induces invariant sub-Riemannian structure (D, g^D) on $Q = G/K$ of contact type.

Invariant sub-Riemannian structure on the Lie group G associated to a Riemannian homogeneous manifold ($M = G/H, g^M$)

The special case of the above result when $K = \{e\}$, i.e. to the principal H -bundle $\pi_M : G \rightarrow M = G/H$ with the reductive distribution $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ and the associated principal left invariant connection $D \subset TG$. in π_M

Theorem

Let $(M = G/H, g^M)$ be a homogeneous Riemannian manifold and $\pi_M : G \rightarrow M = G/H$ the associated principal bundle, g^D the invariant sub-Riemannian metric on D and g^G its extension to a Riemannian metric on G as above.

Then H-geodesics of (D, g^D) coincides with L-geodesics and with horizontal geodesics on the Riemannian metric g^G , and they are horizontal lift of the geodesics of the Riemannian manifold $M = G/H$.

Symmetric sub-Riemannian manifolds

Strichartz (JDG, 1986) defined the notion of **sub-Riemannian symmetric space** as a homogeneous sub-Riemannian manifold $(Q = G/H, D, g^D)$ such that H contains an involutive element σ s.t. $\sigma_*|_{D_0} = -\text{id}$.

He classified 3-dimensional sub-Riemannian symmetric spaces and stated the problem of extension of this classification to higher dimensions.

E. Falbel and C. Gorodski classified symmetric sub-Riemannian manifolds of contact type (1995). W. Respondek and A. J. Maciejewski describes all integrable sub-Riemannian metrics on 3-dimensional Lie groups with integrable H-geodesic flow (2008). They are exhausted by sub-Riemannian symmetric spaces.
Conjecture. Geodesic flow of any sub-Riemannian symmetric space are integrable.

Symmetric sub-Riemannian manifolds associated with a graded semisimple Lie algebra

We describe a class of compact sub-Riemannian symmetric spaces associated with graded semisimple Lie algebras.

Let

$$\mathfrak{g} = \sum_{i=-d}^d \mathfrak{g}_i$$

be a graded complex semisimple Lie algebra of depth $d \geq 2$ and

$$\mathfrak{g} = \mathfrak{g}^{ev} + \mathfrak{g}^{odd}$$

associated symmetric decomposition.

Denote by τ the anti-linear involution which defines the compact real form s.t. $\mathfrak{g}^\tau = \mathfrak{g}_0^\tau + \sum_{i>0} (\mathfrak{g}_{-i} + \mathfrak{g}_i)^\tau$. We set $\mathfrak{h} = \mathfrak{g}_0^\tau$ and denote by H associated subgroup of the compact Lie group G^τ

Associated with \mathfrak{g} flag manifold can be written as $F = G/P = G/G^0 \cdot G^+ = G^\tau/H$. We choose an $ad_{\mathfrak{h}}$ -invariant metric $g^{\mathfrak{m}}$ of the space $\mathfrak{m} = (\mathfrak{g}^{-1} + \mathfrak{g}^1)^\tau$. Then $(\mathfrak{m}, g^{\mathfrak{m}})$ defines an invariant bracket generated sub-Riemannian structure (D, g^D) on the flag manifold $F = G^\tau/H$.

Theorem The flag manifold $F = G^\tau/H$ with invariant sub-Riemannian structure (D, g^D) defined by $(\mathfrak{m}, g^{\mathfrak{m}})$ is sub-Riemannian symmetric space.

Example

Let

$$\mathfrak{g} = \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2, \dim \mathfrak{g}_{\pm 2} = 1$$

be the contact gradation of a complex simple Lie algebra \mathfrak{g} , i.e. the eigenspace decomposition of ad_{H_μ} where H_μ is the coroot, μ is the maximal root of \mathfrak{g} . Then the symmetric space $G^T/G_{e\nu}^T$ is the quaternionic Kähler symmetric space (the Wolf space) and the flag manifold $F = G^T/H$ is the associated twistor space . The distribution D is the holomorphic contact distribution and g^D is unique (up to scaling) invariant sub-Riemannian metric on D . It is the restriction of invariant Kähler-Einstein metric on F (if $\mathfrak{g} \neq \mathfrak{sl}_n(\mathbb{C})$.)

Reduction of nonholonomic geodesic equation on a Lie group to the Lie algebra

Nonholonomic H-geodesics on a Lie group G Let (D, g) be a left invariant sub-Riemannian structure on a Lie group G where $D \subset TG$ be a bracket generated left invariant distribution. The H-geodesics are projection to G of the orbits of the Hamiltonian vector field $\vec{H} = \omega^{-1} \circ dH$ with the Hamiltonian $H(x, p) = g^{-1}(p_D, p_D) = \sum_{j=1}^m f_{X_j}$, where $p_D \in D^*$ is the restriction of a covector p to D , (X_j) is an orthonormal frame of D and $f_X(p) := X^i p_i$ is the function on T^*G defined by a vector field $X \in \mathcal{X}(Q)$

Note that the left invariant functions $\mathcal{F}^L(T^*G) \simeq \mathcal{F}(\mathfrak{g}^*)$ closed under Poisson bracket in $\mathcal{F}(T^*G)$. We identify them with functions on \mathfrak{g}^* . Then the Poisson bracket reduces to the standard Poisson bracket in $\mathcal{F}(\mathfrak{g}^*)$ defined by

$$\{f, h\}(\xi) = \langle \xi, [df_\xi, dh_\xi] \rangle, \quad f, h \in \mathcal{F}(\mathfrak{g}), \quad \xi \in \mathfrak{g}^*$$

and $df_\xi \in T_\xi^* \mathfrak{g}^*$ is identified with an element of $(\mathfrak{g}^*)^* = \mathfrak{g}$. In particular, the Hamiltonian H defines the Hamiltonian vector field \vec{H} such that $\vec{H} \cdot f = \{H, f\}$.

So the geodesic equation reduces to the Hamiltonian equation

$$\dot{\xi} = \vec{H} \cdot \xi.$$

Lagrange equation of geodesics on TG

Let $D \subset TG$ be a left invariant distribution on G , $D_e \subset \mathfrak{g}$. g^D a left invariant metric in D , $g_e^L(X, Y) = \langle X, Y \rangle$, $X, Y \in \mathfrak{g}$.

$TG = G \times \mathfrak{g}$. Then left invariant functions $C^\infty(TG)^L \subset C^\infty(\mathfrak{g}^*)$ forms a Poisson subalgebra. The Euler-Lagrange equation for an invariant $L(g, \dot{g}) = l(X)$, $X = g^{-1}\dot{g} \in \mathfrak{g}$ reduces to Euler equation

$$\frac{d}{dt} \partial_X l = \text{ad}_X^* \partial_X l$$

or

$$\dot{\xi} = \text{ad}_X^* \xi$$

where $\xi = \partial_X l(X) = dl(X) \in T^*\mathfrak{g} = \mathfrak{g}^*$.

In the case of metric m , the Euler equation is

$\dot{\xi} = \text{ad}_X^* \xi$, $\xi = m \circ X$ or

$$\dot{X} = \text{ad}_X^m X$$

where ad_X^m is metric conjugated endomorphism to ad_X .

Theorem (Vershik, Yoashimura, Marsden) The reduced equation for nonholonomic L-geodesics for left invariant sub-Riemannian Lie group (G, D, m) has the form

$$\dot{X} = pr_D \text{ad}_X^* X, X \in \mathfrak{g}$$

or, in cotangent space for $\xi = m \circ X$

$$\dot{\xi} = \text{ad}_{m^{-1}\xi}^* \xi$$

where ad_X^* is the adjoint action.