Shortest and straightest geodesics of an invariant sub-Riemannian metric on a homogeneous manifold

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28 ноября 2018 г.

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Introduction

The efficiency of Riemannian geometry and its important role in applications based on the fact many important equations , arising in mechanics, mathematical physics , biology, economy, information theory etc can be reduced to the geodesic equation associated with a Riemannian metric. Moreover, Riemannian geometry give an effective tool for investigation of geodesic equations and other equations associated with the metric (Laplace, wave, heat and Schrödinger equation, Einstein equation etc).

There are many equivalent definitions of Riemannian geodesics . They are naturally generalised to sub-Riemannian manifold, but become non-equivalent.

We give a review of different definitions of geodesics of a sub-Riemannian manifold and interrelation between them.

Herz indicateed two characterisations of geodesics:

geodesics as shortest curves based on Mopertrui's principle of least action (variational approach)) and

geodesics as straightest curves based on d'Alembert's principle of virtual work.

We recall three variational definitions of geodesics as (locally) shortest curves (Euler-Lagrange, Pontyagin and Hamilton) and three definitions of geodesics as straightest curves (d'Alembert , Levi-Civita-Schouten and Cartan-Tanaka), used in nonholonomic mechanics and discuss their interrelations.

A. Vershik and L. Faddeev showed that shortest geodesics for a generic sub-Riemannian manifolds $(Q, D.g)$ are different from straightest geodesics on a open dense submanifold of Q. They gave first example (compact Lie group with the bi-invariant metric) when shortest geodesics coincides with straightest geodesics.

We generalised this example and consider a big class of sub-Riemannian manifolds associated with principal bundle over a Riemannian manifolds, for which shortest geodesics coincides with straightest geodesics. Using the geometry of flag manifolds, we describe some classes of compact homogeneous sub-Riemannian manifolds (including contact sub-Riemannian manifolds and symmetric sub-Riemannian manifolds) where straightest geodesics coincides with shortest geodesics. Construction of geodesics in these cases reduces to description of Riemannian geodesics of the Riemannian homogeneous manifold or left-invariant metric on a Lie group.

Geodesics as shortest curves. Euler-Lagrange variational definition of geodesics (EL-geodesics) on a Riemannian manifold $(Q.g^Q)$)

Any Lagrangian $L(q, \dot{q})$ (homogeneous degree 1 in velocity $\dot{q}$ function of TQ) defines a variational problem

$$\delta A_L = \delta \int_a^b L(q(t), \dot{q}(t))dt = 0$$

in the space $C(q_0, q_1) = \{q(t),\ t \in [a, b],\ q_0 = \gamma(a), q_1 = \gamma(b)\}$. Solutions are critical points of A_L , i.e. solutions of the Euler-Lagrange equation

$$(EL) \quad \delta L_i := \frac{d}{dt} L_{\dot{q}_i} - L_{q_i} = 0,$$

which defines a vector field $\Gamma^L \in \mathcal{X}(TQ)$.
EL- geodesics are critical point of the length functional $(L = \sqrt{g(\dot{q}, \dot{q})}\ )$ or energy functional $(L = \frac{1}{2} g(q, \dot{q}))$.

EL-geodesic on bracket generated sub-Riemannian manifold (Q, D, g^D)

$D \subset TM$ is a submanifold. EL-geodesics are solution of the variational problem

$$\delta A_L = \delta \int_a^b L(q(t), \dot{q}(t))dt = 0$$

for length functional or energy functional in the space $C^h(q_0, q_1) = \{q(t),\ t \in [a, b],\ \dot{q} \in D, q_0 = \gamma(a), q_1 = \gamma(b)\}$, i.e. a solutiion of EL-equation

$$\delta L_i := \frac{d}{dt} L_{\dot{q}_i} - L_{q_i} - \dot{\lambda}_a \omega_a - \lambda_a(\dot{q} \rfloor \omega_a) = 0$$

where $D = \{\omega_a = 0,\ a = 1, \cdots k\}$ This equation corresponds to a vector field $\Gamma^L \in \mathcal{X}(D \times D^o)$ where $D^0 = \mathrm{Ann}(D) \subset T^*Q$.

Pontryagin optimal control definition of geodesics(P-geodesics) for bracket generated sub-Riemannian manifold (Q, D, g^D)

Let $L(q, \dot{q}) \in C^\infty(Q)$ be a Lagrangian on D, which is either the velocity $L = \sqrt{g^Q(\dot{q}, \dot{q})}$ or the energy $\frac{1}{2} g^Q(\dot{q}, \dot{q})$. and $X_i, \cdots, X_m$ be an orthonormal (local) frame.
Any horizontal curve $q(t) \in C^h(q_0, q_1)$ is a solution of the ODE

$$(*) \qquad \dot{q}^i(t) = \sum_{i=1}^{n} u^i(t) X_i(q^i(t)),\ q(a) = q_0, q(b) = q_1.$$

where vector function $u(t)$ is an admissible control parameter and $A^h(q(t)) = \int_a^b L(q, \dot{q}) dt$ is the cost functional.

Pontryagin definition of geodesics

P-geodesic (resp.,minimal P-geodesic) is a curves $q(t) \in C^h(q_0, q_1)$ which delivers a critical point (respectively, a minimum) for the cost functional

$$A_L = \int_a^b L(q, \dot{q}) dt$$

in the space of admissible control.
P-geodesics coincide with EL-geodesics.
For $D = TQ$, P-geodesics are geodesics of the Riemannian manifold (Q, g^Q).

Hamiltonian sub-Riemannian geodesics (H-geodesics) in (Q, D, g^D)

Let p_D be the restriction of a linear form $p \in T^*_q Q$ to D and $g^*(p,p) = g^{-1}(p_D, p_D) \in C^\infty(T^*Q)$ the cometric.
H-geodesics are projection to Q. of orbits of Hamiltonian vector field $\vec{H} = \Omega^{-1} dH \in \mathfrak{X}(T^*Q)$ with quadratic (degenerate) Hamiltonian $H = \frac{1}{2} g^*(p,p)$, defined by the cometric. Here Ω is the standard symplectic form in T^*Q.

Pontryagin Maximum Principle for bracket generated sub-Riemannian manifold

P-geodesics are exhausted by normal geodesics , which are exactly H-geodesics, and abnormal geodesics, which depend only on D.
By variational (or shortest) geodesics we will understand H-geodesics.

Geodesics as straightest curves. d'Alemebert definition of sub-Riemannian geodesics (dA geodesics) in $(Q, D.g^D)$

To define dA-geodesics , we extend g^D to g^Q. Let $D^0 = Ann(D) \subset T^*Q$ the codistribution. The dA-geodesics are solutions of the equation

$$\frac{d}{dt} L_{\dot{q}_i} - L_{q_i} \equiv 0(\mathrm{mod} D^0)$$

where $L(q, \dot{q}) = L = \sqrt{g(\dot{q}, \dot{q})}$ or $L = \frac{1}{2} g(\dot{q}, \dot{q})$, which is the appropriate projection of the Euler Lagrange geodesic vector field Γ^{g^Q} to D.

In general, the equation of dA-geodesics is neither Lagrangian no Hamiltonian.

Levi-Civita-Schouen definition of Riemannian geodesicsd (L-geodesics) and Schouten partial connection

According to Levi-Civita, Riemannian geodesics (L-geodesics) in (M, g) are autoparallel curves , i.e. solution of the equation

$$\nabla^g_{\dot\gamma}\dot\gamma = \ddot q^i + \Gamma^i_{jk}\dot q_j \dot q^k = 0.$$

Schouten gave the following generalisation of this definition to $(Q.D, g^D)$.
A rigging of on sub-Riemannian manifold (Q, D, g^D) is a complementary to D (vertical) distribution V s.t. $TQ = D + V$.
It defines (by Koszul formula) a partial connection $\nabla^S : \Gamma D \times \Gamma D \to \Gamma D$ (called Schouten connection) in D which preserves g and has zero "torsion"

$$T(X, Y) = \nabla^S_X Y - \nabla^S_Y X - \pi_D[X, Y],\ X, Y \in \Gamma D$$

where π_D is the parallel to V projection to D.

Schouten definition of sub-Riemannian geodesic (L-geodesics)

If $V = D^\perp$ w.r.t. a Riemannian extension g^Q of g^D, then the Schouten partial connection ∇^S is extended to a connection ∇^D in D which is the restriction to D of the Levi-Civita connection ∇^{g^Q}:

$$\nabla^D_X = \pi_D \nabla^{g^Q}_X|_D, \; X \in \mathcal{X}(Q)$$

L-geodesics are geodesics of the Schouten connection , i.t. horizontal curves $\gamma(t)$ which satisfies

$$\nabla^S_{\dot\gamma}\dot\gamma = \pi_D \nabla^{g^Q}_{\dot\gamma}\dot\gamma = 0.$$

Theorem (Schouten, Synge, Vershik-Faddeev) dA- geodesics coincides with L-geodesics and they describe evolution of the free mechanical system with kinetic energy g^Q in configuration space Q with nonholonomic constrains D.

Cartan frame bundle definition of geodesics (C-geodesics)

Let $\pi : P \to Q = P/SO_n$ be the SO_n-principal bundle of orthonormal coframes (isometries $f : T_xQ \to \mathbb{R}^n = V$) with the tautological soldering form

$$\theta : TP \to \mathfrak{so}_n,\ \theta_f(X) := f(\pi_* X).$$

The total space P admits a canonical SO_n-equivariant absolute parallelism

$$\kappa = \theta + \omega : TP \to V + \mathfrak{so}(V),$$

which is an extension of the vertical parallelism (defined by the free action of SO_n on P).
C-geodesics are projection to Q of constant horizontal vector fields from $\kappa^{-1}(V)$.

Generalisation to G-structures of finite type

Given a linear group $G \subset GL(V)$, $V = \mathbb{R}^n$. A G -structure is a G-principal bundle $\pi : P \to Q = P/G$ with a soldering 1-form

$$\theta : TP \to V$$

i.e. a horizontal G-equivariant form with $\ker\ \theta = T^{vert}P$.

Such bundle is identified with a G-bundle of coframes on TQ. If the group G has a finite type k, one can prolong π to a bundle $P^k \to Q$ with absolute parallelism

$$\kappa : TP^{(k)} \to \mathfrak{g}^\infty = V + \mathfrak{g} + \mathfrak{g}^{(1)} + \cdots + \mathfrak{g}^{(k)}.$$

The projection of orbits of constant vector fields from $\kappa^{-1}V$ to Q are generalised C-geodesics for G-structure.
If

$$\mathfrak{g}^{(1)} = \mathfrak{g} \otimes V^* \cap V \otimes S^2 V^* = 0,$$

then $k = 1$ and

$$\kappa : TP \to V + \mathfrak{g}$$

is a Cartan connection.
The theory of such generalised C- geodesics had been developed by J. Slovak, V. Žadnik, A. Čap and B. Doubrov.

Cartan connection

Let $M_0 = L/G$ be a homogeneous n-dimensional manifold.
A Cartan connection of type $M_0 = L/G$ on n-dimensional manifold Q is a principal G-bundle $\pi : P \to Q = P/G$ together with a $\mathfrak{l}$-valued G-equivariant (s.t.$r_g^* \kappa = \mathrm{Ad}_g^{-1} \circ \kappa$, $g \in G$) kernel free 1-form

$$\kappa : TP \to \mathfrak{l}$$

which extends the vertical parallelism $T_p^v P \simeq \mathfrak{g}$.
The form κ defines an absolute parallelism $T_p P \simeq \mathfrak{l}$. Hence, tensor fields on P may be identified with tensor-valued functions.
In particular, the curvature 2-form $K := d\kappa + \frac{1}{2}[\kappa, \kappa]$ is identified with a function with value in $C^2(\mathfrak{l}, \mathfrak{l}) = \mathfrak{l} \otimes \Lambda^2 \mathfrak{l}^*$. A Cartan connection is called normal if the codifferential $\delta^* K = 0$. Then the curvature K is determined by its harmonic component κ_H (satisfying $\delta \kappa_H = 0$) via Bianchi identities.

Nonholonomic C-geodesics on regular sub-Riemannian manifold.The symbol algebra

A distribution $D \subset TQ$ on a manifold defines an increasing filtration

$$0 \subset \mathcal{F}^1 := \mathcal{D}^{-1} = \Gamma D \subset \mathcal{D}^{-2} := \mathcal{D}^{-1} + [\mathcal{D}^{-1}, \mathcal{D}^{-1}] \subset \cdots$$

in $\mathcal{X}(\mathcal{Q})$ and for $x \in Q$ a flag $0 \subset \mathcal{D}_x^{-1} \subset \mathcal{D}_x^{-2} \subset \cdots \subset T_x Q$. The Lie bracket induces in the associated graded space

$$\mathfrak{m}_x = \mathfrak{m}_x^{-1} + \mathfrak{m}_x^{-2} + \cdots + \mathfrak{m}_x^{-k} := \mathcal{D}_x^{-1} + \mathcal{D}_x^{-2}/\mathcal{D}_x^{-1} + \cdots + \mathcal{D}_x^{-k}/\mathcal{D}_x^{-(k-1)}$$

the structure of a graded Lie algebra, called the symbol algebra at a point x.

The distribution $D \subset TQ$ is called bracket generated or totally nonholonomic of depth k if $\mathcal{D}_x^{-k} = T_x Q$ for all $x \in Q$. Then the grades space

$$T_x^{gr} Q = gm_x = \mathcal{D}_x^{-1} + \mathcal{D}_x^{-2}/\mathcal{D}_x^{-1} + \cdots + T_x Q/\mathcal{D}_x^{-(k-1)}$$

is called the graded tangent space at $x \in Q$.
If, moreover $\mathcal{D}^i = \Gamma D^i$, $\forall i < 0$ is the space of sections of a vector bundle D^i and all symbol algebras $\mathfrak{m}_x$ are isomorphic to a fixed graded algebra $\mathfrak{m} = \mathfrak{m}^{-1} + \cdots + \mathfrak{m}^{-k}$ the distribution D is called a regular distribution of type $\mathfrak{m}$.

The natural extension of an Euclidean metric on $\mathfrak{m}^{-1}$ to $\mathfrak{m}$

Let $\mathfrak{m} = \mathfrak{m}^{-1} + \cdots + \mathfrak{m}^{-k}$ be a negatively graded Lie algebra . Then an Euclidean metric $g^{\mathfrak{m}^{-1}}$ on $\mathfrak{m}^{-1}$ has natural extension to the Euclidean metric $g^{\mathfrak{m}}$ on $\mathfrak{m}$ such that the grading subspaces are orthogonal.

Extended symbol of a regular sub-Riemannian manifold of type $\mathfrak{m}$

Let (Q, D, g^D) be a sub-Riemannian manifold, where D is a regular distribution of type

$$\mathfrak{m} = \mathfrak{m}^{-1} + \cdots + \mathfrak{m}^{-k}.$$

The metric g^D induces an Euclidean metric on $\mathfrak{m}^{-1}$, which is extended to an Euclidean metric $g^{\mathfrak{m}}$ on $\mathfrak{m}$.
Denote by $\mathfrak{g}^0 := der(\mathfrak{m}, g^{\mathfrak{m}}) \subset \mathfrak{so}(\mathfrak{m})$ the Lie algebra of skew-symmetric derivations of $\mathfrak{m}$.
The non-positively graded Lie algebra

$$\mathfrak{g} = \mathfrak{g}^0 + \mathfrak{m} = \mathfrak{g}^0 + \mathfrak{m}^{-1} + \cdot + \mathfrak{m}^{-k}$$

is called the extended symbol of the sub-Riemannian manifold (Q, D, g^D) of type $\mathfrak{m}$.

Tanaka structure

Let $D \subset TQ$ be a regular rank m distribution of type $\mathfrak{m}$ and $G^0 \subset \mathrm{Aut}(\mathfrak{m})$ a connected group of graded preserving authomorphisms of $\mathfrak{m}$.
A Tanaka structure (or relative G^0-structure) is a G^0 -principal bundle $\pi : P \to Q = P/G^0$ of coframes $f : D_x \to V = \mathbb{R}^m$ of distribution D.

Sub-Riemannian metric as Tanaka structure

Let (Q, D, g^D) be a sub-Riemannian manifold, where D is a regular distribution of type $\mathfrak{m}$. and $G^0 = \mathrm{Aut}(\mathfrak{m}, g^{\mathfrak{m}}) \subset SO(\mathfrak{m})$ the connected group generated by $\mathrm{der}(\mathfrak{m}, g^{\mathfrak{m}})$. The group G^0 acts simply transitively on the set $Fr(\mathfrak{m}^{-1})$ of oriented orthonormal frames in $\mathfrak{m}^{-1}$. Hence, the bundle

$$\pi : P = Fr(Q) = \cup_{x \in Q} Fr(D_x) \to P/G^0 = Q.$$

of orthonormal frames in D is a G^0-principal bundle of frames , that is a Tanaka G^0-structure.

Tanaka theorems

N. Tanaka generalised the theory of G-structures to Tanaks structures. In particular, he defined the full prolongation

$$\mathfrak{g}^{\infty} = \mathfrak{g} + \mathfrak{g}^{(1)} + \mathfrak{g}^{(2)} + \cdots$$

of a non positively graded Lie algebra.

Theorem (Tanaka) Let $\pi : P \to Q$ be a Tanaka G^0=structure on (Q, D) where D is a regular distribution of type $\mathfrak{m}$. Assume that the full prolongation $\mathfrak{g}^{\infty}$ of the extended symbol algebra $\mathfrak{g} = \mathfrak{g}^0 + \mathfrak{m}$ is finite dimensional. Then there is a canonical bundle $P^{\infty} \to Q$, constructed by successive prolongations, with an absolute parallelism.

If the first prolongation $\mathfrak{g}^{(1)} = 0$, then the bundle $\pi : P \to Q$ carries a Cartan connection $\kappa : TP \to \mathfrak{g}$.

Hence, one can define generalised geodesics as projection to Q of the orbits of constant vector fields.

Theorem (T. Morimoto)

The first prolongation of the graded Lie algebra $\mathfrak{g} = \mathfrak{g}^0 + \mathfrak{m}$, which is the extended symbol algebra of a regular sub-Riemannian manifold, is trivial.
Moreover, any regular sub-Riemannian manifold (Q, D, g^D) admits unique normal Cartan connection

$$\kappa : T(Fr(Q)) \to \mathfrak{g} = \mathfrak{m} + \mathfrak{g}^0.$$

Nonholonomic C-geodesics as projection of constant vector fields

Let (Q, D, g^D) be a regular sub-Riemannian manifold and

$$\kappa : T(Fr(Q)) \to \mathfrak{g} = \mathfrak{m} + \mathfrak{g}^0$$

associated normal Cartan connection.
C-geodesics are the projection to Q of the orbit of constant vector fields from $\kappa^{-1}(\mathfrak{m}^{-1})$.

It is not easy to construct a normal Cartan connection.
In [A- Medvedev- Slovak], it is shown that any rigging V s.t. $TQ = D \oplus V$ with a grading

$$V = V_2 + V_3 + \cdots + \cdots V_k$$

s.t. $D^{-i} = D + V_2 + \cdots + V_i$
defines an isomorphism $grTQ \simeq TQ$ and also an isomorphism of the Tanaka bundle to a G^0-structure with trivial first prolongation,
Moreover, the Cartan connection of the Tanaka structure may be described in terms of the Cartan connection of this G^0-structure

$$\kappa = \theta + \omega : TP \to V + \mathfrak{g}^0$$

(which is the direct sum of the soldering form and the connection form). Hence
Any admissible ridding defines a Cartan connection. The associated C-geodesics are defined as projection to Q of constant vector fields.

Sub-Riemannian geometry on principal bundle

Let $\pi^M : Q \to M = Q/G$ be a G- principal bundle with a connection $\omega : TQ \to \mathfrak{g}$ (a horizontal G-equivariant 1-form)) over a Riemannian manifold (M, g^M). We assume that G is compact.

The metric g^M induces a G -invariant sub-Riemannian metric g^D in D. Using vertical parallelism

$$\mathfrak{g} \simeq T_x^{vert} Q, \ \forall x \in Q,$$

we consider the natural extension of g^D to a Riemannian metric g^Q s.t. $TQ = T^v Q + D$ is orthogonal and the metric on $T_q^v Q \simeq \mathfrak{g}$ corresponds to a bi-invariant metric on $\mathfrak{g}$.

Montgomery theorem

Let $\pi^M : Q \to M = Q/G$ be a principal bundle with a connection ω over (M, g^M) , $D = \ker \omega, g^D$ the associated G-invariant sub-Riemannian structure and g^Q a Riemannian metric, which is the natural extension of g^D.

The following theorem characterises sub-Riemannian H-geodesics of (D, g^D) as horizontal geodesics of the Riemannian manifold (Q, g^Q) and as horizontal lift of geodesics of the base manifold (M, g^M). In other words, it establishes a bijection between (minimal) H- geodesics (up to a G-equivalence) and (minimal) geodesics in M.

Theorem

i) A geodesic $\gamma(t)$ of the metric g^Q with horizontal data $\dot{\gamma}(a) \in D$ is a sub-Riemannian H-geodesic of (D, g^D).

ii) The projection $\bar{\gamma}(t) = \pi^M(\gamma(t))$ of a (minimal) H-geodesic $\gamma(t)$ is a (minimal) geodesic of (M, g^M).

iii) The horizontal lift of a (minimal) geodesic in (M, g^M) is a (minimal) H-geodesic of (D, g^D).

Theorem Let (Q, D, g^D) be the sub-Riemannian manifold associated with a principal G-bundle $\pi_M : Q \to M = Q/G$ with a connection $D \subset TQ$. L-geodesics of the sub-Riemannian manifold $(Q, D = \ker \omega, g^D)$ coincide with H-geodesics.
In particular, they does not depend on the extension of g^D to a Riemannianm metric on Q.
In particular, L-geodesics of sub-Riemannian manifold , associated with a principal bundle , can be written in Hamiltonian form and we may speak about Liouville–Arnold integrability of the associated geodesic flow.

Sub-Riemannian structures associated to a Riemannian homogeneous manifold $M = G/H$

Let $(M = G/H, g^M)$ be a homogeneous Riemannian manifold with reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ and induced on $\mathfrak{m}$ an Ad_H-invariant metric $g^{\mathfrak{m}}$.
Assume that $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{a}$ is a direct sum of two ideals and K, A corresponding normal subgroups. Then

$$\pi_M : Q = G/K \to M = G/H = G/K \cdot A$$

is a principal A-bundle (with the right action of the group A).
The manifold $Q = G/K$ has the reductive decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{n} = \mathfrak{k} + (\mathfrak{a} + \mathfrak{m})$ and $(\mathfrak{m}, g^{\mathfrak{m}})$ defines an invariant sub-Riemannian structure (D, g^D) of codimension $\dim \mathfrak{a}$ in $Q = G/K$. As above, we extend g^D to a Riemannian metric g^Q.

Montgomery theorem establishes the equivalence of the following problems:
i) Description of geodesics of the Riemannian manifold $M - G/H = G/K \cdot A$;
ii) Description of H-geodesics on the associated left invariant sub-Riemannian metric (D, g^D) in $Q = G/K$;
iii) Description of geodesics of the left invariant metric g^Q on $Q = G/K$, which is the bi-invariant extension of the metric g^D.

Theorem

Let $\pi_M : Q = G/K \to M = G/K \cdot A$ be the homogeneous A-principal bundle over a homogeneous Riemannian manifold (M, g^M) with the sub-Riemannian metric (D, g^D) as above. Denote by g^Q the bi-invariant extension of g^D to a G-invariant metric on $Q = G/K$.

Then H-geodesics of the sub-Riemannian metric (D, g^D) coincides with L-geodesics and are horizontal geodesics of the Riemannian metric g^Q and horizontal lifts of geodesics of the base manifold $(M = G/H, g^M)$.

Application to compact homogeneous sub-Riemannian structures of contact type

This result can be applied to regular homogeneous compact contact sub-Riemannian manifolds $(Q = G/K, D = \ker\theta, g^D)$ which described as follows.
Let $(F = G/H, g^F)$ be a flag manifold with an invariant Riemannian metric and reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ and $Z \in \mathfrak{h}$ is s.t. $C_{\mathfrak{g}}(Z) = \mathfrak{h}$ and $\mathfrak{h} = \mathfrak{k} \oplus \mathbb{R}Z$. Then $\pi_F : Q = G/K \to F = G/H$ is a principal T^1-bundle and g^F induces invariant sub-Riemannian structure (D, g^D) on $Q = G/K$ of contact type.

Invariant sub-Riemannian structure on the Lie group G associated to a Riemannian homogeneous manifold $(M = G/H, g^M)$

We specialazed the above construction to the case $K = \{e\}$.

Theorem

Let $(M = G/H, g^M)$ be a homogenelous Riemannian manifold and $\pi_M : G \to M = G/H$ the associated principal bundle, g^D the invariant sub-Riemannian metric on D and g^G its extension to a Riemannian metric on G as above.

Then H-geodesics of (D, g^D) coincides with L-geodesics and with horizontal geodesics on the Riemannian metric g^G, and they are horizontal lift of the geodesics of the Riemannian manifold $M = G/H$.

Symmetric sub-Riemannian manifolds

Strichartz (JDG, 1986) defined the notion of sub-Riemannian symmetric space as a homogeneous sub-Riemannian manifold $(Q = G/H, D, g^D)$ such that H contains an involutive element σ which acts on the subspace D_o of the point $o = eH \in Q$ as $-$id. He classified 3-dimensional sub-Riemannian symmetric spaces and stated the problem of extension of this classification to higher dimensions.
E. Falbel and C. Gorodski classified symmetric sub-Riemannian manifolds of contact type (1995). W. Respondek and A. J. Maciejewski describes all integrable sub-Riemannian metrics on 3-dimensional Lie groups with integrable H-geodesic flow (2008). They are exhausted by sub-Riemannian symmetric spaces.
Conjecture. Geodesic flow of any sub-Riemannian symmetric space are integrable.

Symmetric sub-Riemannian manifolds associated with a graded semisimple Lie algebra

We describe a class of compact sub-Riemannian symmetric spaces associated with graded semisimple Lie algebraa.
Let

$$\mathfrak{g} = \sum_{i=-d}^{d} \mathfrak{g}_i$$

be a graded complex semisimple Lie algebra of depth $d \geq 2$ and

$$\mathfrak{g} = \mathfrak{g}^{ev} + \mathfrak{g}^{odd}$$

associated symmetric decomposition.
Denote by τ the anti-linear involution which defines the compact real form s.t. $\mathfrak{g}^\tau = \mathfrak{g}_0^\tau + \sum_{i>0}(\mathfrak{g}_{-i} + \mathfrak{g}_i)^\tau$ We set $\mathfrak{h} = \mathfrak{g}_0^\tau$ and denote by H associated subgroup of the compact Lie group G^τ

Associated with $\mathfrak{g}$ flag manifold can be written as $F = G/P = G/G^0 \cdot G^+ = G^\tau/H$. We choose an $ad_{\mathfrak{h}}$-invariant metric $g^{\mathfrak{m}}$ on the space $\mathfrak{m} = (\mathfrak{g}^{-1} + \mathfrak{g}^1)^\tau$. Then $(\mathfrak{m}, g^{\mathfrak{m}})$ defines an invariant bracket generated sub-Riemannian structure (D, g^D) on the flag manifold $F = G^\tau/H$.

Theorem The flag manifold $F = G^\tau/H$ with invariant sub-Riemannian structure (D, g^D) defined by $(\mathfrak{m}, g^{\mathfrak{m}})$ is a sub-Riemannian symmetric space.

Example

Let

$$\mathfrak{g} = \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_9 + \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2, \, \dim \mathfrak{g}_{\pm 2} = 1$$

be the contact gradation of a complex simple Lie algebra $\mathfrak{g}$, i.e. the eigenspace decomposition of ad $_{H_\mu}$ where H_μ is the coroot associated to the maximal root μ of $\mathfrak{g}$. Then the symmetric space G^τ/G^τ_{ev} is the quaternionic Kähler symmetric space (the Wolf space) and the flag manifold $F = G^\tau/H$ is the associated twistor space . The distribution D is the holomorphic contact distribution and g^D is unique (up to scaling) invariant sub-Riemannian metric on D. It is the restriction of the invariant Kähler-Einstein metric on F (if $\mathfrak{g} \neq \mathfrak{sl}_n(\mathbb{C})$.)