Special Vinberg cones and their application to supergravity

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Abstract

The talk is based on joint works with V. Cortes; and A. Spiro and A. Marrani.

E. B. Vinberg gave a description of homogeneous convex cones as cones of hermitian positively defined matrices in matrix T-algebra M_n of order n. The diagonal entries of such matrix are real numbers, but off-diagonal elements a_{ij} belong to different euclidean vector spaces V_{ij}. To determine the matrix multiplication, an isometric bilinear multiplication

$$V_{ij} \times V_{jk} \to V_{ik}$$

such that

$$|a_{ij} \cdot a_{jk}| = |a_{ij}| \cdot |a_{jk}|$$

must be defined. Unfortunately, such isometric multiplication $V \times U \to W$ is known only in two cases:

i) When the euclidean spaces have the same dimension (hence, isomorphic to V). Then the problem is equivalent to the classification of division algebras in V and was solved by A. Hurwitz (1898).

ii) When $\dim U = \dim W$. Then the problem reduces to i) a description of $\mathbb{Z}_2$-graded modules $S = S^0 + S^1 = U + W$ over the Clifford algebra $Cl(V)$ and ii) classification of $Spin(V)$-invariant metrics in S. It was solved by M.F.Atiyah, R. Bott and A.S.Shapiro (1964) and , respectively A-Cortes (1997). Due to this, the explicit classification of homogeneous Vinberg cones is known only in two cases :

i) When all Euclidean vector spaces V_{ij} are isomorphic (to one of division algebra $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$) (Then $\mathcal{V}$ is a symmetric self-adjoint cone) ;

ii) when T-algebra is the order 3 matrix algebra such that

$$V_{12} = V_{21} = V,\ V_{23} = V_{32} = S^0,\ V_{13} = V_{31} = S^1$$

where $S^0 + S^1$ is a graded $CL(V)$-module. The associated homogeneous convex cone is called special Vinberg cone.

Fortunately, only special Vinberg cones $\mathcal{V}$ are important for supergravity. They describe the target space of the scalar mass multiplet in $d = 5$ $N = 2$ Supergravity. The dimensional reduction to dimension $d = 4$ and then to dimension $d = 3$ associates with spacial Vinberg cone $\mathcal{V}$ a special Kähler

homogeneous manifold $\mathcal{K} = r(\mathcal{V})$ and then a special quaternionic Kähler homogeneous manifold $\mathcal{Q} = q(\mathcal{V}) = c \circ r(\mathcal{V})$, which are target spaces of the scalar multiplet in $d = 4$ and $d = 3$ Supergravity, respectively.
We describe a generalisation of the theory of special Vinberg cone to indefinite case.
We calculate the inversion of the quadratic map, associated to the fundamental cubic polynomial (the natural generalization of the determinant of order 3 matrix), which describes the determinant cubic hypersrface in a special Vinberg cone $\mathcal{V}$.
This allows to obtained the explicit formula for the Bekenstein-Hawking entropy of BPS (Bogomol'nui - Prasad-Sommerfield) black holes in $d = 4$ Supergravity for any homogeneous target manifold $\mathcal{K}$. Before it was known only for symmetric cones (that is $Herm_3^+(\mathbb{K})$, $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$).