Geometric Analysis of Metric Legendre Foliated Cocycles on Contact Manifolds via SODE Structure

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Introduction

In recent years, an increasing consideration has been devoted to the qualitative analysis of systems of (non-) autonomous second (higher) order ordinary (partial) differential equations fields through some associated geometric structures. Second order ordinary differential equations (SODE) are of special significance mainly due to their extensive applications in various domains of mathematics, science and engineering. A remarkably type of SODE is the one which can be deduced from a variational principle. In this research, a thoroughgoing structural investigation of the transverse Legendre foliated cocycles on contact manifolds is presented.

Introduction

For this goal, by applying Spencer theory of formal integrability, sufficient conditions for the metric associated with the given SODE structure are designated to extend to a transverse metric for the lifted Legendre foliated cocycle on the tangent space of an arbitrary contact manifold. Indeed, the concept of formal integrability is applied as a noteworthy reformulation of the inverse problem of the calculus of variations in terms of a partial differential operator which acts on semi-basic 1-forms. Consequently, this expression of the Helmholtz metrizability conditions, enables us to construct a transverse metric on the tangent bundle of a given contact manifold which leads to creation of the specific type of metric Legendre foliated cocycles which are entirely compatible with SODE structure.

Introduction

Moreover, the local structure of metric Legendre foliations is exhaustively analyzed by applying two significant local invariants existing on the tangent bundle of a Legendre foliation of the contact manifold (M, η); One of them is a symmetric 2-form and the other one is a symmetric 3- form. Mainly, it is proved that under some particular circumstances the behaviour of the Legendre foliation on the contact manifold (M, η) is locally the same as the foliation defined by the complementary orthogonal distribution in TTM° whose leaves are the c-indicatrix bundle over M.

Helmholtz Conditions and Formal Integrability

In this paper, the notion of the formal integrability is applied as a powerful tool and a significant reformulation of the inverse problem of the calculus of variations in terms of a partial differential operator that acts on semi-basic 1-forms. This expression of the Helmholtz conditions is fundamentally based on Frölicher-Nijenhuis formalism and is extensively fruitful since it provides a noteworthy setting to apply Spencer theory in order to investigate the formal integrability of Helmholtz conditions. Moreover, the only existing obstruction regarding this approach is due to the curvature tensor of the induced nonlinear connection.

The main goal of the current research is thoroughgoing study of metric Legendre foliations on contact manifolds via the global Helmholtz conditions, declared in terms of a semi-basic 1-form, that characterize when a semispray is locally Lagrangian.
The inverse problem of the calculus of variations can be explicitly expressed as follows: Under what conditions the solutions of a system of second order differential equations (SODE), on an arbitrary m-dimensional manifold M,

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \dot{x}\right) = 0 \quad , \quad i \in \{1, \cdots, m\} \tag{1}$$

can be deduced from a variational principle?

In other words, are among the solutions of the Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right)\frac{\partial L}{\partial x^i} = 0 \quad , \quad i \in \{1, \cdots, m\}. \tag{2}$$

for some Lagrangian function L. Literally, one privileged standpoint regarding the problem mentioned above, applies the Helmholtz conditions, which are necessary and sufficient conditions for the existence of a multiplier matrix $g_{ij}(x, \dot{x})$ such that for some Lagrangian function $L(x, \dot{x})$, the following identity holds:

$$g_{ij}(x, \dot{x})\left(\frac{d^2x^j}{dt^2} + 2G^i(x, \dot{x})\right) = \frac{d}{dt}\left(\frac{\partial \mathrm{L}}{\partial \dot{x}^i}\right) - \frac{\partial \mathrm{L}}{\partial x^i}, \tag{3}$$

It is noticeable that the multiplier matrix g_{ij} identically induces a symmetric $\binom{0}{2}$-type tensor field g along the tangent bundle projection.

A remarkable standpoint to the inverse problem of the calculus of variations applies the Helmholtz conditions, which are necessary and sufficient conditions for the existence of a multiplier matrix $g_{ij}(x, \dot{x})$ such that the relation (3) holds for some Lagrangian function $L(x, \dot{x})$. The Helmholtz conditions can be expressed as follows:

$$g_{ij} = g_{ji}, \qquad \frac{\partial g_{ij}}{\partial y^k} = \frac{\partial g_{ik}}{\partial y^j}, \tag{4}$$

$$\nabla g_{ij} = 0, \qquad g_{ik} R^k_j = g_{jk} R^k_i. \tag{5}$$

It is noticeable that conditions (4) are necessary and sufficient conditions for the existence of a Lagrange function which has as Hessian the matrix multiplier g_{ij}. Moreover, the conditions (5) represent the compatibility among the multiplier matrix and the given SODE and induced geometric structures such as: The Douglas tensor (Jacobi endomorphism) Φ and the dynamical covariant derivative.

Legendre Foliations on Contact Manifolds

Let M be a real $(2n+1)$-dimensional smooth manifold which carries a 1-form η satisfying $\eta \wedge (d\eta)^n \neq 0$ everywhere on M, where the exponent represents the nth exterior power. Then $\big(M, \eta\big)$ is called a contact manifold with the contact form η. Then a global vector field ξ, called the characteristic vector field or Reeb vector field on the contact manifold $\big(M, \eta\big)$, is defined on M by these conditions: $i_\xi \eta = 1$ and $i_\xi d\eta = 0$. A contact manifold $\big(M, \eta\big)$ admits a natural $2n$-dimensional distribution $\mathcal{H}$ which is defined by the kernel of η. In other words, $\mathcal{H}$ is simply the subbundle of TM on which $\eta = 0$. To be more precise we can write:

$$\Gamma(\mathcal{H}) = \Big\{ X \in \Gamma(TM) : \ \eta(X) = 0 \Big\}$$

The distribution $\mathcal{H}$ is defined by the contact distribution on $\big(M, \eta\big)$.

In the following, we want to relate contact manifolds with the notion of the contact metric manifolds. Let (M, g) be a real $(2n+1)$-dimensional Riemannian manifold endowed with a tensor field φ of the type $\binom{1}{1}$, a 1-form η and a vector field ξ. Then $\big(M, g, \varphi, \xi, \eta\big)$ is denoted by a contact metric manifold if for any $X, Y \in \Gamma(TM)$, the following tensor fields satisfy:

$$\begin{cases} (\mathbf{a}) : \varphi^2 = -I + \eta \otimes \xi, \\ (\mathbf{b}) : \eta(X) = g\big(X, \xi\big), \\ (\mathbf{c}) : g\big(X, \varphi Y\big) = d\eta\big(X, Y\big). \end{cases}$$

Taking into account above relations for any $X, Y \in \Gamma(TM)$ we have:

$$\begin{cases} (\mathbf{d}) : \eta(\xi) = 1 \quad , \qquad (\mathbf{e}) : \varphi(\xi) = 0 \\ (\mathbf{f}) : \eta(\varphi X) = 0 \quad , \quad (\mathbf{g}) : g\big(X, \varphi Y\big) + g\big(Y, \varphi X\big) = 0, \\ (\mathbf{h}) : g\big(\varphi X, \varphi Y\big) = g\big(X, Y\big) - \eta(X)\eta(Y). \end{cases}$$

Proposition

The contact distribution $\mathcal{H}$ on a contact manifold $\big(M, \eta\big)$ is not an integrable distribution.

Theorem

Let $\big(M, \eta\big)$ be a $\big(2n+1\big)$-dimensional contact manifold. Then the maximal dimension of any integrable subbundle of the contact distribution $\mathcal{H}$ is n.

Definition

A Legendre distribution on a $(n+1)$-dimensional contact manifold (M, η) is an n-dimensional subbundle P of the contact distribution such that for all $X, \widetilde{X} \in \Gamma(P)$, we have: $d\eta(X, \widetilde{X}) = 0$. Whenever P is integrable, it defines a Legendre foliation of (M, η) .

Remark

Thus due to above definition, a foliation $\mathcal{F}$ of (M, η) is a Legendre foliation if and only if the distribution $\mathcal{D}$ tangent to $\mathcal{F}$ is an n-subbundle of the $2n$-distribution $\mathcal{H}$.

Now assume that $\mathcal{F}$ is a foliation of codimension n on the manifold M and $\pi: \mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is the second projection. Then the map $f_i = \pi o \varphi_i^{-1}: U_i \longrightarrow \mathbb{R}^n$ is a submersion and on $U_i \cap U_j \neq \emptyset$ the following identity holds: $f_j = \gamma_{ij} o f_i$. Furthermore, the fibres of the submersion f_i are considered as the $\mathcal{F}$-plaques of $(U_i, \mathcal{F})$ and the foliation $\mathcal{F}$ is thoroughly characterized via the submersions f_i and the local diffeomorphisms γ_{ij} of $\mathbb{R}^n$. Overall, a foliation $\mathcal{F}$ of codimension n on M is totally identified via an open cover $\{U_i\}_{i\in I}$ and submersions $f_i: U_i \longrightarrow \mathcal{T}$ over an n-dimensional manifold $\mathcal{T}$ and a diffeomorphism $\gamma_{ij}: f_i(U_i \cap U_j) \longrightarrow f_j(U_i \cap U_j)$ such that $f_j = \gamma_{ij} o f_i$ for $U_i \cap U_j \neq \emptyset$. Then $\{U_i, f_i, \mathcal{T}, \gamma_{ij}\}$ is denoted by a *foliated cocycle* characterizing the foliation $\mathcal{F}$.

Theorem

Let M be a manifold of dimension $m+n$ and $\mathcal{F}=\left\{U_i, f_i, \mathcal{T}, \gamma_{ij}\right\}$ be a foliated cocycle of codimension n on M. Then the distribution $D(\mathcal{F})=span\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^m}\}$ defines canonically a foliated cocycle $\mathcal{F}^{\mathbb{T}}=\left\{\tilde{U}_i, \tilde{f}_i, \tilde{\mathcal{T}}, \tilde{\gamma}_{ij}\right\}$ on the tangent space TM.

Theorem

Let M be a manifold of dimension $m+n$ which is equipped with a foliation $\mathcal{F}$ of codimension n. Then $\mathcal{F}=\left\{U_i, f_i, \mathcal{T}, \gamma_{ij}, g\right\}$ is a metric foliated cocycle if and only if the induced metric on the transverse bundle is holonomy invariant.

Theorem

Let M be a manifold of dimension $m+n$ and $\mathcal{F} = \left\{U_i, f_i, \mathcal{T}, \gamma_{ij}\right\}$ be a foliated cocycle of codimension n on M. Then the distribution $D(\mathcal{F}) = span\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m}\}$ defines canonically a foliated cocycle $\mathcal{F}^{\mathbb{T}} = \left\{\tilde{U}_i, \tilde{f}_i, \tilde{\mathcal{T}}, \tilde{\gamma}_{ij}\right\}$ on the tangent space TM.

Theorem

Let M be a manifold of dimension $m+n$ which is equipped with a foliation $\mathcal{F}$ of codimension n. Then $\mathcal{F} = \left\{U_i, f_i, \mathcal{T}, \gamma_{ij}, g\right\}$ is a metric foliated cocycle if and only if the induced metric on the transverse bundle is holonomy invariant.

The semispray $\mathcal{S} = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$, determines a nonlinear connection N with local coefficients $G^i_j = \frac{\partial G^i}{\partial y^j}$. The nonlinear connection N has the local components as follows:

$(G^i_j) = \begin{bmatrix} G^a_b & G^\alpha_b \\ G^a_\beta & G^\alpha_\beta \end{bmatrix}$ Each of the local components $G^a_b, G^\alpha_b, G^a_\beta, G^\alpha_\beta$ has $x^a, x^\alpha, y^b, y^\beta$ as variables.

The nonlinear connection N defines a local base of its horizontal vector fields given by:

$$\frac{\delta}{\delta x^a} = \frac{\partial}{\partial x^a} - G^b_a \frac{\partial}{\partial y^b} - G^\beta_a \frac{\partial}{\partial y^\beta}$$
$$\frac{\delta}{\delta x^\alpha} = \frac{\partial}{\partial x^\alpha} - G^b_\alpha \frac{\partial}{\partial y^b} - G^\beta_\alpha \frac{\partial}{\partial y^\beta}$$

In this section, by imposing the following four significant conditions we will provide an appropriate setting in order to construct transverse foliated cocycles on the tangent space TM.

Definition

Let $\mathcal{F} = \{U_i, f_i, \mathcal{T}, \gamma_{ij}\}$ be a foliated cocycle of codimension n on M and $\mathcal{F}^{\mathbb{T}} = \{\tilde{U}_i, \tilde{f}_i, \tilde{\mathcal{T}}, \tilde{\gamma}_{ij}\}$ be the foliated cocycle on the tangent space TM. Let $\mathcal{S}$ be a semispray which is locally represented as $\mathcal{S} = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i}$. Then $\mathcal{S}$ is denoted by the **adopted foliated semispray (AFS)** and the metric g is called the **adopted transverse metric (ATM)** if the following four conditions are satisfied:

- The following partial differential operator

$$\mathcal{P}_{\mathrm{L}} = \big(d_{\mathcal{J}}, d_h, d_\Phi, \nabla d\big) : \ \mathrm{Sec}(T_v^*) \longrightarrow \mathrm{Sec}\big(\oplus^{(4)} \Lambda^2 T_v^* \big)$$

  is formally integrable.
- $g_{b\beta} = g(\frac{\partial}{\partial y^b}, \frac{\partial}{\partial y^\beta}) = 0.$

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  is formally integrable.
- $g_{b\beta} = g(\frac{\partial}{\partial y^b}, \frac{\partial}{\partial y^\beta}) = 0$.
- The local functions $(g_{\alpha\beta})$ and $(g^{\alpha\beta})$ are basic functions i.e. they do not depend on the tangent variables (x^a, y^a).

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- The local functions $(g_{\alpha\beta})$ and $(g^{\alpha\beta})$ are basic functions i.e. they do not depend on the tangent variables (x^a, y^a).
- The semispray $\mathcal{S}$ is foliated, namely the following identities hold:

$$(i) : G_b^\alpha = \frac{\partial G^\alpha}{\partial y^b} = 0 \ , \ (ii) : \frac{\partial G^\alpha}{\partial x^b} = 0; \quad \text{or}$$

$$(i') : G_\alpha^b = \frac{\partial G^b}{\partial y^\alpha} = 0 \ , \ (ii') : \frac{\partial G^b}{\partial x^\alpha} = 0.$$

- The following partial differential operator

$$\mathcal{P}_{\mathrm{L}} = \left(d_{\mathcal{J}}, d_h, d_\Phi, \nabla d\right) : \ \mathrm{Sec}(T_v^*) \longrightarrow \mathrm{Sec}\left(\oplus^{(4)} \Lambda^2 T_v^* \right)$$

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Theorem

Let $\mathcal{F} = \{U_i, f_i, \mathcal{T}, \gamma_{ij}\}$ be a foliated cocycle of codimension n on M and $\mathcal{F}^{\mathbb{T}} = \{\tilde{U}_i, \tilde{f}_i, \tilde{\mathcal{T}}, \tilde{\gamma}_{ij}\}$ be the foliated cocycle on the tangent space TM. Suppose that $\mathcal{S} = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i}$ is an adopted foliated semispray (AFS). Then $V\mathcal{F}^{\mathbb{T}}$ and $H\mathcal{F}^{\mathbb{T}}$ induce the nonlinear connection (G^a_b) , $a, b \in \{1, ..., m\}$, on the leaves of $\mathcal{F}^{\mathbb{T}}$.

Theorem

Let $\Big(M, \eta, \mathcal{F} = \{U_i, f_i, \mathcal{T}, \gamma_{ij}\}\Big)$ be a $(2n+1)$-dimensional contact manifold equipped with an n-dimensional Legendre foliated cocycle $\mathcal{F}$ and $\mathcal{F}^{\mathbb{T}} = \{\tilde{U}_i, \tilde{f}_i, \tilde{\mathcal{T}}, \tilde{\gamma}_{ij}\}$ be a foliated cocycle on the tangent space TM . Assume that $\mathcal{S} \in \mathcal{X}\big(TM \backslash \{0\}\big)$ be a semispray which is locally represented by: $\mathcal{S} = y^i \dfrac{\partial}{\partial x^i} - 2G^i(x,y)\dfrac{\partial}{\partial y^i}$. We denote by $\mathcal{D}$ the tangent distribution to $\mathcal{F}$. Then the symmetric bilinear form Π on $\Gamma(\mathcal{D})$:

$$\Pi\big(X, Y\big) = -\big(\mathcal{L}_X \mathcal{L}_Y \eta\big)(\xi), \quad \forall\ X, Y \in \Gamma(\mathcal{D}).$$

is positive definite if and only if $\mathcal{S} = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$ is an adopted foliated semispray (AFS) and g is an adopted transverse metric (ATM).

According to above theorem, it is deduced that:

Corollary

If all the conditions of the theorem are satisfied, then the Legendre foliation $\mathcal{F}$ on the contact manifold (M, η) is identically equivalent to the foliations constructed via the c-indicatrices of the Finsler function F resulted from the metrizability of the spray $\mathcal{S}$.

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Thank you for your kind attention.