Diffieties, Cohomological Physics, and Other Animals

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Darboux integrability for diagonal systems of hydrodynamic type

SERGEY AGAFONOV

We prove that diagonal systems of hydrodynamic type are Darboux integrabe if and only if the Laplace transformation sequences of the system for commuting flows terminate, give geometric interpretation for Darboux integrability of such systems in terms of congruences of lines and in terms of solution orbits with respect to symmetry subalgebras, show that Darboux integrable systems are necessarily semihamiltonian, and discuss known and new examples.

Geometric analysis of metric Legendre foliated cocycles on contact manifolds via SODE structure

FATEMEH AHANGARI

In recent years, an increasing consideration has been devoted to the qualitative analysis of systems of (non-) autonomous second (higher) order ordinary (partial) differential equations fields through some associated geometric structures. Second order ordinary differential equations (SODE) are of special significance mainly due to their extensive applications in various domains of mathematics, science and engineering. A remarkably type of SODE is the one which can be deduced from a variational principle. In this research, a thoroughgoing structural investigation of the transverse Legendre foliated cocycles on contact manifolds is presented. For this goal, by applying Spencer theory of formal integrability, sufficient conditions for the metric associated with the given SODE structure are designated to extend to a transverse metric for the lifted Legendre foliated cocycle on the tangent space of an arbitrary contact manifold. Indeed, the concept of formal integrability is applied as a noteworthy reformulation of the inverse problem of the calculus of variations in terms of a partial differential operator which acts on semi-basic 1-forms. Consequently, this expression of the Helmholtz metrizability conditions, enables us to construct a transverse metric on the tangent bundle of a given contact manifold which leads to creation of the specific type of metric Legendre foliated cocycles which are entirely compatible with SODE structure.

Group classification and conservation laws of the system of equations of two-dimensional shallow water over uneven bottom

ALEXANDER AKSENOV

A system of equations of two-dimensional shallow water over uneven bottom is considered. Overdetermined systems of equations for determining the symmetries and the conservation laws are obtained. The compatibilities of this overdetermined systems of equations are investigated. Symmetry operators and conservation laws are found. The results of the group classification have indicated that the system of equations of two-dimensional shallow water over uneven bottom cannot be linearized by point transformation in contrast to the system of equations of one-dimensional shallow water in the cases of horizontal and inclined bottom profiles.

Joint work with Konstantin Druzhkov.

Special Vinberg cones and their application to supergravity

D.V. Alekseevsky

Abstract

The talk is based on joint works with V. Cortes; and A. Spiro and A. Marrani.

E. B. Vinberg gave a description of homogeneous convex cones as cones of hermitian positively defined matrices in matrix T-algebra M_n of order n. The diagonal entries of such matrix are real numbers, but off-diagonal elements a_{ij} belong to different euclidean vector spaces V_{ij}. To determine the matrix multiplication, an isometric bilinear multiplication

$$V_{ij} \times V_{jk} \to V_{ik}$$

such that

$$|a_{ij} \cdot a_{jk}| = |a_{ij}| \cdot |a_{jk}|$$

must be defined. Unfortunately, such isometric multiplication $V \times U \to W$ is known only in two cases:

i) When the euclidean spaces have the same dimension (hence, isomorphic to V). Then the problem is equivalent to the classification of division algebras in V and was solved by A. Hurwitz (1898).

ii) When $\dim U = \dim W$. Then the problem reduces to i) a description of $\mathbb{Z}_2$-graded modules $S = S^0 + S^1 = U + W$ over the Clifford algebra $Cl(V)$ and ii) classification of $Spin(V)$-invariant metrics in S. It was solved by M.F.Atiyah, R. Bott and A.S.Shapiro (1964) and , respectively A-Cortes (1997). Due to this, the explicit classification of homogeneous Vinberg cones is known only in two cases :

i) When all Euclidean vector spaces V_{ij} are isomorphic (to one of division algebra $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$) (Then $\mathcal{V}$ is a symmetric self-adjoint cone) ;

ii) when T-algebra is the order 3 matrix algebra such that

$$V_{12} = V_{21} = V,\ V_{23} = V_{32} = S^0,\ V_{13} = V_{31} = S^1$$

where $S^0 + S^1$ is a graded $CL(V)$-module. The associated homogeneous convex cone is called special Vinberg cone.

Fortunately, only special Vinberg cones $\mathcal{V}$ are important for supergravity. They describe the target space of the scalar mass multiplet in $d = 5$ $N = 2$ Supergravity. The dimensional reduction to dimension $d = 4$ and then to dimension $d = 3$ associates with spacial Vinberg cone $\mathcal{V}$ a special Kähler

homogeneous manifold $\mathcal{K} = r(\mathcal{V})$ and then a special quaternionic Kähler homogeneous manifold $\mathcal{Q} = q(\mathcal{V}) = c \circ r(\mathcal{V})$, which are target spaces of the scalar multiplet in $d = 4$ and $d = 3$ Supergravity, respectively.
We describe a generalisation of the theory of special Vinberg cone to indefinite case.
We calculate the inversion of the quadratic map, associated to the fundamental cubic polynomial (the natural generalization of the determinant of order 3 matrix), which describes the determinant cubic hypersrface in a special Vinberg cone $\mathcal{V}$.
This allows to obtained the explicit formula for the Bekenstein-Hawking entropy of BPS (Bogomol'nui - Prasad-Sommerfield) black holes in $d = 4$ Supergravity for any homogeneous target manifold $\mathcal{K}$. Before it was known only for symmetric cones (that is $Herm_3^+(\mathbb{K})$, $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$).

Matrix Painlevé equations

IRINA BOBROVA

The interest in the non-commutative extensions of various integrable systems was motivated by needs of modern quantum psychics as well as by a natural attempts of mathematicians to extend various "classical" structures to non-commutative case.

The Painlevé equations satisfy the Painlevé-Kovalevskaya test and possess isomonodromic Lax pairs. We generalize the test to find matrix Painlevé equations and propose a non-abelinization procedure of known Lax pairs for searching Lax representations for these matrix systems.

The talk is based on the following recent papers: arXiv:2012.05639 (V. Adler, V. Sokolov) and arXiv:2107.11680, arXiv:2110.12159 (I. Bobrova, V. Sokolov).

Mathematics of Quantum Computing: Ideas and Reality

ALEXEI BOCHAROV

This talk goes over the basics of quantum computing, gives a high-level view of Shor's quantum-assisted integer factorization algorithm, introduces one of the key designs of Quantum error correction - the toric code - and emphasizes the need for native topological protection of quantum information.

The talk is an introductory overview of quantum computing concepts meant for mathematicians. Basic familiarity with the principles of quantum mechanics is assumed.

Scalar differential invariants of 2-dimensional Killing foliations

DIEGO CATALANO FERRAIOLI

We will present a fundamental system of scalar differential invariants of 4-dimensional semi-Riemannian metrics, which admits a 2-dimensional Abelian Killing algebra with non-null Killing leaves. We show how these invariants can be used to solve the local equivalence problem for metrics of the considered type, and discuss possible applications to the search of new solutions of Einstein equations. This is a joint work with Michal Marvan (Silesian University of Opava, Czech Republic).

Coverings and integrable pseudosymmetries of differential equations

VLADIMIR CHETVERIKOV

Finite-dimensional coverings from systems of differential equations are investigated. This problem is of interest in view of its relationship with the computation of differential substitutions, nonlocal symmetries, recursion operators, and Bäcklund transformations. We show that the distribution specified by the fibers of a covering is determined by an integrable pseudosymmetry of the system. Conversely, every integrable pseudosymmetry of a system defines a covering from this system. The vertical component of the pseudosymmetry is a matrix analog of the evolution differentiation. The corresponding generating matrix satisfies a matrix analog of the linearization of the equation. We also show how the coverings from an equation are related to coverings over the equation. A method for constructing coverings is given and demonstrated by the examples of the Laplace equation and the Kapitsa pendulum system.

On the relation between symplectic structures and variational principles in continuum mechanics

KONSTANTIN DRUZHKOV

The relation between symplectic structures and variational principles of equations in an extended Kovalevskaya form is considered.

It is shown that each symplectic structure of a system of equations in an extended Kovalevskaya form determines a variational principle.

A canonical way to derive variational principle from a symplectic structure is obtained.

The relation between variational principles in Eulerian and Lagrangian variables is discussed.

It is shown that if a system of equations in Lagrangian variables is an Euler-Lagrange system of equations, then the corresponding variational principle has no analogues in Eulerian variables.

Symmetries and differential invariants of the Navier-Stokes system on a space curve

ANNA DUYUNOVA

We consider flows of a viscid medium on a space curve in a constant gravitational field (the Navier-Stokes system). We discuss symmetries and differential invariants of the this system, and give their classification based on symmetries group of the system.

This is a joint work with V. Lychagin and S. Tychkov.

Approximate Noether symmetries and approximate conservation laws

MATTEO GORGONE

Within the framework of the recently introduced consistent approach for approximate Lie symmetries of differential equations, approximate Noether symmetries of variational problems involving small terms are considered. Then, an approximate Noether theorem leading to the construction of approximate conservation laws for models admitting a perturbed Lagrangian function is stated. Some illustrative applications are presented.

Presymplectic gauge PDEs and Batalin-Vilkovisky quantization

MAXIM GRIGORIEV

Gauge PDE is a geometrical object underlying what physicists call a local gauge field theory defined in terms of BV-BRST formalism. Although gauge PDE can be defined as a PDE equipped with extra structures, the generalization is not entirely straightforward as, for instance, two gauge PDEs can be equivalent even if the underlying PDEs are not. As far as Lagrangian gauge systems are concerned the powerful framework is provided by the Batalin-Vilkovisky (BV) formalism on jet-bundles. However, just like in the case of usual PDEs it is difficult to encode the BV extension of the Lagrangian in terms of the intrinsic geometry of the equation manifold while working on jet-bundles is often very restrictive especially in analyzing boundary behavior, e.g., in the context of AdS/CFT correspondence. We show that BV Lagrangian (or its weaker analogs) can be encoded in the compatible graded presymplectic structure on the gauge PDE. In the case of genuine Lagrangian systems this presymplectic structure is related to a certain completion of the canonical BV symplectic structure. A presymplectic gauge PDE gives rise to the BV gauge system through an appropriate generalization of the Alexandrov-Kontsevich-Schwarz-Zaboronsky (AKSZ) sigma-model construction followed by taking the symplectic quotient and resulting in the BV quantization setup or its presymplectic extension. The construction is illustrated on the standard examples of gauge theories with particular emphasis on the Einstein gravity, where this naturally leads to an elegant presymplectic AKSZ representation of the BV extension of the Cartan-Weyl formulation.

q-Analogs of differential operators and their applications

DIMITRI GUREVICH

Their exists a way, based on the notion of Quantum Doubles, to introduce analogs of partial derivatives on the so-called Reflection Equation algebras. Analogously to the classical case it is possible to use these q-derivatives for different applications. I plan to explain their utility for constructing q-analogs of the so-called cut-and-join operators and other problems of "braided differential calculus".

Darboux transformations of the KP hierarchy, its strict version and their reductions

GERARD HELMINCK

All four types of integrable hierarchies mentioned in the title, possess a geometric description of the construction of solutions. In the talk we will discuss which points in the varieties that describe this geometric construction, correspond to solutions that are linked by a Darboux transformation.

Algebra and geometry of Lax representations and Bäcklund transformations for (1+1)-dimensional partial differential and differential-difference equations

Sergei Igonin*

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Bäcklund transformations (BTs) are well known to be a powerful tool for constructing exact solutions for integrable nonlinear partial differential and difference equations, including soliton solutions. We present a review of recent results on algebraic and geometric methods in the theory of BTs for (1+1)-dimensional partial differential and differential-difference equations. The main tools in the methods are zero-curvature representations (Lax representations), gauge transformations, jet spaces (jet bundles), Lie algebras (including infinite-dimensional ones), Lie groups, and their actions on manifolds. Also, we use the theory of coverings of PDEs developed by A.M. Vinogradov and I.S. Krasilshchik [1, 2].

The main topics are the following:

1. Algebraic necessary conditions for existence of a Bäcklund transformation (BT) between two given (1+1)-dimensional evolution partial differential equations (PDEs). Here we consider the most general class of BTs, which are not necessarily of Miura type. The obtained necessary conditions allow us to prove non-existence of BTs between two given equations in many cases.

To obtain these conditions, for (1+1)-dimensional evolution PDEs we find a normal form for zero-curvature representations (ZCRs) with respect to the action of the group of local gauge transformations and define, for a given (1+1)-dimensional evolution PDE $\mathcal{E}$, a family of Lie algebras $F(\mathcal{E})$ whose representations classify all ZCRs of the equation $\mathcal{E}$ up to local gauge transformations. Furthermore, these Lie algebras allow us to prove non-existence of any nontrivial ZCRs for some classes of PDEs.

In our approach, ZCRs may depend on partial derivatives of arbitrary order, which may be higher than the order of the PDE. The algebras $F(\mathcal{E})$ are defined in terms of generators and relations and generalize Wahlquist–Estabrook prolongation Lie algebras, which are responsible for a much smaller class of ZCRs.

The structure of the Lie algebras $F(\mathcal{E})$ has been studied for some classes of (1+1)-dimensional evolution PDEs of orders 2, 3, 5, which include Korteweg–de Vries (KdV), modified KdV, Krichever–Novikov, Kaup–Kupershmidt, Sawada–Kotera, nonlinear Schrödinger, and (multicomponent) Landau–Lifshitz type equations. Among the obtained algebras one finds infinite-dimensional subalgebras of Kac–Moody algebras and infinite-dimensional Lie algebras of certain matrix-valued functions on some algebraic curves.

2. A method to construct BTs of Miura type (differential substitutions) for (1+1)-dimensional evolution PDEs, using zero-curvature representations and actions of Wahlquist–Estabrook prolongation Lie algebras. Our method is a generalization of a result of V.G. Drinfeld and V.V. Sokolov [3] on BTs of Miura type for the KdV equation.

3. A method to construct BTs of Miura type for differential-difference (lattice) equations, using Lie group actions associated with Darboux–Lax representations of such equations. The considered examples include Volterra, Narita–Itoh–Bogoyavlensky, Toda, and Adler–Postnikov lattices. Applying our method to these examples, we obtain new integrable nonlinear differential-difference equations connected with these lattices by BTs of Miura type.

Some results of the talk are based on joint works with G. Manno [4, 5, 6], with G. Berkeley [7], as well as on the paper [8].

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Applications of complexes of differential operators in gauge theories

IGOR KHAVKINE

In mathematical physics, gauge theories are variational PDEs that have gauge symmetries (symmetries locally parametrized by differential operators). Generators of gauge symmetries naturally fit into certain complexes of differential operators. From the point of view of the formal integrability theory of PDEs, these are the corresponding compatibility complexes. I will discuss the structure of these complexes and the possible roles played by their cohomology.

Non-linear homomorphisms of algebras of functions are induced by thick morphisms

HOVHANNES KHUDAVERDIAN

In 2014 Th. Voronov introduced the notion of thick morphisms of (super)manifolds as a tool for constructing L_∞-morphisms of homotopy Poisson algebras. Thick morphisms generalise ordinary smooth maps, but are not maps themselves. Nevertheless, they induce pull-backs on C^∞ functions. These pull-backs are in general non-linear maps between the algebras of functions which are so-called "non-linear homomorphisms". By definition, this means that their differentials are algebra homomorphisms in the usual sense. The following conjecture was formulated: an arbitrary non-linear homomorphism of algebras of smooth functions is generated by some thick morphism. We prove here this conjecture in the class of formal functionals.

Holomorphic relative Hopf modules over the irreducible quantum flag manifolds

ANDREY KRUTOV

We construct covariant q-deformed holomorphic structures for all finitely-generated relative Hopf modules over the irreducible quantum flag manifolds endowed with their Heckenberger-Kolb calculi. In the classical limit these reduce to modules of sections of holomorphic homogeneous vector bundles over irreducible flag manifolds. For the case of simple relative Hopf modules, we show that this covariant holomorphic structure is unique.

The talk is based on F. Díaz García, A. Krutov, R. Ó Buachalla, P. Somberg, K. R. Strung (2021), Holomorphic relative Hopf modules over the irreducible quantum flag manifolds. Lett. Math. Phys. 111(1), 24 p. arXiv:2005.09652

Generalized Atiyah-like sequences from differential operators in graded commutative algebras

JACOB KRYCZKA

In the context of differential calculus over arbitrary graded commutative algebras, we will describe a special class of short exact sequences of functors of differential calculus (in Vinogradov's sense) and discuss how they may be interpreted as higher order Atiyah sequences. Time permitting, we will discuss the corresponding notion of the Atiyah class in this setting.

Boundary topological indices of a pair of vector fields and existence theorems

OLGA KUNAKOVSKAYA

The problem of the existence of solutions of equations of the type

$$F_2(x) = \lambda F_1(x)$$

will be discussed. The method of topological boundary index is proposed. The topological boundary (bi)index $B(F_1, F_2)$ is additive and admits also a local form. The construction for smooth fields F_1, F_2 and some applications one can find in the monograph: Kunakovskaya O.V. Topological indices of a pair of fields (Topologicheskije indexi pary polej). Voronezh, Nauchnaya kniga, 2020. 88 pp., in Russian.

Polynomial graph invariants and the KP hierarchy

SERGEI LANDO

We describe a large family of polynomial graph invariants whose average value is a τ-function for the Kadomtsev-Petviashvili hierarchy of partial differential equations. In particular, this is valid for Stanley's symmetrized chromatic polynomial, as well as for the Abel polynomial for graphs we introduce. The key point here is a Hopf algebra structure on the space spanned by graphs and the behavior of the invariants on its primitive space.

The talk is based on a joint paper with S. Chmutov and M. Kazarian.

Continuum mechanics of media with inner structure

VALENTIN LYCHAGIN

The generalizations of the Navier-Stokes equations on media with inner structures and their applications to molecular media will be discussed.

Metrics admitting projective symmetries

GIOVANNI MANNO

A classical problem is to classify and find normal forms of metrics admitting projective symmetries, i.e., vector fields whose local flow sends (unparametrized) geodesics into (unparametrized) geodesics. In this seminar we discuss this problem for 2 and 3 dimensional metrics and, time permitting, also its complex analogue.

A Manifold view point of Lie Algebras

GIUSEPPE MARMO

Lie algebra brackets play a prominent role in the description of evolution (equations of motion) for any physical system, be it classical or quantum. On the other hand, the advent of general relativity has called for a description of physical systems in a coordinate independent manner. Special relativity has introduced the need for a composition law for velocities whose expression $(v + w)/[1 + (vw/c^2)]$ is alternative to the usual Galilean one $v + w$, which appears to be a "contraction" of previous one when c, the speed of light, goes to infinity. The composition law of special relativity is not Archimedean and is not associative in usual space-time (1+3). However in 1+1 space-time it defines an alternative "linear structure".

What we learn from these observations is that from the point of view of Physics it would be convenient to have a "tensorial" presentation of Lie algebras where not only the binary, bilinear product is represented by a tensor but the linear structure itself be represented by a tensor field. These aspects have been tackled in some previous papers by our research group and, as we shall show, have a strong relation with some of latest published papers by Alexander Vinogradov on the classification of finite dimensional real Lie algebras [Particle-like structure of Lie algebras].

Quantisation of free associative dynamical systems

ALEXANDER MIKHAILOV

Bi-quantisation of stationary KdV hierarchy and Novikov's equations and non-de formation quantisation of the Volterra sub-hierarchy.

Traditional quantisation theories start with classical Hamiltonian systems with variables taking values in commutative algebras and then study their non-commutative deformations, such that the commutators of observables tend to the corresponding Poisson brackets as the (Planck) constant of deformation goes to zero. I am proposing to depart from dynamical systems defined on a free associative algebra $\mathfrak{A}$. In this approach the quantisation problem is reduced to description of two-sided ideals $\mathfrak{J} \subset \mathfrak{A}$ satisfying two conditions: the ideals have to be invariant with respect to the dynamics of the system and to define a complete set of commutation relations in the quotient algebras $\mathfrak{A}_{\mathfrak{J}} = \mathfrak{A} \diagup \mathfrak{J}$.

To illustrate this approach I'll consider the quantisation problem for N-th Novikov equations and the corresponding finite KdV hierarchy. I will show that stationary KdV equations and Novikov's equations admit two compatible quantisations, i.e. two distinct commutation relations between the variables, such that a linear combination of the corresponding commutators i s also a valid quantisation rule leading to the Heisenberg form of quantum equations. The picture is very similar to the bi-Hamiltonian structure in the case of classical integrable equations.

I'll discuss quantisation of the Bogoyavlensky family of integrable N–chains:

$$\frac{du_n}{dt} = \sum_{k=1}^{N} (u_{n+k}u_n - u_n u_{n-k}), \qquad n \in \mathbb{Z}, \tag{1}$$

quantisation of their symmetries and modifications. In particular, I will show t hat odd degree symmetries of the Volterra chain ($N = 1$ in (1)) admit two quantisations, one of them corresponds to known quantisation of the Volterra chain, and another one is new and not deformational.

The talk is partially based on:

- AVM, *Quantisation ideals of nonabelian integrable systems*, arXiv:2009.01838, 2020 (Published in Russ. Math. Surv. v.75:5, pp 199-200, 2020).
- V.M. Buchstaber and AVM, *KdV hierarchies and quantum Novikov's equations*, arXiv:2109.06357, 2021.

Lie algebras of maximal class

DMITRY MILLIONSCHIKOV

Lie algebras of maximal class were introduced by Shalev and Zelmanov in late 90s. They can be thought of as infinite-dimensional analogs of filiform Lie algebras, which should be considered in the category of pro-nilpotent Lie algebras. We will discuss problems of classifying Lie algebras of maximal class, cohomology computations and applications. There is an interesting connection between Lie algebras of the maximum class and lamplighter groups, which have become a popular object of research in recent years.

In memory of Alexandre Vinogradov

MARCO MODUGNO

My seminar is devoted to a memory of Sasha as scientist and dear friend, based on my cooperation and friendship for 36 years.

Automatic Determination of Optimal Systems of Lie Subalgebras

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Abstract

It is known that a Lie group of symmetries (generating a Lie algebra) admitted by a partial differential equation can be used for determining group invariant solutions. By taking different subgroups of the principal Lie group, different sets of group invariant solutions can be characterized, most of which are connected by group transformations. Since a Lie group G (or Lie algebra $\mathfrak{g}$) usually contains infinitely many subgroups (or subalgebras) of the same dimension, a classification of them up to some equivalence relation is necessary. Following Ovsiannikov [1], two subalgebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ of a given finite-dimensional Lie algebra $\mathfrak{g}$ are equivalent if one can find some element g in the Lie group generated by $\mathfrak{g}$ such that $\mathrm{Ad}_g(\mathfrak{g}_1) = \mathfrak{g}_2$, where Ad_g is the adjoint representation of g on $\mathfrak{g}$.

A family of s-dimensional subalgebras $\{\mathfrak{g}_\alpha\}$ is an optimal system if it contains only inequivalent s-dimensional Lie subalgebas, and any s-dimensional subalgebra is equivalent to some element of the family.

An effective computer algebra program allowing for the automatic derivation of optimal systems of Lie subalgebras is presented, and some results in the literature amended.

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Homogeneous geodesics in sub-Riemannian geometry

ALEXEY PODOBRYAEV

We study homogeneous geodesics of sub-Riemannian manifolds, i.e., geodesics that are orbits of one-parametric subgroups of isometries. We obtain a criterion for a geodesic to be homogeneous in terms of its initial momentum. We discuss some examples of geodesic orbit sub-Riemannian manifolds (that means all geodesics are homogeneous) and prove that Carnot groups of step more than 2 can not be geodesic orbit. We prove that the geodesic flow for geodesic orbit sub-Riemannian manifold is integrable in non-commutative sense.

Negative numbers of times of integrable hierarchies

ANDREI POGREBKOV

Time evolutions of the dressing operators of the integrable hierarchies, like Kadomtsev-Petviashvili or Davey-Stewartson, are given by linear differential operators. In the standard situation these operators result from dressing of positive powers of a ∂_x. It is natural to call these semiinfinite hierarchies of integrable (2+1)-dimensional equations as hierarchies with positive numbers of times. Here we develop hierarchies directed to negative numbers of times. Derivation of such systems, as well as of the corresponding hierarchies, is based on the commutator identities. This approach enables introduction of linear differential equations that admit lift up to nonlinear integrable ones by means of the special dressing procedure. Thus one can construct not only nonlinear equations but corresponding Lax pairs, as well. Lax operator of such evolutions coincide with the Lax operator of the "positive" hierarchy. We also derive (1+1)-dimensional reductions of equations of such hierarchies.

On invariant differential ideals and homomorphic representations of functional derivations in differential rings

ANATOLIJ PRYKARPATSKY

We analyze finitely-generated by some differntial-algebraic relationships differential ideals in functional rings, invariant with respect some specially constructed derivations and satisfying the corresponding Lie-algebraic relationships. Taking into account the finite-dimensionality of these ideals, we construct the suitably defined homomorphic Lax type representations of these derivations, which in some cases are reduced to constraints equivalent to differential-algebraic relationships on a generating function. The work in part generalizes the results devised before for proving integrability of the well known generalized hierarchy of the Riemann type equations. We have also reformulated by means of the differential-algebraic terms the well known Dubrovin's integrability criterion of the classical Riemann equations, perturbed by means of some special terms from a suitably constructed differential ring.

Smooth local normal forms of hyperbolic Roussarie vector fields

ALEXEY REMIZOV

In 1975, Robert Roussarie studied a special class of vector fields, whose singular points fill a submanifold of codimension two and the ratio between two non-zero eigenvalues $\lambda_1 : \lambda_2 = 1 : -1$. He established a smooth orbital normal form for such fields at points where $\lambda_{1,2}$ are real and the quadratic part of the field satisfied a certain genericity condition. In this paper, we establish smooth orbital normal forms for such fields at points where this condition fails. Moreover, we prove similar results for vector fields, whose singular points fill a submanifold of codimension two and the ratio between two non-zero eigenvalues $\lambda_1 : \lambda_2 = p : -q$ with arbitrary integers $p, q \geq 1$.

Lie algebroid cohomology: from Der to Lie algebroid non-abelian extensions

VLADIMIR RUBTSOV

After a short historical reminder I consider the extension problem for Lie algebroids over schemes over a field. Given a locally free Lie algebroid Q over a scheme $(X, \mathcal{O})$, and a sheaf of finitely generated Lie $\mathcal{O}$-algebras L, we determine the obstruction to the existence of extensions $0 \to L \to E \to Q \to 0$, and classify the extensions in terms of a suitable Lie algebroid hypercohomology group.

Sub-Riemannian geometry on the group of motions of the plane

YURI SACHKOV

We will discuss old and new results on the standard left-invariant sub-Riemannian structure on the group of motions of the plane:

- Sub-Riemannian geodesics and minimizers,
- Cut locus,
- Spheres,
- Metric lines,
- Isometries, homogeneous and equioptimal geodesics,
- Applications to robotics and vision.

Conservation laws in action: an approach and implementations

ALEXEY SAMOKHIN

Quantities which are conserved in nondissipative media decay in presence of dissipation. The selective rates of such a decay can be found explicitly using the generating functions of the conservation laws. The general approach is illustrated by three examples:

- An arbitrary compact-support initial datum for the KdV equation eventually splits into solitons and a radiation tail. A numerically simple method to predict the number and amplitudes of resulting solitons using only a finite number of conservation laws is given.
- The behavior of the soliton which, while moving in non-dissipative and dispersion-constant medium encounters a finite-width barrier with varying dissipation and/or dispersion; beyond the layer dispersion is constant (but not necessarily of the same value) and dissipation is null. The passed wave either retains the form of a soliton or becomes a multi-soliton. Some rough estimations for a prediction of an output are given using the selective decay of the KdV conserved quantities.
- Some solutions of a system of MHD-equations for incompressible magnetofluids are found using the selective decay and the Taylor trick.

Quasi-differential operators on universal enveloping algebras and their applications

GEORGY SHARYGIN

In my talk I will describe a family of operators on the universal enveloping algebras, first of all on Ugl_n, which were first introduced by Gourevich and Saponov. We will discuss their properties, alternative definitions and relation with the algebra of differential operators on the corresponding Lie group. We shall also speculate on the possible applications of these operators to Vinberg's question to describe argument shift subalgebras in the universal enveloping algebras.

Recursion operators and bi-Hamiltonian representations of cubic evolutionary (2+1)-dimensional systems

MIKHAIL SHEFTEL

We construct all (2+1)-dimensional PDEs which have the Euler-Lagrange form and determine the corresponding Lagrangians. We convert these equations and their Lagrangians to two-component forms and find Hamiltonian representations of all these systems using Dirac's theory of constraints. Integrability properties of one-parameter equations that are cubic in partial derivatives of the unknown are derived by our method of skew factorization of the symmetry condition. Lax pairs and recursion relations for symmetries are determined both for one-component and two-component form. For the integrable cubic one-parameter equations in the two-component form we obtain recursion operators in 2×2 matrix form and bi-Hamiltonian representations, thus discovering new bi-Hamiltonian (2+1) systems.

On super Plücker embedding and cluster algebras

EKATERINA SHEMYAKOVA

There has been active work towards definition of super cluster algebras (Ovsienko, Ovsienko-Shapiro, and Li-Mixco-Ransingh-Srivastava), but the notion is still a mystery. As it is known, the classical Plücker map of a Grassmann manifold into projective space provides one of the model examples for cluster algebras. In the talk, we present our construction of "super Plücker embedding" or Grassmannian of $r|s$-planes in $n|m$-space.

There are two cases. The first one is of completely even planes in a super space, i.e., the Grassmannian $G_{r|0}(n|m)$. It admits a straightforward algebraic construction similar to the classical case. In the second, general case of $r|s$-planes, a more complicated construction is needed.

Our super Plücker map takes the Grassmann supermanifold $G_{r|s}(V)$ to a "weighted projective space" $P_{1,-1}(\Lambda^{r|s}(V) \oplus \Lambda^{s|rs}(\Pi V))$, with weights $+1, -1$. Here $\Lambda^{r|s}(V)$ denotes the $(r|s)$th exterior power of a superspace V and Π is the parity reversion functor. We identify the super analog of Plücker coordinates and show that our map is an embedding. We obtain the super analog of the Plücker relations and consider applications to conjectural super cluster algebras.

Based on a joint work with Th. Voronov.

Generalized Wünsch calculus for parabolic geometries

JAN SLOVÁK

The class of parabolic geometries includes many well known and useful examples, like conformal, CR, quaternionic etc. They all enjoy an affine space of distinguished connections modelled on 1-forms Υ. Long ago, efficient techniques were developed in the conformal geometry to deal with differential operators in terms of these connections, invariantly of the choice. The approach formalized by Volkmar Wünsch constructs operators which depend on the choice of the connection only algebraically in terms of Υ. The aim of this talk is to clarify this technique in a very different picture and to extend it to all parabolic geometries.

Energy functionals and the normal forms of knots and plane curves

ALEXEI SOSSINSKY

I will talk about knot theory, which is a topic that always interested Sasha Vinogradov: he translated the book by Ralph Fox on the subject and wrote an important supplement to the book.

This talk is a brief survey of ongoing research on the energy of knots, but will include a survey my own earlier work [1], [2], [5] and joint work [3], [4] with my former pupils O. Karpenkov and S. Avvakumov. The goal of these papers is to devise an algorithm (based on minimizing an energy functional) that will classify curves on the plane with a small number of double points and knots with a small number of crossings by bringing them to normal form.

The talk will begin with the demonstration of mechanical experiments with wire knots, showing how they automatically switch from any position to normal form and computer experiments with plane curves and knots, showing how our algorithm actually takes them to normal form. Then I will explain our algorithm: for polygonal knots, it minimizes a functional that consists of a summand that tries to straighten out the curve locally and a summand that forbids self-intersections (and the subsequent crossing of one part of the curve by another). In practice, the algorithm always terminates, which is actually amazing, because its termination is not a determinstic fact (i.e., not a theorem) - it is due to probabilistic laws of nature.

Another unexpected result of our study is that the normal form is not always unique - for the eight knot there are two: they depend on the mechanical characteristics of the wire, and in the case of implementation of our algorithm, on the initial shape of the knot.

Recently, I have constructed an algorithm minimizing a functional which consists of three summands, the two summands used before and a third summand based on the writhe of the knot. All the computer experiments with this algorithm have shown that the obtained normal form is unique. This is confirmed by physical experiments with new models of wire knots possessing a torque force due to twisting the wire before the ends of the wire are connected.

Time permitting, I will briefly explain our work on plane curves, a byproduct of which was our solution of the Euler elasticae problem, discuss some biological applications of our study, and formulate some conjectures about the further development of our approach.

References:

[1] Mechanical normal forms of knots and flat knots, Russian J. Math. Physics, Vol.18, no.2, 216-226 (2011).

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[3] [Jointly with S.Avvakumov, O.Karpenkov] Euler elasticae in the plane and the Whitney-Graustein theorem, Russian J. Math. Physics 20 (3), 257267 (2013).

[4] [Jointly with S. Avvakumov] bringing closed polygonal curves in the plane to normal form via local moves, Math. Notes, 103 (3) 466-473 (2018)

[5] Normal forms of unknotted ribbons and DNA, Russian J. Math. Physics, 25, No. 2, 241-247 (2018)

Higher holonomy versus strong homotopy representations

JIM STASHEFF

Given a connection for a smooth vector bundle $p\colon E \to M$, parallel transport with respect to smooth paths in the base space M provides a correspondence between smooth vector bundles with flat connection on M and representations of $\pi_1(M)$. Based in part on earlier groundbreaking work of K.T. Chen, recently this correspondence has been enhanced to the level of smooth paths (not homotopy classes) in the base space M and differential graded vector bundles with generalized flat connections.

Higher Haantjes brackets and integrability

GIORGIO TONDO

We propose a new, infinite class of brackets generalizing the Frölicher-Nijenhuis bracket. This class can be reduced to a family of generalized Nijenhuis torsions recently introduced. In particular, the Haantjes bracket, the first example of our construction, is relevant in the characterization of Haantjes moduli of operators. We also prove that the vanishing of a higher-level Nijenhuis torsion of an operator field is a sufficient condition for the integrability of its eigen-distributions. This result (which does not re-quire any knowledge of the spectral properties of the operator) generalizes the celebrated Haantjes theorem. The same vanishing condition also guarantees that the operator can be written, in a local chart, in a block-diagonal form.

References:

Piergiulio Tempesta and Giorgio Tondo, Higher Haantjes Brackets and Integrability, Commun. Math. Phys. (2021), https://doi.org/10.1007/s00220-021-04233-5, arXiv:1809.05908

The quotient of the Navier-Stokes system on a space curve

SERGEY TYCHKOV

We consider flows of a viscid medium on a space curve in a constant gravitational field (the Navier-Stokes system). Using differential invariants for this system, we obtain its quotient. The solutions of the quotient equation that are constant along characteristic vector field provide some solutions of the Navier-Stokes system.

This is a joint work with V. Lychagin and A. Duyunova.

Can gravity be repulsive?

GAETANO VILASI

Light carries energy and therefore is the source of a gravitational field. Geometric and quantum field theory an alysis suggest that, in special cases, gravity can be repulsive.

More precisely, it turns out that the Light is the source of non-linear gravitational waves corresponding, from a quantum point of view, to massless particles of spin -1.

These results give a possible solution to the old problem on the lack of gravitational attraction between two la ser beams moving parallel.

n-ary Batalin-Vilkovysky brackets

MICHAEL VINOGRADOV

The talk will describe how, using only the adgebraic definition of a linear differential operator, one can construct the higher Batalin-Vilkovysky brackets.

Alexandre M. Vinogradov's Cohomological Geometry of Partial Differential Equations (from my personal point of view)

LUCA VITAGLIANO

I recollect the story of my scientific relationship with Alexandre Mikhailovic Vinogradov. This story intertwines with the story of the Italian Diffiety School that have been run by Vinogradov for 20 years, from 1999 until 2019. I will give special emphasis on Vinogradov's cohomological/homotopical point of view on PDEs which basically inspired all my research so far.

On thick morphisms of (super)manifolds

THEODORE VORONOV

I will tell about a natural generalization of smooth maps, which came about in relation with homotopy Poisson brackets. The key feature of such "thick" or "microformal" morphisms is that they, like usual maps, induce pull-backs of functions; however, unlike the familiar case, such pull-backs are nonlinear. They are, actually, formal nonlinear differential operators of special sort. Thick morphisms make a formal category (i.e., the composition law is formal), which a formal thickening of the ordinary category of smooth supermanifolds and smooth maps. They provide a nonlinear version of the classical functional-algebraic duality between spaces and algebras, where ordinary algebra homomorphisms are replaced by certain "nonlinear homomorphisms". A quantum version of the theory exists, based on some formal Fourierv integral operators.