Invariants of forth order linear differential operators on 2-dimensional manifolds

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1. Introduction

Let M be an n-dimensional manifold, $A \in \mathbf{Diff}_k(M)$, i.e., A is a k-order linear differential operator on M acting on $C^\infty(M)$, and σ_k its symbol, i.e., $\sigma_k \stackrel{df}{=} A \bmod \mathbf{Diff}_{k-1}(M) \equiv \Sigma_k(M)$.
A stationary Lie algebra of σ_k is trivial at every point when $n \geq 2$ and $k \geq 3$, [3]. Then codimension of orbit of σ_k is

$$c(n,k) = \binom{n+k-1}{k} - n^2.$$

$c(n,k) \geq n$ for all $n \geq 2$ and $k \geq 3$ excepting three cases:

$$n=2, k=3;\ n=2, k=4;\ n=3, k=3.$$

It follows that the field of differential invariants of regular operator is generated by 0-order invariants in non exceptional cases.
The case $n=2$, $k=3$ was investigated in [4]. We consider the case $n=2, k=4$. In this case $c(2,4)=1$. This means that there is a unique independent differential invariant of 0-order for σ_4.

2. Introduction

Essentially, this invariant has long been known. Indeed, in the Hilbert lectures [1], p. 57, two relative invariants for a forth degree homogeneous polynomial of two variables $a_0x^4 + 4\,a_1x^3y + 6\,a_2x^2y^2 + 4\,a_3x\,y^3 + a_4y^4$ are found:

$$\mathcal{R}_2 = a_0a_4 - 4a_1a_3 + 3a_2^2,$$
$$\mathcal{R}_3 = a_0a_2a_4 - a_0a_3^2 - a_1^2a_4 + 2a_1a_2a_3 - a_2^3.$$

These invariants have weights 2 and 3 respectively. Hence,

$$I_0 = \mathcal{R}_3^2/\mathcal{R}_2^3 \tag{1}$$

is an invariant for the the polynomial.

3. First rational differential invariants

Let M be an oriented 2-dimensional manifold, x, y its local coordinates, $A \in \mathbf{Diff}_4(M)$, and σ_4 its regular symbol. Then:

$$A = a_0\partial_x^4 + 4a_1\partial_x^3\partial_y + 6a_2\partial_x^2\partial_y^2 + 4a_3\partial_x\partial_y^3 + a_4\partial_y^4$$
$$+ \text{lower order terms},$$
$$\sigma_4 = a_0\partial_x^4 + 4a_1\partial_x^3 \cdot \partial_y + 6a_2\partial_x^2 \cdot \partial_y^2 + 4a_3\partial_x \cdot \partial_y^3 + a_4\partial_y^4,$$

where all coefficients are smooth functions of x, y.
Thus the function $I_0(A) \in C^\infty(M)$ defined by (1), where coefficients a_i are coefficients of σ_4, is a rational differential invariant of A.
Suppose $dI_0(A) \neq 0$ on M. Then

$$I_1(A) = \left\langle\, dI_0(A)^4,\, \sigma(A) \,\right\rangle \tag{2}$$

is a first order rational differential invariant of A. Here $\langle \cdot\, , \cdot \rangle$ is the convolution of symmetric 4-form and symmetric 4-vector.

4. The bundle of differential operators

Let $\chi_4 : \mathrm{Diff}_4(M) \to M$ be the bundle of operators $A \in \mathbf{Diff}_4(M)$, (x, y, u^α) its local canonical coordinates, $\alpha = (\alpha_1, \alpha_2)$, and $0 \le |\alpha| = \alpha_1 + \alpha_2 \le 4$.
Then the section $S_A : M \to \mathrm{Diff}_4(M)$ generated by operator $A = \sum_{0 \le |\alpha| \le 4} a^\alpha(x, y) \partial_x^{\alpha_1} \partial_y^{\alpha_2}$, has the form $u^\alpha = a^\alpha(x, y)$.
Let $\pi_l : J^l(\chi_4) \to M$, $l = 0, 1, 2, \ldots$, be the bundles of l-jets of sections of χ_4. The bundles χ_4 and π_l are natural in the sense that the action of the diffeomorphism group $\mathcal{G}(M)$ of M is lifted to automorphisms

$$\varphi^{(l)} : [A]_p^l \mapsto [\varphi_*(A)]_{\varphi(p)}^l, \; \varphi \in \mathcal{G}(M)$$

of these bundles in the natural way.
A rational function $I \in C^\infty\big(J^l(\chi_4)\big)$ invariant w.r.t. these automorphisms $\varphi^{(l)}$ is called an l- order *natural invariant.*
Functions $I_1 \in C^\infty\big(J^0(\chi_4)\big)$ and $I_2 \in C^\infty\big(J^1(\chi_4)\big)$ defined by

$$(S_A)^*(I_0) = I_0(A), \quad (j^1 S_A)^*(I_1) = I_1(A), \quad \forall A \in \mathbf{Diff}_4(M)$$

are natural invariants.

5. The bundle of differential operators

Let $\mathcal{S}_2 = \{\theta_2 \in J^2\chi_4 \,|\, (\widehat{dI_0} \wedge \widehat{dI_1})_{\theta_2} = 0\}$. Then $\widehat{dI_0} \wedge \widehat{dI_1} \neq 0$ on the domain $J^2\chi_4 \setminus \mathcal{S}_2$.

Let $\mathcal{J}$ be a natural invariant, then

$$\widehat{d\mathcal{J}} = \mathcal{J}_1 \widehat{dJ_1} + \mathcal{J}_2 \widehat{dJ_2},$$

where functions $\mathcal{J}_i$ are natural invariants (*Tresse derivatives*). We will denote them by $\dfrac{d\mathcal{J}}{dJ_i}$.

The 4-order total differential operator

$$\square : C^\infty\big(J^l(\chi_4)\big) \longrightarrow C^\infty\big(J^{l+4}(\chi_4)\big),\ l = 0, 1, \ldots,$$

is defined by the formula

$$(j_{4+l}S_A)^*\big(\square(f)\big) = A\big((j_l S_A)^*(f)\big),$$

for all $f \in C^\infty\big(J^l(\chi_4)\big)$ and all operators $A \in \mathbf{Diff}_4(M)$. In jet-coordinates of π_l, this operator has the form

$$\square = \sum_{0 \le |\alpha| \le 4} u^\alpha \frac{d^{|\alpha|}}{d_x^{\alpha_1} d_y^{\alpha_2}}.$$

6. Constant type operators

Let V be a 2-dimensional vector space and let $\varpi \subset S^k(V)$ be a regular $\mathrm{GL}(V)$-orbit. Recall, that:

1. a symbol σ_4 has a *constant type* ϖ if for any point $q \in M$ and any isomorphism $\varphi : T_q(M) \to V$ the image $\varphi_*(\sigma) \in S^k(V)$ belongs to ϖ;
2. an operator $A \in \mathbf{Diff}_4(M)$ has the *constant type* ϖ if its symbol $\sigma_4(A)$ has the constant type ϖ.

Remark that a symbol $\sigma \in \Sigma_4(M)$ has a constant type if and only if its zero order invariant $I_0(\sigma)$ is constant.

7. The field of natural invariants of non constant type operators

Theorem *The field of all natural rational invariants of non constant operators* $A \in \mathbf{Diff}_4(M)$ *is generated by the invariants* I_0, I_1, *and Tresse derivatives*

$$\frac{d^{|\beta|} J_\alpha}{dI_0^{\beta_1} dI_1^{\beta_2}}$$

of invariants

$$J_\alpha = \Box(I_0^{\alpha_1} \cdot I_1^{\alpha_2}), \quad 0 \leq |\alpha| \leq 4.$$

The field of rational natural invariants separates regular orbits in the jet spaces of differential operators of non constant type.

8. The Wagner connection.

Theorem *A regular symbol $\sigma \in \Sigma_4$ has a constant type if and only if it has a linear connection ∇^σ in the tangent bundle to M such that for all vector fields X on M*

$$\nabla^\sigma_X(\sigma) = 0. \tag{3}$$

We call this connection by a Wagner connection.
A regular symbol σ can be reduced to the following forms by transformation of coordinates of M:

$$\sigma = \partial_x \cdot \partial_y \cdot \left(\alpha_0 \partial_x^2 + 2\alpha_1 \partial_x \cdot \partial_y + \alpha_2 \partial_y^2\right), \text{ or} \tag{4}$$

$$\sigma = (\partial_x^2 + \partial_y^2) \cdot \left(\alpha_0 \partial_x^2 + 2\alpha_1 \partial_x \cdot \partial_y + \alpha_2 \partial_y^2\right). \tag{5}$$

Let σ be defined by (4). Then non zero components Γ^i_{jk} of its Wagner connection are defined from (3) by the formulas:

$$\Gamma^1_{1\,1} = (-3\alpha_2 \partial_x \alpha_0 + \alpha_0 \partial_x \alpha_2)/(8\alpha_0\alpha_2),$$
$$\Gamma^1_{2\,1} = (-3\alpha_2 \partial_y \alpha_0 + \alpha_0 \partial_y \alpha_2)/(8\alpha_0\alpha_2),$$
$$\Gamma^2_{1\,2} = (\alpha_2 \partial_x \alpha_0 - 3\alpha_0 \partial_x \alpha_2)/(8\alpha_0\alpha_2),$$
$$\Gamma^2_{2\,2} = (\alpha_2 \partial_y \alpha_0 - 3\alpha_0 \partial_y \alpha_2)/(8\alpha_0\alpha_2).$$

9. The Wagner connection.

Let M be a connected and simply connected manifold, $\sigma \in \Sigma_4$ a regular symbol, and ∇^σ the complete Wagner connection.
Let the torsion tensor T^σ of ∇^σ is parallel, i.e., $d_{\nabla^\sigma}(T^\sigma) = 0$.
Then the 2-dimensional vector space $\mathfrak{g}^\sigma$ of all parallel vector fields on M, is a Lie algebra with respect to the bracket

$$X, Y \in \mathfrak{g}^\sigma \longrightarrow T^\sigma(X, Y) \in \mathfrak{g}^\sigma.$$

Theorem. *Let $\sigma \in \Sigma_4(M)$ be a regular symbol and ∇^σ be its Wagner connection with parallel torsion tensor T^σ. Then:*

1. *Symbol σ is locally equivalent to the following one*

   $$\sigma = c_0\partial_x^4 + 4c_1\partial_x^3 \cdot \partial_y + 6c_2\partial_x^2 \cdot \partial_y^2 + 4c_3\partial_x^3 \cdot \partial_y + c_4\partial_y^4, \quad c_i \in \mathbb{R},$$

   if and only if $T^\sigma = 0$.
2. *Symbol σ is locally equivalent to the symbol*

   $$\sigma = c_0e^{4y}\,\partial_x^4 + 4c_1e^{3y}\,\partial_x^3 \cdot \partial_y + 6c_2e^{2y}\,\partial_x^2 \cdot \partial_y^2 + 4c_3e^y\,\partial_x \cdot \partial_y^3 + c_4\partial_y^4, \quad c_i \in \mathbb{R},$$

   if and only if $T^\sigma \neq 0$.

10. Symbols and quantization

Let $\Sigma^{\cdot} = \oplus_{k\geq 0}\Sigma^{k}(M)$ be the graded algebra of symmetric differential forms and let ∇ be a Wagner connection of a regular symbol from $\Sigma_4(M)$. Then taking symmetrization of the covariant differential $d_\nabla : \Sigma^k(M) \longrightarrow \Sigma^k(M) \otimes \Omega^1(M)$, we get operators

$$d^s_\nabla : \Sigma^k(M) \longrightarrow \Sigma^{k+1}(M).$$

Let $\sigma_k \in \Sigma_k(M)$ be a regular symbol. We define a differential operator $\mathcal{Q}(\sigma_k) \in \mathbf{Diff}_k(M)$ as follows:

$$\mathcal{Q}(\sigma_k)(h) \overset{\text{def}}{=} \frac{1}{k!} \left\langle \sigma_k, \left(d^s_\nabla\right)^k (h) \right\rangle$$

where $h \in C^\infty(M)$, $\left(d^s_\nabla\right)^k (h) \in \Sigma^k(M)$, and $\langle \cdot\,,\cdot\rangle$ is the natural convolution $\Sigma_k(M) \otimes \Sigma^k(M) \to C^\infty(M)$.

Remark that the symbol of operator $\mathcal{Q}(\sigma_k)$ equals σ_k.

We call this operator $\mathcal{Q}(\sigma_k)$ a *quantization of symbol* σ_k.

11. Natural decomposition

Let now $A \in \mathbf{Diff}_4(M)$ and $\sigma_4(A)$ be its symbol. Then operator

$$A - \mathcal{Q}\big(\sigma_4(A)\big)$$

has order 3. Let $\sigma_3(A)$ be its symbol.
Then operator $A - \mathcal{Q}\big(\sigma_4(A)\big) - \mathcal{Q}\big(\sigma_3(A)\big)$ has order 2.
Repeating this process we get subsymbols $\sigma_i(A) \in \Sigma_i(M)$, $0 \le i \le 3$, such that

$$A = \mathcal{Q}\big(\sigma_{(4)}(A)\big),$$

where

$$\sigma_{(4)}(A) = \sigma_4(A) + \sigma_3(A) + \ldots + \sigma_0(A)$$

is a total symbol of the operator, and

$$\mathcal{Q}\big(\sigma_{(4)}(A)\big) = \mathcal{Q}\big(\sigma_4(A)\big) + \mathcal{Q}\big(\sigma_3(A)\big) + \ldots + \mathcal{Q}\big(\sigma_0(A)\big).$$

12. Differential invariants of constant type operators

Let $\pi_4 : S^4T(M) \to M$ be the bundle of symmetric 4 -vectors (symbols) and let $\nu_l \in \Sigma_l(\pi_4)$ be the universal symbol (of order 0).

We denote by $\mathcal{O}_0 \subset J^0(\pi_4)$ the domain of regular symbols. The symbols having the constant type ϖ constitute a subbundle

$$\pi_4^{\varpi} : E^{\varpi} \longrightarrow M$$

of the bundle $\pi_4|_{\mathcal{O}_0} : \mathcal{O}_0 \to M$ of regular symbols. Then the Wagner connection defines a total covariant differential

$$\widehat{d}_{\varpi} : \Sigma_1(\pi_4^{\varpi}) \longrightarrow \Sigma_1(\pi_4^{\varpi}) \otimes \Omega^1(\pi_4^{\varpi}),$$

over $\mathcal{O}_0$, and, by the construction $\widehat{d}_{\varpi}(\nu_4) = 0$.

Let $T^{\varpi} \in \Omega^2(\pi_4^{\varpi}) \otimes \Sigma_1(\pi_4^{\varpi})$ be the total torsion of the connection and $\theta^{\varpi} \in \Omega^1(\pi_4^{\varpi})$ be the total torsion form.

(Recall that torsion form of the Wagner connection is defined by the formula $\theta^{\sigma}(X) = \mathrm{Tr}\big(Y \to T^{\sigma}(X, Y)\big)$, where T^{σ} is the torsion tensor.)

13. Differential invariants of constant type operators

Let us apply the total differential of the dual (to Wagner) connection $\widehat{d}^{*}_{\varpi} : \Omega^1(\pi_4^{\varpi}) \to \Omega^1(\pi_4^{\varpi}) \otimes \Omega^1(\pi_4^{\varpi})$, we get the following tensor

$$\widehat{d}^{*}_{\varpi}(\theta^{\varpi}) \in \Omega^1(\pi_4^{\varpi}) \otimes \Omega^1(\pi_4^{\varpi}).$$

Taking the symmetric g^{ϖ} and antisymmetric a^{ϖ} parts of this tensor, we get tensors

$$g^{\varpi} \in \Sigma^2(\pi_4^{\varpi}), \quad a^{\varpi} \in \Omega^2(\pi_4^{\varpi}).$$

Assuming that tensor g^{ϖ} is non degenerated, we get a total operator

$$A^{\varpi} \in \Sigma_1(\pi_4^{\varpi}) \otimes \Omega^1(\pi_4^{\varpi}),$$

instead of a^{ϖ}, and horizontal 1-forms

$$\theta_1^{\varpi} = \theta^{\varpi}, \quad \theta_2^{\varpi} = A^{\varpi}(\theta_1^{\varpi}). \tag{6}$$

14. Differential invariants of constant type operators

Let (e_1^ϖ, e_2^ϖ) be the frame of horizontal vector fields $e_i^\varpi \in \Sigma_1(\pi_4^\varpi)$ dual to coframe $(\theta_1^\varpi, \theta_2^\varpi)$.
Let $\chi_4^\varpi : \mathrm{Diff}_4(M) \to M$ be the bundle of scalar differential operator of order 4, having symbols of constant type ϖ, $\mathbf{Diff}_4^\varpi(M)$ its module of smooth sections, $A \in \mathbf{Diff}_4^\varpi(M)$,

$$\sigma_{(4)}(A) = \sigma_4 + \sigma_3 + \ldots + \sigma_0$$

the total symbol of A, and $\nu_4,\ \nu_3,\ \ldots,\ \nu_0$ corresponding universal symbols. Then

$$\nu_i = \sum_{|\alpha|=i} J_\alpha^\varpi (e_1^\varpi)^{\alpha_1} \cdot (e_2^\varpi)^{\alpha_2}$$

Teorem *The field of natural differential invariants of linear scalar differential operators of order 4 having constant type* ϖ *is generated by the basic invariants* J_α^ϖ, $|\alpha| \leq 4$, *and invariant derivatives* e_i^ϖ, $i = 1, 2$.

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