Non-linear homomorphisms of algebra of functions and thick morphisms

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Contents

Papers that talk is based on are

[1] Th.Th. Voronov "Nonlinear pullbacks" of functions and L_∞-morphisms for homotopy Poisson structures. J. Geom. Phys. 111 (2017), 94-110. arXiv:1409.6475

[2] Th.Th. Voronov Microformal geometry and homotopy algebras. Proc. Steklov Inst. Math. 302 (2018), 88-129. arXiv:1411.6720

[3] H.M.Khudaverdian, Th.Th.Voronov. Thick morphisms, higher Koszul brackets, and L_∞-algebroids. arXiv:1808.10049

[4] H.M.Khudaverdian, Th,Th,Voronov. Thick morphisms of supermanifolds, quantum mechanics and spinor representation. J. Geom. Phys. 113 (2019), DOI: 10.1016/j.geomphys.2019.103540, arXiv:1909.00290

[4] H.M.Khudaverdian Non-linear homomorphisms of algebra of functions are induced by thick morphisms,arXiv:2006.03417

Abstract...

In 2014, Voronov introduced the notion of thick morphisms of (super)manifolds as a tool for constructing L_∞-morphisms of homotopy Poisson algebras. Thick morphisms generalise ordinary smooth maps, but are not maps themselves. Nevertheless, they induce pull-backs on C^∞ functions. These pull-backs are in general non-linear maps between the algebras of functions which are so-called “non-linear homomorphisms”. By definition, this means that their differentials are algebra homomorphisms in the usual sense. The following conjecture was formulated: an arbitrary non-linear homomorphism of algebras of smooth functions is generated by some thick morphism. We prove here this conjecture.

Standard pull-back

A map (morphism) $\varphi\colon\ M \to N$
defines the linear map, the pull-back

$$\varphi^*\colon\ C^\infty(N) \to C^\infty(M)\,, \quad \varphi^*(g) = g(\varphi(x)) \tag{1}$$

which is homomorphism of algebras of functions.

$$\begin{cases} \varphi^*(\lambda f + \mu g) = \lambda\varphi^*(f) + \mu\varphi^*(g) \\ \varphi^*(f\cdot g) = \varphi^*(f)\,\varphi^*(g) \end{cases},$$

Thick morphisms $\Phi\colon M \Rrightarrow N$

A thick morphism $\Phi\colon M \Rrightarrow N$ defines a non-linear map on algebras of functions

$$\Phi^*\colon C^\infty(N) \to C^\infty(M).$$

(This notion provides a natural way to construct L_∞ morphisms for homotopy Poisson algebras)

One important property of thick morphisms

Functionals $\Phi^*\colon\ C^\infty(N) \to C^\infty(M)$ are non-linear however their differentials are usual homomorphisms induced by pull-backs: for arbitrary smooth functions g and h

$$\Phi^*(g+\varepsilon h) - \Phi^*(g) = \varepsilon h(y^a_g(x))\,, \quad (\varepsilon^2 = 0)\,.$$

(Ted Voronov, 2014.) We will call these maps non-linear homomorphisms.

Definition of non-linear homomorphism

Definition

Let $\mathbf{A}, \mathbf{B}$ be two algebras.
A map L from an algebra $\mathbf{A}$ to an algebra $\mathbf{B}$ is called *a non-linear homomorphism* if at an arbitrary element of algebra $\mathbf{A}$ its derivative is a homomorphism of the algebra $\mathbf{A}$ to the algebra $\mathbf{B}$.

Conjecture

Theorem

A map $\Phi^*(g)$ from $C^\infty(N)$ to $C^\infty(M)$ induced by thick morphism $M \Rrightarrow N$ is non-linear homomorphism.

Is it true that every non-linear homomorphism from algebra $C^\infty(N)$ to algebra $C^\infty(M)$ is induced by some thick morphism $M \Rrightarrow N$?

We prove here this conjecture in the class of formal functionals.

Formal functional

We say that $L = L(g)$ is formal functional from $C^\infty(N)$ to $C^\infty(M)$ if for every $g \in C^\infty(N)$

$$L(g) = L_0(x) + L_1(x,g) + L_2(x,g) + \dots + L_n(x,g) + \dots ,$$

where every summand $L_r(x,g)$ $(r = 0,1,2,\dots)$ is functional on g of order r in g with values in $C^\infty(M)$

$$L_r(x,g) = \int L(x,y_1,\dots,y_r) g(y_1)\dots g(y_r) dy_1 \dots dy_r$$

The kernel $L(x,y_1,\dots,y_r)$ can be generalised function

Example of functional on smooth functions on **R**

Example

$L(x,g) =$

$$L_0(x)+L_1(x,g)+L_2(x,g) = L_0(x)+g(y)\big|_{y=f(x)}+g(y)g'(y)\big|_{y=f(x)},$$

$L_0(x)$ has order 0 in g,

$$L_1(x,g) = g(f(x)) = \int \delta(y-f(x))g(y)dy \text{ has order 1 in } g,$$

$$L_2(x,g) = h(x)g(y)g'(y)\big|_{y=f(x)} =$$

$$\int h(x)\delta(y_1-f(x))\delta'(y_2-f(x))g(y_1)g(y_2)dy_1dy_2 \text{ has order 2 in } g.$$

In other words formal functional $L = L(x, g)$ is a sequence $\{L_r(x, g)\}$ of functionals, where $L_r(x, g)$ is functional of order r in g with values in smooth functions on N

Pull-back of functions on N to functions on M which corresponds to thick morphism $\Phi\colon M \Rrightarrow N$ is an example of formal functional
Now we will consider thick morphisms. They produce examples of formal functionals.

Definition of thick morphism $\Phi_S\colon M \Rrightarrow N$ (T.Voronov)

$$\underbrace{M}_{x^i\text{- loc.coord.}}\,,\qquad \underbrace{N}_{y^a\text{- loc.coord.}}\,,$$

Consider also cotangent bundles T^*M and T^*N.

$$\underbrace{T^*M}_{x^i,p_j\text{- loc.coord.}}\,,\qquad \underbrace{T^*N}_{y^a,q_b\text{- loc.coord.}}\,,$$

p_i are components of momenta which are conjugate to x^i, respectively q_a are components of momenta which are conjugate to y^a.

Action $S(x,q)$, generating function of thick morphism

Consider an 'action' $S(x,q)$, which is a formal function, power series in q

$$S(x,q) = S_0(x) + S_1^a(x) q_a + S_2^{ab}(x) q_a q_b + \dots + S_r^{a_1 \dots a_r}(x) q_{a_1} \dots q_{a_r} + \dots,$$

where x^i are local coordinates on M and q_a are coordinates of momenta in T^*N.
Thick morphism $\Phi = \Phi_S \colon \quad M \Rrightarrow N$ can be defined by the 'action' $S(x,q)$ in the following way

(**Remark** Later we will explain why we call function $S(x,q)$ an 'action'.)

Thick morphism Φ_S generated by action S

To the thick morphism Φ_S corresponds pull-back, the formal functional

$$\Phi_S^*(g)\colon C^\infty(N) \to C^\infty(M) \qquad (*)$$

such that for every smooth function g,

$$f(x) = g(y) + S(x,q) - y^a q_a \qquad (**)$$

with

$$y^a(x) = \frac{\partial S(x,q)}{\partial q_a}, \quad q_a = \frac{\partial g(y)}{\partial y^a} \qquad (***)$$

All equations are formal. In particular

$$y^a(x) = y^a(x,g) = y_0^a(x) + y_1^a(x,g) + y_2^a(x,g) + \dots$$

here every term $y_r^a(x,g)$ is a smooth map of order r in g.

Formal functional $\Phi^*_S(g)$ in equation (*) assignes to function $g(y)$ a function $f(x)$ which depends on x .

Indeed one can see that function (**) does not depend on y^a and q_a:

$$\frac{\partial}{\partial y^b}\left(g(y)+S(x,q)-y^a q_a\right)=\frac{\partial g(y)}{\partial y^b}-q_b=0\,,$$

$$\frac{\partial}{\partial q_b}\left(g(y)+S(x,q)-y^a q_a\right)=\frac{\partial S(x,q)}{\partial q_b}-y^b=0\,,$$

One can see that equations (*),(**) and (***) define in a recurrent way formal functional

Expression for formal map $y^a(x)$

$$y^a(x,g) = \frac{\partial}{\partial q_a}\left[S_0(x) + S_1^a(x)q_a + S_2^{ab}(x)q_a q_b + \dots\right] =$$

$$S_1^a(x) + 2S_2^{ab}(x)q_b + \dots =$$

$$\underbrace{S_1^a(x)}_{\text{term of order 0 in } g} \qquad +$$

$$\underbrace{2S_2^{ab}(x)g_b^*(x)}_{\text{term of order 1 in } g} \quad + \text{terms of order} \geq 2 \text{ in } g\,,$$

where

$$g_b^*(x) = \frac{\partial g(y)}{\partial y^b}\Big|_{y^a = S_1^a(x)}$$

Explicit expression for formal functional $\Phi^*_S(g)$ up to order 3 in g

If action, generating function of thick morphism is equal to

$$S = S(x,q) = S_0(x) + S_1^a(x)q_a + S_2^{ab}(x)q_aq_b + S_3^{abc}(x)q_aq_bq_c + \dots,$$

then

$$\Phi^*_S(g) = \underbrace{S_0(x)}_{\text{term of order 0 in } g} + \underbrace{g\left(S^a(x)\right)}_{\text{term of order 1 in } g} + \underbrace{S^{ab}(x)g^*_b(x)g^*_b(x)}_{\text{terms of order 2 in } g} +$$

$$\underbrace{S^{abc}(x)g^*_c(x)g^*_b(x)g^*_a(x) + 2S^{ac}S^{bd}(x)g^*_{ab}(x)g^*_d(x)g^*_c(x)}_{\text{terms of order 3 in } g} +$$

$$+\text{terms of order} \geq 4 \text{ in } g$$

Reconstructing an action

Example

Consider thick morphism Φ_S with action

$$S(x,q) = S_0(x) + S_1^a(x)q_a + S_2^{ab}(x)q_aq_b + S_3^{abc}(x)q_aq_bq_c + \dots$$

if function $g = g(y)$ is linear function, $g(y) = l_a y^a$, then

$$\Phi_S^*(g) = g(y) + S(x,q) - y^a q_a = y^a (l_a - q_a) + S(x,q)\,.$$

$q_a = \frac{\partial g(y)}{\partial y^a} = l_a \Rightarrow \Phi_S^*(g) = S(x,l)\,. =$

Value of pull-back on linear function reconstructs the action.

Example of 'degenerate' thick morphism

Example

Let

$$S(x,q) = S_1^a(x)q_a\,.$$

Then

$$\Phi_S^*(g) = g(y) + S(x,q) - y^a q_a = g(y) + \left(S_1^a(x) - y^a\right) q_a\,,$$

$y^a = \frac{\partial S(x,q)}{\partial q_a} = S_1^a(x)$,

$$\Phi_S^*(g) = g(y) + S(x,q) - y^a q_a = g\left(S_1^a(x)\right)$$

In this case thick morphism is nothing but usual map $y^a(x) = S_1^a(x)$.

This is not a good example.

Recall what is it action of mechanical system

Let $L = L(q, \dot{q})$ be Lagrangian of the system ($H = H(x, p)$ be Hamiltonian of this system).
A function $W_t(x, y)$ such that

$$W_t(x,y) = \int_0^t \left[L(q,\dot{q})\big|_{q=q(\tau)} \right] d\tau\,,$$

is called action. Here $q = q(\tau)$ is solution of Euler-Lagrange equations which obeys boundary conditions $q(0) = x, q(t) = y$.
One can consider a function $S = S_t(x, q)$ which is Legendre transformation for this action:

$$S_t(x,q) = W_t(x,y) - y^a q_a$$

Example

Free particle

$$L = \frac{m\dot{q}^2}{2}\,, \quad H = \frac{p^2}{2m}$$

$$W_t(x,y) = \frac{m(x-y)^2}{2t}$$

$$S_t(x,q) = xq + \frac{q^2 t}{2m}$$

Harmonic oscillator

$$L = \frac{m\dot{q}^2}{2} - \frac{mw^2q^2}{2}\,, \quad H = \frac{p^2}{2m} + \frac{mw^2x^2{:}wq}{2}$$

$$W_t(x,y) = \frac{mw(x^2+y^2)}{2}\operatorname{ctg} wt - \frac{mwyx}{\sin wt}$$

$$S_t(x,q) = \frac{xq}{\cos wt} + \left(\frac{q^2}{2mw} + \frac{mwx^2}{2}\right)\operatorname{tg} wt\,.$$

Why generating function $S(x,q)$ of thick morphism is called 'action'

Theorem

Consider the one-parametric group of thick (diffeo)morphism $\Phi_t\colon M \Rrightarrow M$ generated by $S(t,x,q)$. For an arbitrary function $g=g(x)$ consider

$$f(t,x)=\Phi_t^*(g)$$

The function $f(t,x)$ obeys the Hamilton-Jacobi equation:

$$\frac{\partial f(t,x)}{\partial t}=H\left(x,\frac{\partial f}{\partial x}\right), \quad f(t,x)\big|_{t=0}=g(x).$$

The pull-back by thick diffeomorphism maps initial conditions to the solution of differential equation.

Cotangent bundle with canonical symplectic structure

Consider cotangent bundle T^*M with canonical Poisson bracket

$$(f(x,p),g(x,p)) = \frac{\partial f(x,p)}{\partial p_i}\frac{\partial g(x,p)}{\partial x^i} - \frac{\partial f(x,p)}{\partial x^i}\frac{\partial g(x,p)}{\partial p_i}$$

A formal Hamiltonian $H(x,p)$

$$H_M(x,p) = H_{0(M)}(x) + H^i_{1(M)}(x)p_i + H^{ij}_{2(M)}(x)p_ip_j + H^{ijk}_{3(M)}(x)p_ip_jp_k + \cdots +$$

defines on algebra $C^\infty(M)$ the collection of 'brackets' $\{\langle f_1,\ldots,f_r\rangle_r\}$ $(r = 0,1,2,3,\ldots)$ such that

Collection of brackets on M

$$\langle\emptyset\rangle_0 = H(x,p)\big|_{p=0} = H_0(x) \qquad \langle f\rangle_1 = (H, f_1)\big|_{p=0} = H_1^i(x)\frac{\partial f(x)}{\partial x^i}\,,$$

$$\langle f_1, f_2\rangle_2 = ((H, f_1), f_2)\big|_{p=0} = H_2^{ij}(x)\frac{\partial f_1(x)}{\partial x^i}\,,\frac{\partial f_2(x)}{\partial x^j}\,,$$

and so on

$$\langle f_1, f_2, \dots, f_r\rangle_r = \underbrace{(\dots(}_{r \text{ times}}\ H, f_1), f_2)\dots f_r)\big|_{p=0} =$$

$$H_r^{i_1\dots i_r}(x)\frac{\partial f_1(x)}{\partial x^{i_1}}\cdots\frac{\partial f_r(x)}{\partial x^{i_r}}\,.$$

We say that this collection of brackets on $C^\infty(M)$ is defined by master-Hamiltonian $H(x,p)$ which is function on T^*M.

Collection of brackets on N

Respectively formal Hamiltonian $H_N(y,q)$, function on T^*N,

$$H_N(y,q) = H_{0(N)}(x) + H^a_{1(N)}(x)p_a + H^{ab}_{2(N)}(x)q_aq_b + H^{ijk}_{3(N)}(x)p_ip_jp_k + \cdots$$

defines on algebra $C^\infty(N)$ the collection of 'brackets' $(r = 0,1,2,3,\dots)$

$$\langle g_1, g_2, \dots, g_k\rangle_r = \underbrace{(\dots(}_{k \text{ times}} \; H, f_1), f_2) \dots f_k)\big|_{p=0} =$$

$$H_k^{a_1\dots a_k}(x)\frac{\partial g_1(y)}{\partial x^{a_1}} \dots \frac{\partial g_r(y)}{\partial x^{a_r}}\,.$$

Theorem (Voronov, 2014)

Theorem

Let collection of brackets on $C^\infty(M)$ and $C^\infty(N)$ are defined by master-Hamiltonians $H_M(x,p)$ on T^*M and $H_N(y,q)$ on T^*N respectively. Let $S(x,q)$ be a function such that the following Hamilton-Jacobi like equation holds:

$$H_M\left(x^i, \frac{\partial S(x,q)}{\partial x^j}\right) = H_N\left(\frac{\partial S(x,q)}{\partial q_a}, q_b\right).$$

then formal functional Φ_S^* defines non-linear map which connects these brackets.

Remark In the 'real life', M, N are supermanifolds, collections of brackets are homotopy Poisson (Schouten) brackets and Hamiltonians define homological vector fields on space of functions.

Revenons a nos moutons

Now we will return to formal functionals.
Let $L = L(x,g)$ be a formal functional on $C^\infty(N)$ with values in $C^\infty(M)$

$$L(x,g) = L_0(x) + L_1(x,g) + L_2(x,g) + \cdots + L_r(x,g) + \dots,$$

for every $g \in C^\infty(M)$,
for every m the functional $L_m(x,g)$ has order m in g, and it takes values in smooth functions,

$$L_m(x,g) = \int L_m(x,y_1,\dots,y_m)g(y_1)\dots g(y_m)dy_1\dots dy_m$$

We prove that formal functionals which are non-linear homomorphisms are pull-backs induced by thick morphisms.

Recall notion of non-linear homomorphism

Recalling of non-linear homomorphism

Formal functional $L = L(x,g)$ is non-linear homomorphism if for arbitrary smooth function g there exists map $K^a = K^a(x,g)$ such that for every smooth function h

$$L(x,g+\varepsilon h) - L(x,g) = \varepsilon h\left(K^a(x,g)\right), \quad (\varepsilon^2 = 0),$$

$K^a(x,g)$ is a formal map

$$K^a(x,g) = K_0^a(x) + K_1^a(x,g) + \dots + K_r^a(x,g) + \dots,$$

where $K_m^a(x,g)$ has order m over g:

$$K_m^a(x,g) = \int K(x,y_1,\dots,y_m) g(y_1)\dots g(y_m) dy_1 \dots dy_m .$$

($K_0^a(x)$ is genuine map.)

Action associated with formal functional

Consider for every formal functional $L(x,g)$

$$L(x,g) = L_0(x) + L_1(x,g) + L_2(x,g) + \dots$$

a formal function which is equal to the value of the functional L on the linear function $g = q_a y^a$. We come to formal function

$$S_L(x,q) = L(x,g)\big|_{g=q_a y^a} = S_0(x) + S_1^a(x) q_a + S_2^{ab}(x) q_a q_b + \dots,$$

$$S_0(x) = L_0(x)\,,\ S_1^a(x) = L_1(x,y^a)\,,\ S^{ab}(x) = L_2^{\text{polarised}}(x,y^a,y^b)\,,\dots,$$

We call $S_L(x,q)$—action associated with formal functional.

Thick morphism induced by action associated with formal functional

Recall that if formal functional L is equal to pull-back of thick morphism, then action associated with formal functional L coincides with action of thick morphism.

$$\text{Let } \Phi = \Phi_S \colon M \Rrightarrow N, \text{ be a thick morphism}.$$

Then

$$L(x,g) = \Phi_S^*(g) \to S_L = S. \quad \text{(See example above)}.$$

We will prove the following theorem:

Theorem

Let $L = L(x, g)$ be a formal functional, and let $S(x, q)$ be action associated with this formal functional,

$$L(x, g) = L_0(x) + L_1(x, g) + L_2(x, g) + \cdots + L_r(x, g) + \ldots ,$$

$$S_L(x, q) = S_0(x) + S_1(x, q) + \cdots + S_r(x, q) + \ldots$$

where $S_k(x, q) = L_k(x, g)\big|_{g=q_a y^a}$.

$$\text{Then } L(x, g) = \Phi^*_{S_L}(g) .$$

in the case if $L(x, g)$ is non-linear homomorphism.

The sequence of formal functionals

Consider the sequence of formal functionals

$$L_1(x) = \Phi_1^*(g) = \Phi^*_{S_0(x)+S_1(x,q)}(g)\,,$$

$$L_2(x) = \Phi_2^*(g) = \Phi^*_{S_0(x)+S_1(x,q)+S_2(x,q)}(g)\,,$$

$$L_3(x) = \Phi_3^*(g) = \Phi^*_{S_0(x)+S_1(x,q)+S_2^{ab}(x)q_a q_b+S_3(x,q)}(g)\,,$$

and so on:

$$L_k(x) = \Phi_k^*(g) = \Phi^*_{S_0(x)+S_1(x,q)+\cdots+S_k(x,q)}(g)\,,$$

where $S_m(x,q) = S_m^{a_1\ldots a_m}(x)q_{a_1}\ldots q_{a_m}$ We will show that

$$L(x,g) = \Phi^*_\infty(g)$$

Step by step we will show that for every $m = 1, 2, 3, \ldots$ formal functional $L(x, g)$ coincides with functional $\Phi_m^*(g)$ up order m in g:

$$L(x, g) = \Phi_1^*(g) + \text{terms of order} \geq 2 \text{ in } g\,, \tag{1}$$

Then we will climb inductive ladder: Suppose we already proved

$$L(x, g) = \Phi_m^*(g) + \text{terms of order} \geq m + 1 \text{ in } g\,. \tag{2}$$

on the base of it we will prove that

$$L(x, g) = \Phi_{m+1}^*(g) + \text{terms of order} \geq m + 2 \text{ in } g\,, \tag{3}$$

Proof of basis of induction

Prove (1).
Recall that

$$L(x,g) = L_0(x) + L_1(x,g) + \cdots + L_r(x,g) + \ldots$$

$L_0(x) = S_0(x)$, $L_1(x, y^a) = S_1^a(x)$, ...
Put $g = 0$, we have

$$L(x,g)\Big|_{g=0} = L_0(x) = S_0(x)\,, \Rightarrow$$

$$L(x,g) = S_0(x) + L_1(x,g) + \text{terms of order} \geq 2 \text{ in } g\,, \qquad (1a)$$

First step (finishing)

Differentiate equation (1a),

$$L(x,g+\varepsilon h)-L(x,g)=L_1(x,g+\varepsilon h)-L_1(x,g)+\text{terms of order} \geq 2 \text{ in } g\,.$$

Since $L(x,g)$ is non-linear homomorphism

$$L(x,g+\varepsilon h)-L(x,g)=\varepsilon h\left(K_0^a(x)+K_1^a(x,g)+\cdots+K_r^a(x,g)+\dots\right)\,, \tag{2a}$$

comparing these expressions we come to

$$\varepsilon L_1(x,h)=\varepsilon h\left(K_0^a(x)\right)\Rightarrow L_1(x,g)=g\left(K_0^a(x)\right)\,.$$

Put $g=y^a$. Then $L_1(x,y^a)=S_1^a(x)=K_0^a(x)$. Hence

$$L(x,g)=S_0(x)+g\left(S_1(x)\right)+\text{terms of order} \geq 2 \text{ in } g \text{ .this proves (1)}$$

(Recall that

$\Phi_1^*(g)=\Phi^*_{S_0(x)+S_1^a(x)q_a}=S_0(x)+g(S_1(x))+$terms of order ≥ 2 in g)

Inductive step: Proof of (3) on base of (2)

Equation (2) implies that

$$L(x,g) = \Phi_m^*(g) + L_{m+1}(x,g) + \text{terms of order} \geq m+2 \text{ in } g\,, \qquad (3a)$$

where $\Phi_m = \Phi_{S_0(x)+\dots+S_m^{a_1\dots a_m}(x)q_{a_1}\dots q_{a_m}}$ Using the fact that $L(x,g)$ is non-linear homomorphism we will show that the last term $L_{m+1}(x,g)$ is $m+1$- linear and it has the following appearance:

$$L_{m+1}(x,g) = T_{m+1}^{a_1\dots a_{m+1}} g_{a_1}^*(x)\dots g_{a_{m+1}}^*(x)\,, \qquad (4)$$

where

$$g_a^*(x) = \frac{\partial}{\partial y^a}\Big|_{y^a = S_1^a(x)}.$$

This will be the central point of our proof.

It follows from (4) that

$$L(x,g) = \Phi^*_m(g) + T^{a_1\dots a_{m+1}}_{m+1} g^*_{a_1}(x)\dots g^*_{a_{m+1}}(x) + +\text{terms of order} \geq m+2 \text{ in } g$$

and

$$S^{a_1\dots a_{m+1}}_{m+1} = T^{a_1\dots a_{m+1}}_{m+1}$$

This implies (3).

It remains to prove equation (4).

Differentiate equation (3a) bearing in mind the condition (2a) on non-linear homomorphism:

$$L(x,g+\varepsilon h)-L(x,g)=\Phi^*_m(g+\varepsilon h)-\Phi^*_m(g)+$$

$$L_{m+1}(x,g+\varepsilon h)-L_{m+1}(g)+\text{terms of order}\geq m+2 \text{ in } g=$$

$$\varepsilon h\left(K^a_0(x)+\dots+K^a_r(x,g)+\dots\right),$$

where all $K_m(x,g)$ are formal maps of order m in g $(m=1,2,3,\dots)$ ($K_0(x)$ is a genuine map)

Functionals $L(x,g)$ and $\Phi^*_m(g)$ coincide up to terms of order m, hence collecting the terms of the same order on g we come to relation

$$L_{m+1}(x, g+\varepsilon h) - L_{m+1}(x, g) =$$
$$\varepsilon h\left(K_0^a(x) + K_m^a(x, g)\right) - \varepsilon h\left(K_0^a(x)\right)$$

i.e.

$$(m+1)L_{m+1}^{\text{polarised}}\left(\underbrace{g, \ldots, g}_{m \text{ times}}, \varepsilon h\right) = \frac{\partial h(y)}{\partial y^a}\Big|_{y^a = K_0^a(x)} K_m^a(x, g)$$

LHS of this expression is symmetric with respect to swapping g and h, and LHS of this expression depends linearly on first derivatives of h. This implies that it is linear:

$$(m+1)L_{m+1}^{\text{polarised}}(g_1 \ldots, g_m, h) = T^{a_1 \ldots a_m a_{m+1}} g_{a_1} \ldots g_{a_m} h_{a_{m+1}}$$

(we denote by $g_a^*(x)$, the function $g_a^*(x) = \frac{\partial g(y)}{\partial y^a}\Big|_{y^a = S_1^a(x)}$.)