# Overdetermined systems of PDEs related to representations of semi-simple Lie algebras

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# Outline

# Projective geometry of surfaces

- It is well-known that the projective geometry of surfaces is closely related to the following system of PDEs:

$$u_{xx} = A_1 u_x + B_1 u_y + C_1 u,$$
$$u_{yy} = A_2 u_x + B_2 u_y + C_2 u,$$

  where $u = u(x, y)$, $A_i, B_i, C_i$ are also functions of $x, y$, and we assume that the compatibility conditions are satisfied. Then this system possesses 4-dim solution space: each solution is parametrized by the values of $u$, $u_x$, $u_y$, $u_{xy}$ at a point.
- Let $\{u_0, u_1, u_2, u_3\}$ be an arbitrary basis in the solution space of this system. Then $(x, y) \to [u_0 : u_1 : u_2 : u_3]$ defines a surface in $P^3$ with the second fundamental form of signature $(1, 1)$, which is uniquely defined by the above system up to projective transformations.
- It is easy to see that any such surface in $P^3$ can be encoded in this way. If in addition $x, y$ are chosen as asymptotic coordinates, then we further get $A_1 = B_2 = 0$.

# Projective geometry of surfaces

- Compatibility conditions on the remaining 4 coefficients can be used to derive may interesting non-linear integrable PDEs (E.V. Ferapontov, Integrable systems in projective differential geometry).
- The trivial system $u_{xx} = u_{yy} = 0$ corresponds to the quadrics of indefinite signature. Each such quadric can be viewed as an image of the Segre embedding $P^1 \times P^1 \to P^3$:

$$([x_0 : x_1], [y_0 : y_1]) \mapsto [x_0 y_0 : x_0 y_1 : x_1 y_0 : x_1 y_1]$$

- It can also be viewed as a projectivization of the space of rank 1 matrices of size $2 \times 2$, which is the only closed orbit of the action of $PSL(2) \times PSL(2)$ on $P(\mathrm{Mat}_{2,2})$:

$$(X, Y).[A] = [XAY^{-1}].$$

## Representations of $\mathfrak{sl}_2$

- Let $\mathfrak{g}$ be a semisimple Lie algebra and $V$ its finite-dimensional irreducible representation. The nearest goal is to encode this data in the form of a (trivial) system of linear PDEs (of finite type).
- Start with $\mathfrak{g} = \mathfrak{sl}_2$ and $V = V_k$ an $(k+1)$-dimensional irreducible representation. Then the corresponding "system of PDEs" is just $u^{(k+1)} = 0$, $u = u(x)$.
- Here $V = \langle 1, x, \dots, x^k \rangle$ is the solution space of this equation. The symmetry algebra is

$$\begin{aligned} \mathfrak{sl}_2 = &\langle \partial_x, 2x\partial_x + kx\partial_u, x^2\partial_x + kxu\partial_u \rangle \\ &+ \langle u\partial_u \rangle \\ &+ \langle x^i\partial_u \mid i = 0, \dots, k \rangle. \end{aligned}$$

- In more geometric terms, we have a vector bundle $\mathcal{O}(k)$ over $P^1$. The semisimple Lie algebra $\mathfrak{g}$ naturally acts on $P^1$. This action is prolonged to $\mathcal{O}(k)$ and preserves the space of "solutions" $V$, which is identified with a subspace in the space of sections of $\mathcal{O}(k)$.

- $\mathfrak{g} = \mathfrak{sl}_2 \times \mathfrak{sl}_2$, $V = V_k \otimes V_l$:

$$u_x^{(k+1)} = 0, \quad u_y^{(l+1)} = 0$$

  where $u = u(x, y)$. Here the vector bundle is $\mathcal{O}(k) \otimes \mathcal{O}(l)$ over $P^1 \times P^1$. This can be naturally generalized to any number of copies of $P^1$.

- $\mathfrak{g} = \mathfrak{sl}_{n+1}$, $V = S^k(\mathbb{R}^{n+1})$:

$$\partial_{x_{i_1} x_{i_2} \dots x_{i_{k+1}}} u = 0, \quad \text{for all } 1 \le i_1, i_2, \dots, i_{k+1} \le n.$$

  Here $u = u(x_1, \dots, x_n)$ and the vector bundle is $k$-th symmetric power of the canonical line bundle on $P^n$.

# Other examples

- $\mathfrak{g} = \mathfrak{sl}_3$, $V$ is an adjoint representation of $\mathfrak{sl}_3$. Let $X, Y, Z$ be the basis of the Heisenberg Lie algebra on $\mathbb{R}^3$:

$$\begin{aligned} X &= \partial_x, \\ Y &= \partial_y + x\partial_z, \\ Z &= [X, Y] = \partial_z \end{aligned}$$

- The corresponding system of PDEs is:

$$X^2 u = Y^2 u = 0,$$

  where $u = u(x, y, z)$.

- It has an 8-dimensional solution space:

$$1, \quad x, \quad y, \quad z, \quad xy, \quad xz, \quad y(z - xy), \quad z(z - xy).$$

## General construction

- Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra, $\mathfrak{g}_+ \subset \mathfrak{g}$ ($\mathfrak{g}_- \subset \mathfrak{g}$) the subalgebra of "strictly upper (lower) triangular matrices" in $\mathfrak{g}$. More rigorously, $\mathfrak{g}_+$ is a nilradical of a Borel subalgebra in $\mathfrak{g}$, so that $\mathfrak{g}$ is decomposed as

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+.$$

- Realize $\mathfrak{g}_-$ as a Lie algebra of left-invariant vector fields on the corresponding Lie group $G_-$ and denote by $X_i = X_{-\alpha_i}$ vector fields that correspond to the basis $\{\alpha_1, \dots, \alpha_l\}$ in the root system of $\mathfrak{g}$.
- Let $\{\lambda_1, \dots, \lambda_l\}$ be the corresponding system of fundamental weights and $\lambda = k_1\lambda_1 + \dots + k_l\lambda_l$ the highest root of the representation $V = V_\lambda$. Then the corresponding system of PDEs is:

$$X_i^{k_i+1} u = 0, \quad i = 1, \dots, l.$$

Here $u \in C^\infty(G_-)$.

## More examples

- In case of $\mathfrak{g} = \mathfrak{sp}(4)$ we have:

$$\begin{aligned} X_1 &= \partial_t, \\ X_2 &= \partial x + z\partial_y + t\partial_z \end{aligned}$$

spanning the contact distribution on $J^2(\mathbb{R}, \mathbb{R})$. Systems

$$X_1^{k_1+1} u = X_2^{k_2+1} u = 0 \quad \text{for } u = u(x, y, z, t)$$

have solution spaces that correspond to irreducible representations of $\mathfrak{g}$.

# Rational homogeneous varieties

- Let $\{u_0, \dots, u_k\}$ be a basis of the solution space for one of the above systems. Here $k+1$ is the dimension of the solution space.
- Then $[u_0 : u_1 : \cdots : u_k]$ is a projective variety in $P^k$ defined modulo projective transformations.
- It is called a *rational homogeneous variety* and coincides with the orbit of the Lie group $G$ on the highest weight of the representation $V$. It is the only closed orbit of $G$ acting on $P(V)$.
- It is an isomorphic to a parabolic homogeneous space $G/P$. Here $P$ is a parabolic subgroup in $G$.

# Classical examples

- Rational normal curve $[1 : x : x^2 : \cdots : x^k] \subset P^k$ corresponds to the trivial ODE $u^{(k+1)} = 0$. It is an embedding of $P^1$ to $P^k$.
- Segre variety $[x_i y_k]$ corresponds to the system $u_{x_i x_j} = 0$, $u_{y_k y_l} = 0$. It is an embedding of $P^r \times P^s \to P^{rs+r+s}$.
- Veronese variety $[x_i x_j]$ corresponds to the system $u_{x_i x_j x_k} = 0$. It is an embedding of $P^r \to P^{r(r+3)/2}$.
- Adjoint variety of $\mathfrak{sl}_3$ consists of all rank 1 nilpotent 3x3 matrices. It corresponds to the system $X^2 u = Y^2 = 0$, where $X = \partial_x$ and $Y = \partial_y + x\partial_z$. It is isomorphic to $SL(3)/ST(3) = \mathrm{Flag}(\mathbb{R}^3)$.

# Filtered manifolds

- Let $\Delta_\alpha u = 0$ be the linear system of PDEs corresponding to the representation $V$ of a semisimple Lie algebra $\mathfrak{g}$. We can assume that all operators $\Delta_\alpha$ have only terms of highest order, if we use the language of *filtered manifolds*.
- Independent variables for this system are coordinates on the parabolic homogeneous space $M = G/P$, which is a *filtered manifold*. It is equipped with a bracket-generating vector distribution $D \subset TM$, whose brackets defined the filtration on $M$:

$$0 \subset T^{-1}M = D \subset T^{-2}M = D + [D, D] \subset \cdots \subset T^{-\mu}M = TM.$$

  The length of this filtration is called the *depth* of a filtered manifold $M$.

## Weighted order of differential operators

- We consider vector fields $X \subset T^{-i}M$ as linear operators of weighted degree $i$. This can be encoded by assigning weights to variables on $M$.
- For example, $M = J^1(\mathbb{R}, \mathbb{R})$ is a filtered manifold of depth 2, where $D$ is the contact structure on $J^1$. Choosing the coordinates $(x, y, y' = z)$, we have the contact form $\omega = dy - z\,dx$ and vector fields

$$X = \partial_z,$$
$$Y = \partial_x + z\partial_y,$$

spanning the contact distribution. They are first order operators. The bracket $[X, Y] = \partial_y$ is already the second order operator. This is equivalent to say that variables $x, z$ have degree $-1$ and $y$ has degree $-2$.

# Compatibility of systems with a given symbol

- Consider now a *deformed linear homogeneous system* with the same symbol:

$$\Delta_\alpha u = L_\alpha(u)$$

, where $\deg L_\alpha < \deg \Delta_\alpha$. We say that it is compatible if its solution space has the same dimension $= \dim V$ as the initial system.

- For example, the deformed system for the adjoint variety of $\mathfrak{sl}_3$ has the form:

$$\begin{aligned} X^2 u &= A_1 X u + B_1 Y u + C_1 u, \\ Y^2 u &= A_2 X u + B_2 Y u + C_2 u, \end{aligned}$$

where $u = u(x, y, z)$ and $A_i$, $B_i$, $C_i$ are also functions of $x, y, z$.

- Since $X^2 Y^2 \neq Y^2 X^2$, checking explicitly the compatibility of this system requires differentiating it along $X$, $Y$ four times!

## Rigid vs non-rigid symbols

- We say that the deformation is *trivial* if it is equivalent to the symbol via some (local) transformation of the vector bundle:

$$(x, u) \mapsto (\lambda(x), \mu(x)u).$$

- **Main question**. When a system defined by a $\mathfrak{g}$-representation $V$ admits non-trivial deformations? We call such systems *non-rigid*.
- Example: the system $X^2 u = Y^2 u = 0$ does admit a non-trivial deformation:

$$X^2 u = \epsilon Y u,$$
$$Y^2 u = 0.$$

for any $\epsilon \neq 0$. It is easy to check that this system still has 8-dimensional solution space, but smaller symmetry algebra than the original system with $\epsilon = 0$.

## Deformations of the rational homogeneous varieties

- Let $\{u_0, u_1 \cdots, u_k\}$ be the basis of the solution space of the non-trivial deformation. Then $[u_0 : u_1 : \cdots : u_k]$ is again a well-defined submanifold in $P^k$, which defines a so-called *non-trivial deformation of a rational homogeneous variety* that has the same 2nd, 3rd, ... fundamental forms as the initial variety.
- For example, solving the above system we get an explicit deformation of the adjoint variety of $\mathfrak{sl}_3$:

$$1, x, y+\epsilon x^2/2, z+\epsilon x^3/6, z-xy, xz+\epsilon x^4/12,$$
$$y(z-xy)+\epsilon x^2(z/2-xy/6)+\epsilon^2 x^5/60,$$
$$z(z-xy)+\epsilon x^3(z/6-xy/12)+\epsilon^2 x^6/360.$$

- Such deformations are actively studied in algebraic geometry (Griffiths–Harris, Landsberg–Robles, Hwang–Yamaguchi) via the methods of local differential geometry.

- **Theorem (Landsberg–Robles, D.–Machida–Morimoto).** Invariants of deformations of rational homogeneous varieties are governed the following Lie algebra cohomology:

$$H^1_+(\mathfrak{g}_-, \mathfrak{sl}(V)/\mathfrak{g}),$$

  which can be effectively computed via Kostant's theorem. The online tool is available at `http://web.math.muni.cz/~silhan/lie/`.
- If $\mathfrak{g}^{\mathbb{C}}$ is simple, then such cohomology is non-zero only if $G/P$ is one of
  1. $P^\ell$ (under Veronese embeddings);
  2. $Q^\ell$ (quadratic hypersurface in $P^{\ell+1}$);
  3. $\mathrm{Flag}_{1,\ell}(\mathbb{C}^{\ell+1})$ (the adjoint variety of $\mathfrak{sl}_{\ell+1}$)

  or their reembeddings.

## Corresponding systems of PDEs

- $P^\ell$ (Veronese embeddings)

$$\partial_{x_{i_1} x_{i_2} \dots x_{i_{k+1}}} u = 0, \quad \text{for all } i_1, i_2, \dots, i_{k+1} = 1, \dots, \ell.$$

- $Q^\ell$ (quadratic hypersurface in $P^{\ell+1}$)

$$\partial_{x_i x_j} u = 0, \quad 1 \le i < j \le \ell,$$
$$\partial_{x_1 x_1} u = \partial_{x_2 x_2} u = \dots = \partial_{x_l x_l} u$$

  for quadratic surfaces of positive signature or similar for indefinite signature.

- $\mathrm{Flag}_{1,\ell}(\mathbb{C}^{\ell+1})$ (the adjoint variety of $\mathfrak{sl}_{\ell+1}$)

$$X_i X_j u = 0, \quad Y_i Y_j u = 0, \qquad 1 \le i \le j \le \ell,$$

  where

$$X_i = \partial_{x_i},$$
$$Y_i = \partial_{y_i} + x_i \partial_z.$$

## Discussion and open questions

- The current talk is based on the paper D.–Machida–Morimoto, Extrinsic geometry and linear differential equations (SIGMA, 2021). This paper also treats systems of PDEs with an arbitrary symbol. However, the computation of the corresponding cohomology space works best for the semisimple case.
- Systematic analysis of potential integrable equations coming from the compatibility conditions for the above cases is open.
- A particularly interesting case in the adjoint orbit of $\mathfrak{sl}_3$, as it seems to be a non-holonomic analog of the projective geometry of surfaces with a lot of similar geometric constructions.