Bi-Hamiltonian systems and projective geometry

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Contents

Provide a correspondence (and then a classification) between

- bi-Hamiltonian systems whose bi-Hamiltonian pair has the structure of a 'trio' of compatible operators:

  $$A_1 = P_1 + R_i, \quad i = 2, 3; \qquad A_2 = Q_1$$

  (i: order of the operator) and
- triples of algebraic varieties in the space of (projective) lines, *i.e.* the Plücker embedding,

  $$\mathrm{Gr}(2, \mathbb{R}^{n+1}) \hookrightarrow \mathbb{P}(\wedge^2 \mathbb{R}^{n+1})$$

  of the projective space with coordinates $[u^0, \dots, u^n]$ where $(u^1, \dots, u^n)$ are the field variables.

Hamiltonian PDEs

An evolutionary system of PDEs

$$F = u^i_t - f^i(t, x, u^j, u^j_x, u^j_{xx}, \dots) = 0$$

admits a Hamiltonian formulation if there exist A, $\mathcal{H} = \int h\,dx$ such that

$$u^i_t = A^{ij}\left(\frac{\delta \mathcal{H}}{\delta u^j}\right)$$

where $A = (A^{ij})$ is a Hamiltonian operator, i.e. a matrix of differential operators $A^{ij} = A^{ij\sigma}\partial_\sigma$, where $\partial_\sigma = \partial_x \circ \cdots \circ \partial_x$ (total x-derivatives σ times), such that

$$\{F, G\}_A = \int \frac{\delta F}{\delta u^i} A^{ij\sigma}\partial_\sigma \frac{\delta G}{\delta u^j}\,dx$$

is a Poisson bracket (skew-symmetric and Jacobi).

Bi-Hamiltonian systems

Bi-Hamiltonian systems have two Hamiltonian formulations by two compatible Hamiltonian operators A_1, A_2, where:

- skew-symmetry of the Poisson bracket is the skew-adjointness of A_1, A_2;
- the Jacobi property of the Poisson bracket is the vanishing of the Schouten bracket, $[A_1, A_1] = 0$, $[A_2, A_2] = 0$;
- the operators are required to be compatible: $[A_1, A_2] = 0$.

Bi-Hamiltonian systems are considered to be integrable (F. Magri, 1978).

First-order homogeneous operators

First-order homogeneous operators were introduced in 1983 by Dubrovin and Novikov:

$$\boxed{A_1^{ij} = g^{ij}(\mathbf{u})\partial_x + b_k^{ij}(\mathbf{u})u_x^k}$$

They are form-invariant with respect to point transformations of the type:

$$\bar{u}^i = U^i(u^j).$$

where $u^i = u^i(t,x)$, $i,j = 1,\dots,n$ (n-components).

Homogeneity: $\deg \partial_x = 1$.

Canonical form: $A_1^{ij} = \eta^{ij}\partial_x$.

Higher-order homogeneous operators

Higher order homogeneous operators were introduced in 1984 by Dubrovin and Novikov. We consider here second-order and third-order homogeneous operators:

$$
\begin{aligned}
A_2^{ij} =& g_2^{ij}(\mathbf{u})\partial_x^2 + b_{2\,k}^{ij}(\mathbf{u})u_x^k\partial_x \\
&+ c_{2\,k}^{ij}(\mathbf{u})u_{xx}^k + c_{2\,km}^{ij}(\mathbf{u})u_x^k u_x^m,
\end{aligned}
$$

$$
\begin{aligned}
A_3^{ij} =& g_3^{ij}(\mathbf{u})\partial_x^3 + b_{3\,k}^{ij}(\mathbf{u})u_x^k\partial_x^2 \\
&+ [c_{3\,k}^{ij}(\mathbf{u})u_{xx}^k + c_{3\,km}^{ij}(\mathbf{u})u_x^k u_x^m]\partial_x \\
&+ d_{3\,k}^{ij}(\mathbf{u})u_{xxx}^k + d_{3\,km}^{ij}(\mathbf{u})u_x^k u_{xx}^m + d_{3\,kmn}^{ij}(\mathbf{u})u_x^k u_x^m u_x^n.
\end{aligned}
$$

Bi-Hamiltonian systems of KdV-type

Many bi-Hamiltonian systems are indeed compatible triples of Hamiltonian operators P_1, Q_1, R_2 introduced by Olver and Rosenau (1996):

$$A_1 = P_1, \qquad A_2 = Q_1 + R_i \quad i = 2, 3, \qquad \text{where}$$

$$[R_i, P_1] = 0, \quad [R_i, Q_1] = 0, \quad [P_1, Q_1] = 0.$$

Examples:

- with second-order operators R_2: AKNS, 2-component Camassa-Holm, Kaup–Broer (Kuperschmidt 1984), etc..
- with third-order operators R_3: KdV, Camassa–Holm, dispersive water waves (Antonowicz–Fordy 1989), coupled Harry–Dym, etc..

Examples and classification

A classification of bi-Hamiltonian hierarchies which are defined by a triple of mutually compatible Hamiltonian operators was provided by Lorenzoni, Savoldi, V. (JPA 2017).

Examples: scalar case. We have one third-order operator R_3, two first order operators P_1, Q_1:

$$[R_3, P_1] = [R_3, Q_1] = [P_1, Q_1] = 0$$
$$P_1 = \partial_x, \qquad Q_1 = 2u\partial_x + u_x, \quad R_3 = \partial_x^3.$$

KdV hierarchy (Magri (1978)):

$$\Pi_\lambda = Q_1 + \epsilon^2 R_3 - \lambda P_1 = 2u\partial_x + u_x - \lambda\partial_x + \epsilon^2\partial_x^3$$

Camassa–Holm hierarchy:

$$\tilde{\Pi}_\lambda = Q_1 - \lambda(P_1 + \epsilon^2 R_3) = 2u\partial_x + u_x - \lambda(\partial_x + \epsilon^2\partial_x^3).$$

Example: 2-component case. We have one second-order operator R_2 and two first-order operators P_1, Q_1, all of them mutually compatible:

$$P_1 = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}, \qquad Q_1 = \begin{pmatrix} 2u\partial_x + u_x & v\partial_x \\ \partial_x v & -2\partial_x \end{pmatrix},$$
$$R_2 = \begin{pmatrix} 0 & -\partial_x^2 \\ \partial_x^2 & 0 \end{pmatrix}$$

- $\Pi_\lambda = Q_1 + \epsilon^2 R_2 - \lambda P_1$ AKNS (or two-boson) hierarchy;
- $\tilde{\Pi}_\lambda = Q_1 - \lambda(P_1 + \epsilon^2 R_2)$ two-component Camassa-Holm hierarchy.

Canonical forms of homogeneous Hamiltonian operators

In the non-degenerate case ($\det(g^{ij}) \neq 0$) the second and third order operators admit canonical forms by means of a point transformation (Potemin '86, '97; Potemin–Balandin, '01; Doyle '95)

$$R_2^{ij} = \partial_x \circ g_2^{ij} \circ \partial_x,$$

$$R_3^{ij} = \partial_x \circ (g_3^{ij} \partial_x + c_{3\,k}^{ij} u_x^k) \circ \partial_x,$$

Projective invariance of compatible triples

Consider a reciprocal transformations of projective type:

$$d\tilde{x} = \Delta dx, \quad \tilde{u}^i = S^i(u^j) = (S^i_j u^j + S^i_0)/\Delta$$

where $\Delta = S^0_j u^j + S^0_0$. Then,

- R_2 and R_3 transform into new second-order and third-order homogenous Hamiltonian operators in canonical

$$R_2^{ij} = \partial_x g_2^{ij} \partial_x, \quad R_3^{ij} = \partial_x \circ (g_3^{ij} \partial_x + c_3^{ij}{}_k u^k_x) \circ \partial_x;$$

- P_1 (or Q_1) transform into new non-local first order homogeneous Hamiltonian operators (Ferapontov 1991):

$$P_1 = g^{ij} \partial_x + \Gamma^{ij}_k u^k_x + u^i_x \partial_x^{-1} w^j_k u^k_x + w^i_h u^h_x \partial_x^{-1} u^j_x$$

The problem

Problem: projective classification and geometric significance of triples! Initiated in Lorenzoni, Savoldi, V. JPA 2017, here we discuss results from Lorenzoni, V. (2023)
https://arxiv.org/abs/2311.13932
Here we classify triples

$$A_1 = P_1 + R_2, \qquad A_2 = Q_1,$$

where

- R_2 is a constant coefficient second-order operator:

$$R_2 = \eta^{ij}\partial_x^2, \quad \text{where} \quad \eta^{ij} = -\eta^{ji}, \quad \det(\eta^{ij}) \neq 0;$$

- P_1, Q_1 are Ferapontov operators of localizable type:

$$P_1 = g^{ij}\partial_x + \Gamma^{ij}_k u^k_x + w^i_k u^k_x \partial_x^{-1} u^j_x + u^i_x \partial_x^{-1} w^j_k u^k_x$$

Digression: Plücker's line geometry

Two points $U, V \in \mathbb{P}(\mathbb{C}^{n+1})$,

$$U = [u^1, \ldots, u^{n+1}], \qquad V = [v^1, \ldots, v^{n+1}]$$

define a line with coordinates $p^{\lambda\mu} = \det \begin{vmatrix} u^\lambda & u^\mu \\ v^\lambda & v^\mu \end{vmatrix}$ inside the projective space: $\mathbb{P}(\wedge^2 \mathbb{C}^{n+1})$ (Plücker's embedding).

Any 3-form $\omega \in \wedge^3 \mathbb{C}^{n+1*}$ defines the following system of linear equations in Plücker's space:

$$i_L(\omega) = 0, \qquad L \in \wedge^2 \mathbb{C}^{n+1};$$

in coordinates, $L = p^{\lambda\mu} \partial_\lambda \wedge \partial_\mu$ and the system is: $\omega_{\lambda\mu\nu} p^{\mu\nu} = 0$.

The algebraic variety of R_2

(Vergallo, V., 2022) The second-order operator R_2 yields the three-form

$$\omega_2 = \eta_{ij} du^0 \wedge du^i \wedge du^j, \qquad \eta_{ij} = (\eta^{ij})^{-1}.$$

Intersecting the corresponding linear system with the Grassmannian

$$\mathbb{G}(2, \mathbb{C}^{n+1}) \subset \mathbb{P}(\wedge^2 \mathbb{C}^{n+1})$$

we obtain, in the generic case, a linear line congruence, an algebraic variety of dimension $n-1$:

$$X_{\omega_2} = \mathbb{G}(2, \mathbb{C}^{n+1}) \cap \{i_L \omega_2 = 0\}.$$

It is remarkable that they are Fano varieties (of index 3).

Compatibility: $[P_1, R_2] = 0$.

(Lorenzoni, V. 2023) The compatibility of the Hamiltonian operators: $[P_1, R_2] = 0$ is equivalent to the conditions:

$$w^i_j \quad \text{are constant;} \tag{1}$$

$$w^i_l \eta^{lk} + w^k_l \eta^{li} = 0; \tag{2}$$

$$\Gamma^{ij}_l \eta^{lk} + \Gamma^{kj}_l \eta^{li} = 0; \tag{3}$$

$$\Gamma^{ki}_l \eta^{lj} + \Gamma^{ij}_l \eta^{lk} + \Gamma^{jk}_l \eta^{li} = 0; \tag{4}$$

$$\Gamma^{sj}_p \Gamma^{ir}_s - \Gamma^{sr}_p \Gamma^{ij}_s = 0; \tag{5}$$

$$\frac{\partial \Gamma^{kj}_l}{\partial u^s} = -\delta^j_s w^k_l - w^j_s \delta^k_l. \tag{6}$$

Consequences of compatiblity

- Condition (5) implies that Γ^{ij}_k define a Frobenius algebra structure on the tangent space of the field variables;
- Condition (4) implies that η_{ij} and Γ^{ij}_k define a cyclic Frobenius algebra (Buchstaber, Mikhailov 2023). Note that (3) is invariance of the 2- form η with respect to the Frobenius structure.
- Set $\bar{g}_{ab} = \eta_{jb}\eta_{ia}g^{ij}$. Condition (4) is also equivalent to

$$\bar{g}_{bc,a} + \bar{g}_{ca,b} + \bar{g}_{ab,c} = 0,$$

hence $\bar{g}_{ab}$ is the Monge form of a quadratic line complex.

Example: Kaup–Broer system

Kupershmidt '85. The trio is defined by

$$P_1 = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} 2\partial_x & \partial_x u^1 \\ u^1\partial_x & u^2\partial_x + \partial_x u^2 \end{pmatrix}, \tag{7}$$

$$R_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \partial_x^2. \tag{8}$$

The corresponding Monge metrics are

$$(\bar{g}_{1,ab}) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad (\bar{g}_{2,ab}) = \begin{pmatrix} 2u^2 & -u^1 \\ -u^1 & 2 \end{pmatrix}. \tag{9}$$

Monge metrics in detail

Lie's form of Plücker's coordinates:

$$u^1du^2 - u^2du^1, \qquad du^1, \qquad du^2$$

Monge metrics are quadratic forms in the above coordinates. In particular,

$$Q(\bar{g}_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad Q(\bar{g}_2) = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \tag{10}$$

Note that $\mathrm{rk}(Q(\bar{g}_1)) = 2$ and $\mathrm{rk}(Q(\bar{g}_2)) = 3$. We have, for example,

$$\bar{g}_{2,ab}du^a du^b = -2(u^1du^2 - u^2du^1)du^1 + 2du^2du^2,$$

Classification: $n = 2$

The compatibility conditions $[P_1, R_2] = 0$ can be completely solved. The Monge metric of P_1:

$$\bar{g}_{11} = c_0(u^2)^2 + c_3 u^2 + c_4,$$
$$\bar{g}_{12} = -c_0 u^1 u^2 - \frac{1}{2}c_3 u^1 - \frac{1}{2}c_1 u^2 + c_5,$$
$$\bar{g}_{22} = c_0(u^1)^2 + c_1 u^1 + c_2$$

The metric of P_1:

$$g^{11} = c_0(u^1)^2 + c_1 u^1 + c_2,$$
$$g^{12} = c_0 u^1 u^2 + \frac{1}{2}c_3 u^1 + \frac{1}{2}c_1 u^2 + c_5$$
$$g^{22} = c_0(u^2)^2 + c_3 u^2 + c_4.$$

The above metric is linear for every value of the parameters: every two metrics in that space yield compatible operators!

Classification $n = 2$, the form of P_1

It turns out that the leading coefficient matrix (g^{ij}) of P_1 is completely determined by a generic Monge metric. Note that, in Plücker's space:

$$Q(\bar{g}_{ij}) = \begin{pmatrix} c_0 & -\frac{1}{2}c_3 & \frac{1}{2}c_1 \\ -\frac{1}{2}c_3 & c_4 & c_5 \\ \frac{1}{2}c_1 & c_5 & c_2 \end{pmatrix}$$

P_1 takes the form:

$$P_1^{ij} = g^{ij}\partial_x + \Gamma^{ij}_k u^k_x - c_0 u^i_x \partial_x^{-1} u^j_x;$$

note that if $c_0 = 0$ then we recover results previously obtained (Lorenzoni, Savoldi, V. JPA 2018).

Classification $n = 2$, fixing P_1

- Fix P_1 in the previous class. There are two natural choices, a quadratic line complex of rank 2 and rank 3.
- R_2 is stabilized up to a multiplicative constant;
- an arbitrary Q_1 from the previous class can be added, no extra compatibility conditions required.

Note: when $n = 2$ compatibility reduces to P_1 being determined by an arbitrary Monge metric.

Projective correspondence theorem, $n = 2$

Theorem. If $n = 2$, then there is a bijective correspondence between

- trios of mutually compatible localizable first-order homogeneous Hamiltonian operators P_1, Q_1 and $R_2 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\partial_x^2$, and
- pairs of conics $\mathcal{C}_1$, $\mathcal{C}_2$ of rank at least 2.

A classification is then achieved by considering the action of $SL(3, \mathbb{C})$ on the projective space $\mathbb{P}(\mathbb{C}^3)$, and the canonical forms of pairs of conics (Weierstrass 1858, 1868).

Classification: $n = 4$

Fix a second-order operator, for example

$$R_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \partial_x^2.$$

- We have a complete list of solutions of $[P_1, R_2] = 0$, with 288 cases (including the degenerate cases).
- There are local and localizable non-local first-order operators P_1 of Ferapontov type.
- Compatibility conditions do not reduce to (g^{ij}) being determined by a Monge metric.

Classification, $n = 4$: examples

It is not so meaningful to write down the list of solutions of $[P_1, R_2] = 0$. Moreover, to construct a trio, we need to solve the further equation $[P_1, Q_2] = 0$. Let us give an example, where

$$P_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \partial_x$$

and then find, in the set of solutions of $[R_2, Q_1] = 0$, those that are compatible with P_1: $[P_1, Q_1] = 0$. We found 64 cases of both local and localizable non-local first-order operators P_1 of Ferapontov type.

Classification, $n = 4$, a local example

Note that we use the formula $\Gamma^{kj}_l = -w^k_l u^j - w^j_s u^s \delta^k_l + b^{kj}_l$.

$$(g^{ij}) = \begin{pmatrix} 2b^{11}_2 u^2 + c_{55} & c_{54} & b^{11}_2 u^4 + b^{13}_1 u^1 - c_{49} & b^{13}_1 u^2 - c_{34} \\ c_{54} & 0 & b^{13}_1 u^2 - c_{34} & 0 \\ b^{11}_2 u^4 + b^{13}_1 u^1 - c_{49} & b^{13}_1 u^2 - c_{34} & 2b^{13}_1 u^3 + c_{46} & 2b^{13}_1 u^4 + c_{31} \\ b^{13}_1 u^2 - c_{34} & 0 & 2b^{13}_1 u^4 + c_{31} & 0 \end{pmatrix}$$

The free parameters are $b^{11}_2, b^{13}_1, c_{31}, c_{34}, c_{46}, c_{49}, c_{54}, c_{55}$.
Nonzero coefficients in the Christoffel symbols are determined by the only nonzero constants b^{ij}_k:

$$\Gamma^{11}_2 = b^{11}_2, \quad \Gamma^{13}_1 = b^{13}_1, \quad \Gamma^{14}_2 = b^{13}_1, \quad \Gamma^{23}_2 = b^{13}_1,$$
$$\Gamma^{31}_4 = b^{11}_2, \quad \Gamma^{33}_3 = b^{13}_1, \quad \Gamma^{34}_4 = b^{13}_1, \quad \Gamma^{43}_4 = b^{13}_1.$$

Local cases seem to have already been discovered by Strachan and Szablikowski (SAM 2014).

Classification, $n = 4$, a nonlocal example – 1

$$
(g^{ij}) = \begin{pmatrix}
0 & c_{54} - (u^1)^2 w^2_1 & 0 & -(c_{34} + u^1 u^3 w^2_1) \\
c_{54} - (u^1)^2 w^2_1 & 2b^{22}_1 u^1 + c_{53} - 2u^1 u^2 w^2_1 & -(c_{34} + u^1 u^3 w^2_1) & b^{22}_1 u^3 - c_{33} - u^1 u^4 w^2_1 - u^2 u^3 w^2_1 \\
0 & -(c_{34} + u^1 u^3 w^2_1) & 0 & c_{31} - (u^3)^2 w^2_1 \\
-(c_{34} + u^1 u^3 w^2_1) & b^{22}_1 u^3 - c_{33} - u^1 u^4 w^2_1 - u^2 u^3 w^2_1 & c_{31} - (u^3)^2 w^2_1 & c_{28} - 2u^3 u^4 w^2_1
\end{pmatrix}
$$

The nonlocal part is defined by the free parameter w^2_1 (with the requirement $w^2_1 \neq 0$) and the equations

$$
w^4_3 = w^2_1, \qquad w^i_j = 0 \quad \text{otherwise.} \tag{11}
$$

Classification, $n = 4$, a nonlocal example – 2

The free parameters are

$$b^{22}_1, w_{21}, c_{28}, c_{31}, c_{33}, c_{34}, c_{53}, c_{54} \tag{12}$$

The only nonzero constants b^{ij}_k are

$$b^{22}_1, \quad b^{42}_3 = b^{22}_1. \tag{13}$$

The nonzero Christoffel symbols are

$$\begin{array}{llll}
\Gamma^{12}_1 = -u^1 w^2_1, & \Gamma^{14}_1 = -u^3 w^2_1, & \Gamma^{21}_1 = -u^1 w^2_1, & \Gamma^{22}_1 = b^{22}_1 - u^2 w^2_1, \\
\Gamma^{22}_2 = -u^1 w^2_1, & \Gamma^{23}_1 = -u^3 w^2_1 & \Gamma^{24}_1 = -u^4 w^2_1, & \Gamma^{24}_2 = -u^3 w^2_1, \\
\Gamma^{32}_3 = -u^1 w^2_1, & \Gamma^{34}_3 = -u^3 w^2_1, & \Gamma^{41}_3 = -u^1 w^2_1, & \Gamma^{42}_3 = b^{22}_1 - u^2 w^2_1 \\
\Gamma^{42}_4 = -u^1 w^2_1, & \Gamma^{43}_3 = -u^3 w^2_1, & \Gamma^{44}_3 = -u^4 w^2_1, & \Gamma^{44}_4 = -u^3 w^2_1.
\end{array}$$

Projective correspondence theorem, any n

There is a bijective correspondence between:

- bi-Hamiltonian trios of Hamiltonian operators as discussed:

$$A_1 = P_1, \qquad A_2 = Q_1 + R_2$$

- trios of two quadratic line complexes $\mathcal{P}_1$, $\mathcal{Q}_1$ and one linear line congruence $\mathcal{R}_2$ induced by a constant 3-form; compatibility constrains the varieties in a way that is yet to be understood.

Compatibility $[P_1, R_3] = 0$.

Some results are available in the case $[P_1, R_3] = 0$ (Lorenzoni, V. Cont. Math. AMS 2024).

- The Christoffel symbols define a commutative Frobenius algebra (without unity) on the tangent space of the field variables.
- The operator P_1 turn out to be a local one if $n \geqslant 3$.
- If $n \geqslant 3$, then any commutative Frobenius algebra determines a solution P_1 of $[P_1, R_3] = 0$.

Note: in the local case item 1 was independently proved by Bolsinov, Konyaev, Matveev.

Conclusions

Implications of the projective geometric interpretation of bi-Hamiltonian systems of KdV-type are unknown (at the moment), but the geometry is nice.

Thank you!

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