Normal forms of irreducible $\mathfrak{sl}_n$-valued zero curvature representations

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1 Normal forms of $\mathfrak{sl}_2$-valued ZCR

Normal form for C	Normal form for A
$J_1 = \begin{pmatrix} r & 0 \\ 0 & -r \end{pmatrix}$	$N_1 = \begin{pmatrix} . & 1 \\ . & . \end{pmatrix}$
$J_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	$N_2 = \begin{pmatrix} 0 & . \\ . & 0 \end{pmatrix}$

Table 1.

2 Normal forms of $\mathfrak{sl}_3$-valued ZCR

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$$P_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$P_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad P_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad P_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

$$J_1 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & -\lambda_1 - \lambda_2 \end{pmatrix}; \quad \lambda_1 \neq \lambda_2, \qquad W_1 = \begin{pmatrix} w_1 & 0 & 0 \\ 0 & w_2 & 0 \\ 0 & 0 & (w_1 w_2)^{-1} \end{pmatrix},$$

$$J_2 = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & -2\lambda \end{pmatrix}; \quad \lambda \neq 0, \qquad W_2 = \begin{pmatrix} w_{11} & w_{12} & 0 \\ w_{21} & w_{22} & 0 \\ 0 & 0 & w_{33}^{-1} \end{pmatrix},$$

$$J_3 = \begin{pmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 0 & -2\lambda \end{pmatrix}; \quad \lambda \neq 0, \qquad W_3 = \begin{pmatrix} w_1 & 0 & 0 \\ w_2 & w_1 & 0 \\ 0 & 0 & w_1^{-2} \end{pmatrix},$$

$$J_4 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad W_4 = \begin{pmatrix} w_1 & 0 & 0 \\ w_2 & w_1 & w_3 \\ w_4 & 0 & w_1^{-2} \end{pmatrix},$$

$$J_5 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad W_5 = \begin{pmatrix} 1 & 0 & 0 \\ w_2 & 1 & 0 \\ w_3 & w_2 & 1 \end{pmatrix},$$

where $w_{33} = w_{11}w_{22} - w_{12}w_{21}$.

Table 1: Jordan normal forms and the corresponding stabilizers

Normal form for C	Normal form for A
$J_1 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & -\lambda_1 - \lambda_2 \end{pmatrix}$ $\lambda_1 \neq \lambda_2$	$N_1^1 = \begin{pmatrix} . & 1 & . \\ . & . & 1 \\ . & . & . \end{pmatrix}$
$J_2 = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & -2\lambda \end{pmatrix}$ $\lambda \neq 0$	$N_2^1 = \begin{pmatrix} 0 & 1 & 0 \\ . & . & 1 \\ . & . & . \end{pmatrix}$
$J_3 = \begin{pmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 0 & -2\lambda \end{pmatrix}$ $\lambda \neq 0$	$N_3^1 = \begin{pmatrix} . & . & 1 \\ . & . & 0 \\ . & . & . \end{pmatrix}, \quad N_3^2 = \begin{pmatrix} 0 & . & 0 \\ . & . & 1 \\ . & 0 & . \end{pmatrix}$
$J_4 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$N_4^1 = \begin{pmatrix} 0 & . & 0 \\ . & . & 1 \\ . & 0 & . \end{pmatrix}, \quad N_4^2 = \begin{pmatrix} 0 & 0 & 1 \\ . & 0 & 0 \\ . & . & 0 \end{pmatrix}$
$J_5 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$N_5^1 = \begin{pmatrix} 0 & 0 & . \\ . & . & . \\ . & . & . \end{pmatrix}, \quad N_5^2 = \begin{pmatrix} 0 & . & 0 \\ . & . & . \\ . & 0 & . \end{pmatrix}$

Table 2.

3 Normal forms of $\mathfrak{sl}_4$-valued ZCR

3.1 Subalgebras of $\mathfrak{sl}_4$

$$\begin{pmatrix} . & 0 & 0 & 0 \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \end{pmatrix} \text{ or } \begin{pmatrix} . & . & . & . \\ 0 & . & . & . \\ 0 & . & . & . \\ 0 & . & . & . \end{pmatrix} \text{ or } \begin{pmatrix} . & . & 0 & 0 \\ . & . & 0 & 0 \\ . & . & . & . \\ . & . & . & . \end{pmatrix} \text{ or } \begin{pmatrix} . & . & . & . \\ . & . & . & . \\ 0 & 0 & . & . \\ 0 & 0 & . & . \end{pmatrix}$$

+ permutations

3.2 Table of Jordan normal forms and corresponding stabilizers

Jordan normal form of C	Stabilizer W
$J_{11}=\begin{pmatrix}\lambda_1&0&0&0\\0&\lambda_2&0&0\\0&0&\lambda_3&0\\0&0&0&\lambda_4\end{pmatrix}$ $\lambda_1\neq\lambda_2\neq\lambda_3,\ \lambda_4=-\lambda_1-\lambda_2-\lambda_3$	$W_{11}=\begin{pmatrix}w_{11}&0&0&0\\0&w_{22}&0&0\\0&0&w_{33}&0\\0&0&0&w_{44}\end{pmatrix}$ $w_{44}=1/w_{11}w_{22}w_{33}$
$J_{12}=\begin{pmatrix}\lambda_1&0&0&0\\0&\lambda_1&0&0\\0&0&\lambda_2&0\\0&0&0&\lambda_3\end{pmatrix}$ $\lambda_1\neq\lambda_2,\quad \lambda_3=-2\lambda_1-\lambda_2$	$W_{12}=\begin{pmatrix}w_{11}&w_{12}&0&0\\w_{21}&w_{22}&0&0\\0&0&w_{33}&0\\0&0&0&w_{44}\end{pmatrix}$ $w_{44}=1/(w_{11}w_{22}-w_{12}w_{21})w_{33}$
$J_{13}=\begin{pmatrix}\lambda&0&0&0\\0&\lambda&0&0\\0&0&\lambda&0\\0&0&0&-3\lambda\end{pmatrix}$ $\lambda\neq 0$	$W_{13}=\begin{pmatrix}w_{11}&w_{12}&w_{13}&0\\w_{21}&w_{22}&w_{23}&0\\w_{31}&w_{32}&w_{33}&0\\0&0&0&w_{44}\end{pmatrix}$ $w_{44}=1/det(W_{3\times 3})$
$J_{21}=\begin{pmatrix}\lambda_1&0&0&0\\1&\lambda_1&0&0\\0&0&\lambda_2&0\\0&0&0&\lambda_3\end{pmatrix}$ $\lambda_3=-2\lambda_1-\lambda_2$	$W_{21}=\begin{pmatrix}w_{11}&0&0&0\\w_{21}&w_{11}&0&0\\0&0&w_{33}&0\\0&0&0&w_{44}\end{pmatrix}$ $w_{44}=1/w_{11}^2w_{33}$
$J_{22}=\begin{pmatrix}\lambda&0&0&0\\1&\lambda&0&0\\0&0&\lambda&0\\0&0&0&-3\lambda\end{pmatrix}$	$W_{22}=\begin{pmatrix}w_{11}&0&0&0\\w_{21}&w_{11}&w_{23}&0\\w_{31}&0&w_{33}&0\\0&0&0&w_{44}\end{pmatrix}$ $w_{44}=1/w_{11}^2w_{33}$
$J_{23}=\begin{pmatrix}0&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$	$W_{23}=\begin{pmatrix}w_{11}&0&0&0\\w_{21}&w_{11}&w_{23}&w_{24}\\w_{31}&0&w_{33}&w_{34}\\w_{41}&0&w_{43}&w_{44}\end{pmatrix}$
$J_{31}=\begin{pmatrix}\lambda_1&0&0&0\\1&\lambda_1&0&0\\0&0&-\lambda_1&0\\0&0&1&-\lambda_1\end{pmatrix}$	$W_{31}=\begin{pmatrix}w_{11}&0&0&0\\w_{21}&w_{11}&0&0\\0&0&1/w_{11}&0\\0&0&w_{43}&1/w_{11}\end{pmatrix}$
$J_{32}=\begin{pmatrix}0&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&1&0\end{pmatrix}$	$W_{32}=\begin{pmatrix}w_{11}&0&w_{13}&0\\w_{21}&w_{11}&w_{23}&w_{13}\\w_{31}&0&w_{33}&0\\w_{41}&w_{31}&w_{43}&w_{33}\end{pmatrix}$

Jordan normal form of C	Stabilizer W
$J_{41} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 1 & \lambda_1 & 0 & 0 \\ 0 & 1 & \lambda_1 & 0 \\ 0 & 0 & 0 & -3\lambda_1 \end{pmatrix}$	$W_{41} = \begin{pmatrix} w_{11} & 0 & 0 & 0 \\ w_{21} & w_{11} & 0 & 0 \\ w_{31} & w_{21} & w_{11} & 0 \\ 0 & 0 & 0 & 1/w_{11}^3 \end{pmatrix}$
$J_{42} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$W_{42} = \begin{pmatrix} w_{11} & 0 & 0 & 0 \\ w_{21} & w_{11} & 0 & 0 \\ w_{31} & w_{21} & w_{11} & w_{34} \\ w_{41} & 0 & 0 & 1/w_{11}^3 \end{pmatrix}$
$J_{51} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	$W_{51} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ w_{21} & 1 & 0 & 0 \\ w_{31} & w_{21} & 1 & 0 \\ w_{41} & w_{31} & w_{21} & 1 \end{pmatrix}$

Table 3.

3.3 Case J_{11}

Symmetries: $P_1, \ldots, P_{23}, -A^\top$

If $a_{12} \neq 0, a_{23} \neq 0, a_{34} \neq 0$ then

$$N_{11}^1 = \begin{pmatrix} . & 1 & . & . \\ . & . & 1 & . \\ . & . & . & 1 \\ . & . & . & . \end{pmatrix}.$$

Otherwise, we transform conjuctive normal form

$$\bigwedge_{\sigma_i \in \sigma} (a_{\sigma_i(1)\sigma_i(2)} = 0 \vee a_{\sigma_i(2)\sigma_i(3)} = 0 \vee a_{\sigma_i(3)\sigma_i(4)} = 0) \tag{1}$$

to its disjunctive normal form. In (1) we made "logic summation" over σ, the set of all permutations of the set $\{1, \ldots, 4\}$. After simplification using De Morgan laws we obtained that the only case (up to conjugation) which does not belongs to subalgebra of $\mathfrak{sl}_4$ is $a_{12} = 0 \wedge a_{13} = 0 \wedge a_{21} = 0 \wedge a_{23} = 0 \wedge a_{31} = 0 \wedge a_{32} = 0$.

If $a_{14} \neq 0, a_{24} \neq 0, a_{34} \neq 0$ then

$$N_{11}^2 = \begin{pmatrix} . & 0 & 0 & 1 \\ 0 & . & 0 & 1 \\ 0 & 0 & . & 1 \\ . & . & . & . \end{pmatrix}.$$

Obviously, all other cases belongs to subalgebra of $\mathfrak{sl}_4$.

3.4 Case J_{12}

Symmetries: $P_1, P_1(-A^\top)P_1^{-1}, P_6, P_6(-A^\top)P_6^{-1}, P_7, P_7(-A^\top)P_7^{-1}, -A^\top$.

$K = a_{13}D_x a_{23} - a_{23}D_x a_{13} + a_{11}a_{13}a_{23} - a_{21}a_{13}^2 + a_{12}a_{23}^2 - a_{22}a_{13}a_{23}$,
$L = a_{32}D_x a_{31} - a_{31}D_x a_{32} + a_{11}a_{32}a_{31} + a_{21}a_{32}^2 - a_{12}a_{31}^2 - a_{22}a_{32}a_{31}$,
$M = a_{14}D_x a_{24} - a_{24}D_x a_{14} + a_{11}a_{14}a_{24} - a_{21}a_{14}^2 + a_{12}a_{24}^2 - a_{22}a_{14}a_{24}$,
$N = a_{42}D_x a_{41} - a_{41}D_x a_{42} + a_{11}a_{42}a_{41} + a_{21}a_{42}^2 - a_{12}a_{41}^2 - a_{22}a_{42}a_{41}$,
$R_1 = a_{14}a_{31} + a_{24}a_{32}$,
$R_2 = a_{13}a_{41} + a_{23}a_{42}$,
$R_3 = a_{14}a_{41} + a_{42}a_{24}$,
$R_4 = a_{13}a_{31} + a_{23}a_{32}$,
$R_5 = a_{13}a_{24} - a_{23}a_{14}$,
$R_6 = a_{42}a_{31} - a_{41}a_{32}$.

If $K \neq 0$ and $a_{34} \neq 0$, then

$$N_{12}^1 = \begin{pmatrix} 0 & 1 & 0 & . \\ . & . & 1 & . \\ . & . & . & 1 \\ . & . & . & . \end{pmatrix} \quad \sim P_1(N_{12}^1)P_1^{-1} \quad \sim (-N_{12}^1)^\top \quad \sim P_1(-N_{12}^1)^\top P_1^{-1}$$

$K \neq 0$	$M \neq 0$	$L \neq 0$	$N \neq 0$
$a_{34} \neq 0$	$a_{43} \neq 0$	$a_{43} \neq 0$	$a_{34} \neq 0$

Otherwise $(M = 0, L = 0, a_{34} = 0)$ or $(K = 0, N = 0, a_{43} = 0)$ or $(a_{34} = 0, a_{43} = 0)$ or $(K = 0, L = 0, M = 0, N = 0)$.

If $K = 0, L = 0, M = 0, N = 0$, then

$$N_{12}^2 = \begin{pmatrix} . & 0 & . & 0 \\ 0 & . & . & . \\ 0 & 1 & . & . \\ 1 & 1 & . & . \end{pmatrix} \quad \sim P_1(N_{12}^2)P_1^{-1} \quad \sim (-N_{12}^2)^\top \quad \sim P_1(-N_{12}^2)^\top P_1^{-1}$$

$R_1 \neq 0$	$R_2 \neq 0$	$R_2 \neq 0$	$R_1 \neq 0$
$R_3 \neq 0$	$R_4 \neq 0$	$R_3 \neq 0$	$R_4 \neq 0$
$R_6 \neq 0$	$R_6 \neq 0$	$R_5 \neq 0$	$R_5 \neq 0$

If $a_{34}=0, a_{43}=0$, then

$$N_{12}^3 = \begin{pmatrix} . & 1 & 0 & . \\ . & . & 1 & 0 \\ . & . & . & 0 \\ . & 1 & 0 & . \end{pmatrix} \sim P_1(N_{12}^2)P_1^{-1} \quad \sim (-N_{12}^2)^\top \quad \sim P_1(-N_{12}^2)^\top P_1^{-1}$$

$K \neq 0$	$M \neq 0$	$L \neq 0$	$N \neq 0$
$R_2 \neq 0$	$R_1 \neq 0$	$R_1 \neq 0$	$R_2 \neq 0$
$R_5 \neq 0$	$R_5 \neq 0$	$R_6 \neq 0$	$R_6 \neq 0$

If $(M=0, L=0, a_{34}=0)$ or $(K=0, N=0, a_{43}=0)$, then the normal form is N_{12}^2 again.

All others cases lies in subalgebra of $\mathfrak{s}l_4$.

3.5 Case J_{21}

Symmetries: $P_1, P_6(-A^\top)P_6^{-1}, P_7(-A^\top)P_7^{-1}$

If $a_{13} \neq 0$ and $a_{42} \neq 0$, then

$$N_{21}^1 = \begin{pmatrix} . & . & 1 & . \\ . & . & 0 & . \\ . & . & . & . \\ . & 1 & . & . \end{pmatrix} \sim \quad P_1(N_{21}^1)P_1^{-1} \sim \quad P_6(-N_{21}^1)^\top P_6^{-1} \sim \quad P_7(-N_{21}^1)^\top P_7^{-1}$$

$a_{13} \neq 0$	$a_{14} \neq 0$	$a_{32} \neq 0$	$a_{42} \neq 0$
$a_{42} \neq 0$	$a_{32} \neq 0$	$a_{14} \neq 0$	$a_{13} \neq 0$

We solved $\widehat{a}_{13}=1$ for w_{11}, $\widehat{a}_{23}=0$ for w_{21}, $\widehat{a}_{42}=1$ for w_{33}.

Otherwise $(a_{13}=0, a_{32}=0)$ or $(a_{14}=0, a_{42}=0)$ or $(a_{13}=0, a_{14}=0)$ or $(a_{32}=0, a_{42}=0)$.

If $a_{13}=0, a_{32}=0$ and $a_{42} \neq 0, a_{31} \neq 0$, then

$$N_{21}^2 = \begin{pmatrix} . & . & 0 & . \\ . & . & . & . \\ 1 & 0 & . & . \\ 0 & 1 & . & . \end{pmatrix} \sim \quad P_1(N_{21}^2)P_1^{-1} \sim \quad P_6(-N_{21}^2)^\top P_6^{-1} \sim \quad P_7(-N_{21}^2)^\top P_7^{-1}$$

$a_{13}=0$	$a_{14}=0$	$a_{32}=0$	$a_{42}=0$
$a_{32}=0$	$a_{42}=0$	$a_{13}=0$	$a_{14}=0$
$a_{42} \neq 0$	$a_{32} \neq 0$	$a_{14} \neq 0$	$a_{13} \neq 0$
$a_{31} \neq 0$	$a_{41} \neq 0$	$a_{23} \neq 0$	$a_{24} \neq 0$

We solved $\widehat{a}_{31} = 1$ for w_{11}, $\widehat{a}_{41} = 0$ for w_{21}, $\widehat{a}_{42} = 1$ for w_{33}.

If $a_{13} = 0, a_{14} = 0$ and $a_{42} \neq 0, a_{31} \neq 0$, then

$$N_{21}^3 = \begin{pmatrix} . & . & 0 & 0 \\ . & . & . & . \\ . & 1 & . & . \\ 0 & 1 & . & . \end{pmatrix} \sim \quad P_1(N_{21}^3)P_1^{-1} \sim \quad P_6(-N_{21}^3)^\top P_6^{-1} \sim \quad P_7(-N_{21}^3)^\top P_7^{-1}$$

$a_{13} = 0$	$a_{14} = 0$	$a_{32} = 0$	$a_{42} = 0$
$a_{14} = 0$	$a_{13} = 0$	$a_{42} = 0$	$a_{32} = 0$
$a_{42} \neq 0$	$a_{32} \neq 0$	$a_{14} \neq 0$	$a_{13} \neq 0$
$a_{32} \neq 0$	$a_{42} \neq 0$	$a_{13} \neq 0$	$a_{14} \neq 0$

We solved $\widehat{a}_{42} = 1$ for w_{11}, $\widehat{a}_{41} = 0$ for w_{21}, $\widehat{a}_{32} = 1$ for w_{33}.

Otherwise $a_{13} = 0, a_{14} = 0, a_{32} = 0, a_{42} = 0$. If $a_{12} \neq 0, a_{23} \neq 0, a_{41} \neq 0$, then

$$N_{21}^4 = \begin{pmatrix} 0 & . & 0 & 0 \\ . & . & 1 & . \\ . & 0 & . & . \\ 1 & 0 & . & . \end{pmatrix} \sim \quad P_1(N_{21}^4)P_1^{-1} \sim \quad P_6(-N_{21}^4)^\top P_6^{-1} \sim \quad P_7(-N_{21}^4)^\top P_7^{-1}$$

$a_{12} \neq 0$	$a_{12} \neq 0$	$a_{12} \neq 0$	$a_{12} \neq 0$
$a_{23} \neq 0$	$a_{24} \neq 0$	$a_{31} \neq 0$	$a_{41} \neq 0$
$a_{41} \neq 0$	$a_{31} \neq 0$	$a_{24} \neq 0$	$a_{23} \neq 0$

We solved $\widehat{a}_{23} = 1$ for w_{11}, $\widehat{a}_{11} = 0$ for w_{21}, $\widehat{a}_{41} = 1$ for w_{33}.

Otherwise $a_{13} = 0, a_{32} = 0, a_{14} = 0, a_{42} = 0$ and $(a_{12} = 0$ or $(a_{24} = 0, a_{41} = 0)$ or $(a_{23} = 0, a_{31} = 0))$.

If $a_{24} = 0, a_{41} = 0, a_{12} \neq 0, a_{31} \neq 0, a_{43} \neq 0$, then

$$N_{21}^4 = \begin{pmatrix} 0 & . & 0 & 0 \\ . & . & . & 0 \\ 1 & 0 & . & . \\ 0 & 0 & 1 & . \end{pmatrix} \sim \quad P_1(N_{21}^4)P_1^{-1} \sim \quad P_6(-N_{21}^4)^\top P_6^{-1} \sim \quad P_7(-N_{21}^4)^\top P_7^{-1}$$

$a_{24} = 0$	$a_{23} = 0$	$a_{24} = 0$	$a_{23} = 0$
$a_{41} = 0$	$a_{31} = 0$	$a_{41} = 0$	$a_{31} = 0$
$a_{12} \neq 0$	$a_{12} \neq 0$	$a_{12} \neq 0$	$a_{12} \neq 0$
$a_{31} \neq 0$	$a_{41} \neq 0$	$a_{23} \neq 0$	$a_{24} \neq 0$
$a_{43} \neq 0$	$a_{34} \neq 0$	$a_{34} \neq 0$	$a_{43} \neq 0$

We solved $\widehat{a}_{31} = 1$ for w_{11}, $\widehat{a}_{11} = 0$ for w_{21}, $\widehat{a}_{43} = 1$ for w_{33}.

All others cases lies in subalgebra of $\mathfrak{sl}_4$.

3.6 Case J_{22}

Symmetries: $P_6^\top$

3.7 Case J_{31}

Symmetries: $P_7^\top, P_{16}, P_{23}^\top$

$$N_{31}^1 = \begin{pmatrix} . & . & 0 & 1 \\ . & . & . & 0 \\ . & . & . & . \\ . & . & . & . \end{pmatrix} \sim \quad P_7(N_{31}^1)^\top P_7^{-1} \sim \quad P_{16}(N_{31}^1)P_{16}^{-1} \sim \quad P_{23}(N_{31}^1)^\top P_{23}^{-1}$$

$$a_{14} \neq 0 \qquad a_{32} \neq 0 \qquad a_{32} \neq 0 \qquad a_{14} \neq 0$$

$\mathbf{a_{14} = 0, a_{32} = 0}$:

$$N_{31}^2 = \begin{pmatrix} . & . & 1 & 0 \\ . & . & 0 & . \\ . & 0 & 0 & . \\ . & . & . & . \end{pmatrix} \sim \quad P_7(N_{31}^2)^\top P_7^{-1} \sim \quad P_{16}(N_{31}^2)P_{16}^{-1} \sim \quad P_{23}(N_{31}^2)^\top P_{23}^{-1}$$

$$\begin{matrix} a_{13} \neq 0 & a_{42} \neq 0 & a_{12} \neq 0 & a_{12} \neq 0 \\ a_{34} \neq 0 & a_{34} \neq 0 & a_{31} \neq 0 & a_{24} \neq 0 \end{matrix}$$

$\mathbf{a_{14} = 0, a_{32} = 0, a_{34} = 0, a_{12} = 0 \rightarrow}$ **subalgebra**

3.8 Case J_{41}

Symmetries: $P_{14}^\top$

$$N_{41}^1 = \begin{pmatrix} . & . & . & 1 \\ . & . & . & 0 \\ . & . & . & 0 \\ . & . & . & . \end{pmatrix} \sim \quad P_{14}(N_{41}^1)^\top P_{14}^{-1}$$

$$a_{14} \neq 0 \qquad a_{43} \neq 0$$

$\mathbf{a_{14} = 0, a_{43} = 0}$:

$$N_{41}^2 = \begin{pmatrix} . & 0 & . & 0 \\ . & . & . & 1 \\ . & . & 0 & . \\ . & . & 0 & . \end{pmatrix} \sim \quad P_{14}(N_{41}^2)^\top P_{14}^{-1}$$

$$\begin{array}{ll} a_{13} \neq 0 & \qquad a_{13} \neq 0 \\ a_{24} \neq 0 & \qquad a_{42} \neq 0 \end{array}$$

$\mathbf{a_{14} = 0, a_{43} = 0, a_{13} = 0}$:

$$N_{41}^3 = \begin{pmatrix} . & . & 0 & 0 \\ . & . & . & 1 \\ . & 0 & . & 0 \\ . & . & 0 & . \end{pmatrix} \sim \quad P_{14}(N_{41}^3)^\top P_{14}^{-1}$$

$$\begin{array}{ll} a_{12} \neq 0 & \qquad a_{23} \neq 0 \\ a_{24} \neq 0 & \qquad a_{42} \neq 0 \end{array}$$

$\mathbf{a_{14} = 0, a_{43} = 0, a_{13} = 0, a_{24} = 0, a_{42} = 0}$:

$$N_{41}^4 = \begin{pmatrix} 0 & . & 0 & 0 \\ 0 & . & . & 0 \\ . & . & . & . \\ 1 & 0 & 0 & . \end{pmatrix} \sim \quad P_{14}(N_{41}^4)^\top P_{14}^{-1}$$

$$\begin{array}{ll} a_{12} \neq 0 & \qquad a_{23} \neq 0 \\ a_{41} \neq 0 & \qquad a_{34} \neq 0 \\ a_{23} \neq 0 & \qquad a_{12} \neq 0 \end{array}$$

All others lies in subalgebra of sl_4.

3.9 Case J_{51}

Symmetries: $P_{23}^\top$

$$N_{51}^1 = \begin{pmatrix} 0 & 0 & 0 & . \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \end{pmatrix} \sim \quad P_{23}(N_{51}^1)^\top P_{23}^{-1}$$

$$\begin{array}{ll} a_{14} \neq 0 & \qquad a_{14} \neq 0 \end{array}$$

$\mathbf{a_{14} = 0}$:

$$N_{51}^2 = \begin{pmatrix} 0 & 0 & . & 0 \\ . & . & . & . \\ . & . & . & . \\ . & . & 0 & . \end{pmatrix} \sim \quad P_{23}(N_{51}^2)^\top P_{23}^{-1}$$

$$a_{13} \neq 0 \qquad\qquad a_{24} \neq 0$$

$\mathbf{a_{14} = 0, a_{13} = 0, a_{24} = 0}$:

$$N_{51}^3 = \begin{pmatrix} 0 & . & 0 & 0 \\ 0 & . & . & 0 \\ . & . & . & . \\ . & 0 & . & . \end{pmatrix} \sim \quad P_{23}(N_{51}^3)^\top P_{23}^{-1}$$

$$\begin{matrix} a_{12} \neq 0 & \qquad & a_{34} \neq 0 \\ a_{23} \neq 0 & \qquad & a_{23} \neq 0 \end{matrix}$$

All others lies in subalgebra of sl_4.