Gradient catastrophes for a generalized Burgers equation on a finite interval

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The problem

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The Burgers equation

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Boundary problem (BP) $u(\alpha,t) = l(t), \quad u(\beta,t) = r(t)$

Below $l(t) = A, \quad r(t) = B$ are constants

Time-invariant (smooth) solutions

For $n = 1$

$$u(x,t) \tag{3}$$

$$\tag{4}$$

$$\tag{5}$$

$$\tag{6}$$

$$\tag{7}$$

Time-invariant (smooth) solutions

For $n = 1$

$$u(x,t) = c, \tag{3}$$

(4)

(5)

(6)

(7)

Time-invariant (smooth) solutions

For $n = 1$

$$
\begin{aligned}
u(x,t) &= c, && (3)\\
&= -\varepsilon^2 a \tanh(ax + c), && (4)\\
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\end{aligned}
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For $n > 1$ stationary solutions are given by

$$\begin{aligned} x &= C_1 + \varepsilon^2(n+1)\int \frac{dy}{C_2 + \alpha y^{n+1}}, \\ y &= C. \end{aligned}$$

$\mathcal{L}^2$-estimate of decay rate

A solution of the (generalized) Burgers equation

$$u_t = u_{xx} - \alpha u^n u_x \tag{8}$$

with zero boundary conditions

$$u(t,a) = u(t,b) = 0, \quad u(0,x)|_{[a,b]} = f(x) \tag{9}$$

monotonically tends to zero as $t \to \infty$ in $\mathcal{L}^2$ norm since

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$$\frac{\partial}{\partial t}\int_a^b u^2 dx = \int_a^b 2uu_t dx = 2\int_a^b u(u_{xx} - \alpha u^n u_x)dx \quad =$$

$$2\int_a^b u du_x + \frac{-\alpha}{n+1}u^{n+1}\Big|_a^b = 2uu_x|_a^b - 2\int_a^b u_x^2 dx = -2\int_a^b u_x^2 dx \quad \leqslant \quad 0$$

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The greater u_x the faster the convergence.

Equation for the difference

Hypothesis:
When the boundary conditions are non-zero but constant

$$u(0,x)|_{[a,b]} = f(x) \quad u(t,a) = f(a) = A, \quad u(t,b) = f(b) = B, \tag{10}$$

one may expect the solution to converge to the respective stationary invariant solution, ie, to $\mu(x)$,

$$\mu_{xx} - \alpha\mu^n\mu_x = 0, \quad \mu(a) = A, \quad \mu(b) = B \tag{11}$$

Such a solution exists and is of one of the above listed forms depending on the combination of A and B.

The answer to this hypothesis is complex.

Equation, continued

Put $\nu(t,x) = u(t,x) - \mu(x)$, ie, $u(t,x) = \nu(t,x) + \mu(x)$.
Substituting into the generalized Burgers we get

$$u_t = (\nu(t,x) + \mu(x))_t = \nu(t,x)_t = u_{xx} - \alpha u^n u_x = \tag{12}$$
$$(\nu(t,x) + \mu(x))_{xx} - \alpha(\nu(t,x) + \mu(x))^n(\nu(t,x) + \mu(x))_x.$$

In the case $n = 1$ and $\alpha = 2$,
$\nu_{xx} - 2\nu\nu_x + [\mu_{xx} - 2\mu\mu_x] - 2\{\nu_x\mu + \nu\mu_x\}$.
Since $\mu_{xx} - 2\mu\mu_x = 0$ by definition of μ,

$$\nu_t = \nu_{xx} - 2\nu\nu_x - 2(\nu\mu)_x. \tag{13}$$

Boundary conditions for ν are zero by definition of ν.

$\mathcal{L}^2$-estimate for decay rate of difference

The rate of ν by analogy with (10):

$$\langle \nu_t \rangle|_{\mathcal{L}^2} = \frac{\partial}{\partial t}\int_a^b \nu^2 dx =$$

$$\int_a^b 2\nu\nu_t dx = 2\int_a^b \nu(\nu_{xx} - 2\nu\nu_x - 2(\nu\mu)_x)dx =$$

$$2\int_a^b \nu d\nu_x - \frac{4}{3}\nu^3\Big|_a^b - 4\int_a^b \nu d(\nu\mu) =$$

$$2\nu\nu_x|_a^b - 2\int_a^b \nu_x^2 dx - 4\nu(\nu\mu)|_a^b + 4\int_a^b \mu\nu\nu_x dx =$$

$$-2\int_a^b \nu_x^2 dx + 2\int_a^b \mu d\nu^2 = -2\int_a^b \nu_x^2 dx + 2\int_a^b \mu_x\nu^2 dx - 2\,\nu^2\mu|_a^b =$$

$$-2\int_a^b \left(\nu_x^2 + \mu_x\nu^2\right) dx.$$

$\mathcal{L}^2$-estimate for decay rate of difference, continued

The monotony of $\mathcal{L}^2$-convergence is not guaranteed; but it takes place, in the case $\mu_x \geqslant 0$ (increasing initial profile). In the case $n > 1$ the corresponding conditions are less transparent; for instance when $n = 2$

$$\frac{\partial}{\partial t} \int_a^b \nu^2 dx = -2 \int_a^b \left(\nu_x^2 - \mu_x \nu^2 (\mu_x - \nu_x) \right) dx.$$

So $\mu_x(\mu_x - \nu_x) > 0$ guarantees the deviation ν decay. When $\langle \nu_t \rangle|_{\mathcal{L}^2} \leqslant 0$ fails the difference ν doesn't necessarily tend to zero. Usually such evolution ends in a

1. catastrophe,

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1. catastrophe, resulting in a Heaviside-type break at the interval's end;
2. decay to a smooth invariant solution;
3. a frozen multi-oscillation, a piecewise-smooth invariant solution.

Time for tilting

For a general quasilinear transport equation ($x \in \mathbb{R}$)

$$w_t + f(w)w_x = 0 \tag{14}$$

the moment of gradient catastrophe t_c can be defined as follows. Let $w = \varphi(x)$ be an initial profile. The solution of this problem may be given in a parametric form $w = \varphi(\xi),\ x = \xi + \mathcal{F}(\xi)t$ where $\mathcal{F} = f(\varphi(\xi))$.

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The characteristics of the form $x = \xi + \mathcal{F}(\xi)t$ intersect in the case $\varphi'(\xi) < 0$ thus resulting in many-valued w (the tilting of a wave or a gradient catastrophe). If the inequality holds on a finite interval there exist a minimal value of time, t_c, when this problem arises. One may determine t_c by the formula

$$t_c = -1/\mathcal{F}'(\xi_c)$$

where $|\mathcal{F}(\xi_c)| = \max_{[\alpha,\beta]} |\mathcal{F}'(\xi)|$ while $\mathcal{F}'(\xi) < 0$.

Dispersive shock, example 1; n=1

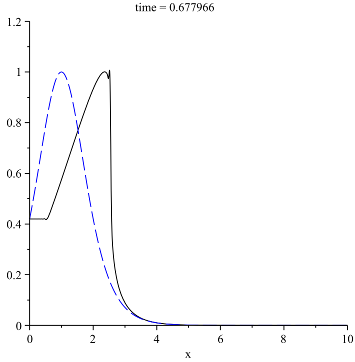

Figure : Shock strucks at $\approx t_c$: Burgers, n=1, IVBP
$u(x,0) = \mathrm{sech}^2(x-1), u(0,t) = \mathrm{sech}^2(1), u(10,t) = \mathrm{sech}^2(9)$

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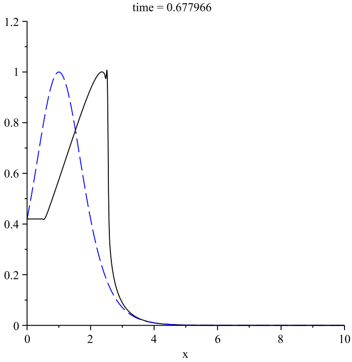

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Heaviside-type gap develops

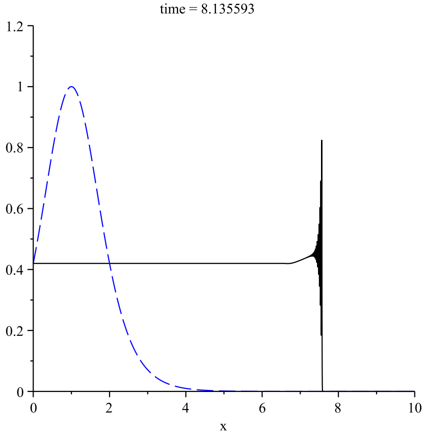


Figure : Multi-oscillations move to a Heaviside-type break $\tanh^2(1) - \tanh^2(9)$ at $x = 10$; $t \approx 8$.

Dispersive shock, example 2; n=1

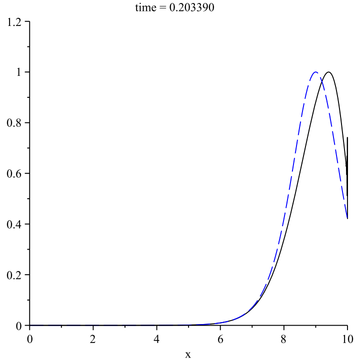
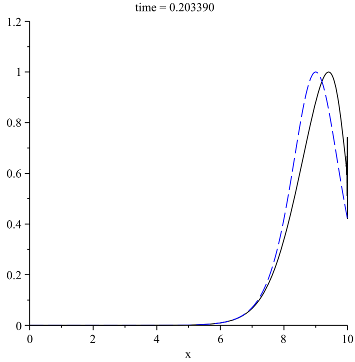


Figure : Shock strucks at $t \approx 0.2 \ll t_c$: Burgers, n=1, IVBP
$u(x,0) = \mathrm{sech}^2(x-9)$, $u(0,t) = \mathrm{sech}^2(9)$, $u(10,t) = \mathrm{sech}^2(1)$

Dispersive shock, example 2; n=1



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Heaviside-type gap develops

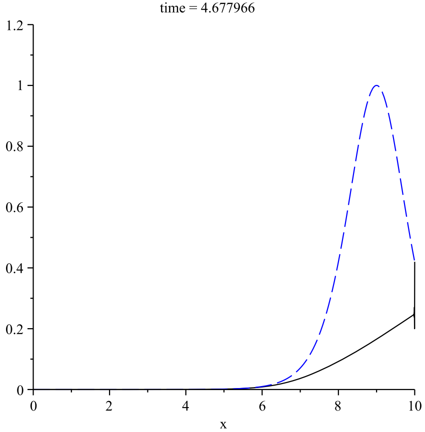

Figure : Multi-oscillations move to a Heaviside-type break $-\tanh^2(1) + \tanh^2(9)$ at $x = 10$; $t \approx 4$.

Dispersive shock, example 1; n=2

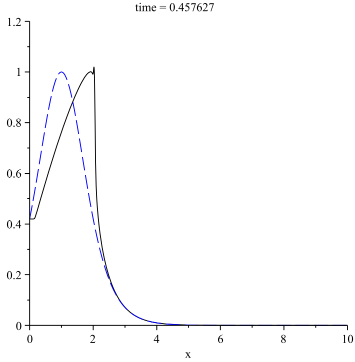

Figure : Start of gradient catastrophe at $t_c \approx 0.45$. Dash line is the initial profile $\operatorname{sech}^2(x-1)$. n=2; Burgers, n=2, $\varepsilon = 0.02$

Dispersive shock, example 1; n=2

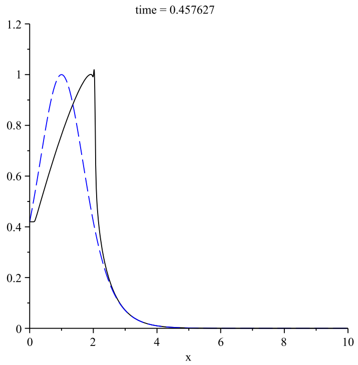

Figure : Start of gradient catastrophe at $t_c \approx 0.45$. Dash line is the initial profile $\mathrm{sech}^2(x-1)$. n=2; Burgers, n=2, $\varepsilon = 0.02$

Heaviside-type gap develops, example 3

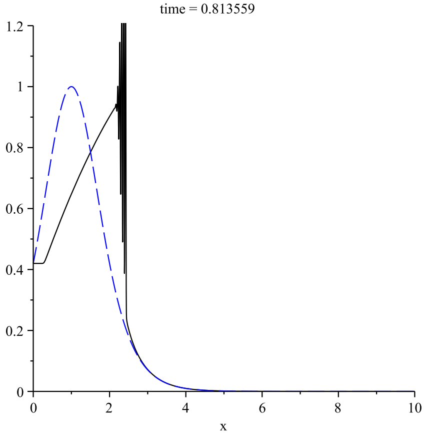

Figure : Multi-oscillations move to a Heaviside-type break $-\tanh^2(1) + \tanh^2(9)$ at $x = 10$; $t \approx 0.8$.

Heaviside-type gap develops, example 4

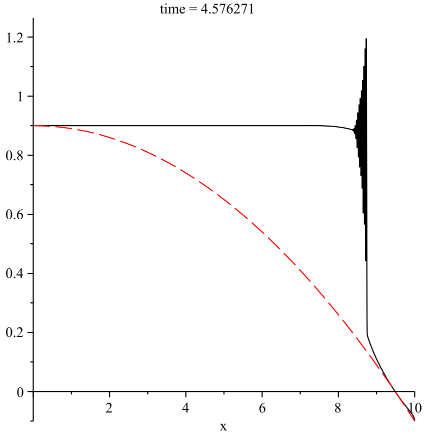

Figure : Multi-oscillations move to a Heaviside-type break at $x = 10$; $t \approx 4.6$.Dash line is the initial profile $-0.01x^2 + 0.9$. n=2

More: Two-humped catastrophe

ANIMATION, click here

Examples of decay

Here are two examples of a decay towards a *decreasing* invariant solution. In both cases the initial profile is chosen in a vicinity of this solution and the right-hand side of (9) is negative.

Consider the equation $u_t = \varepsilon^2 u_{xx} - 2uu_x$.

Here are two examples of a decay towards a *decreasing* invariant solution. In both cases the initial profile is chosen in a vicinity of this solution and the right-hand side of (9) is negative.

Consider the equation $u_t = \varepsilon^2 u_{xx} - 2uu_x$.

Ex. 1. Choose IVBP:
$u(x,0) = -\varepsilon^2 \tanh(x) + 1.6\varepsilon \sin(2\pi x)$, $u(0,t) = 0$, $u(1,t) = -\varepsilon^2 \tanh(1)$; $\varepsilon = 0.05$.
Here $\mu = -\varepsilon^2 \tanh(x)$ is a decreasing invariant solution, $\nu = 1.6\varepsilon \sin(2\pi x)$ — the perturbation.
Asymptotics at $t \to \infty$ coincides with μ, see the following graph. The dissipation reigns in and no catastrophe develops.
The explanation can be seen in next graph, where the typical graph of integrand $\nu_x^2 + \mu_x \nu^2$ in (9) is given at $t = 2$; clearly $\langle \nu_t \rangle|_{\mathcal{L}^2} < 0$.

Decay, example 1; n=1

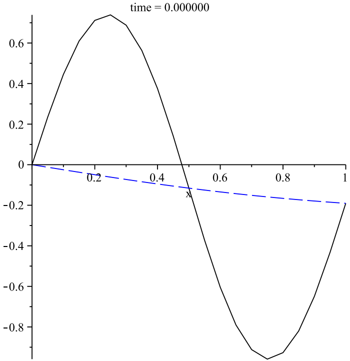

Figure : Initial profile $-\varepsilon^2 \tanh(x) + 1.6\varepsilon \sin(2\pi x)$. Asymptotic limit (dash line) is the invariant solution $-\varepsilon^2 \tanh(x)$

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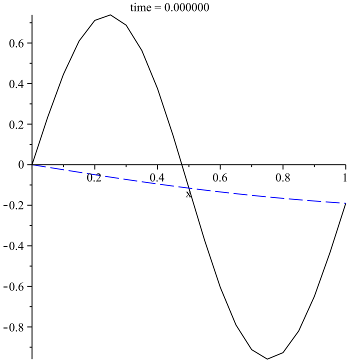

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$\mathcal{L}^2$ estimate for the decay rate

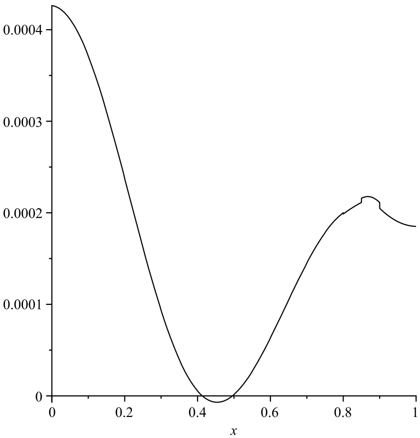

Figure : The graph of the integrand $\nu_x^2 + \mu_x \nu^2$ for previous evolution at $t = 2$.

No shocks

Equation $u_t = \varepsilon^2 u_{xx} - 2uu_x$.

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Ex. 2. IVBP:
$u(1,t) = -\varepsilon^2 \tanh(1) + \varepsilon((\mathrm{sech}^2(1)),$
$u(0,t) = \varepsilon, u(x,0) = -\varepsilon^2 \tanh(x) + \varepsilon((\mathrm{sech}^2(x)).$
The initial profile $u(x,0)$ gives an impression of being in vicinity of the invariant solution $-\varepsilon^2 \tanh(x)$ as it is modestly perturbed by $\varepsilon((\mathrm{sech}^2(x))$. In fact it tends to another (decreasing) invariant solution $2.06\varepsilon^2 \tanh(-2.06x + 2.1)$, see next graph.

Decay, example 2

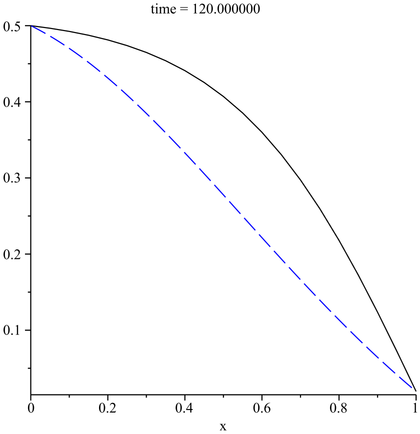


Figure : Initial profile $-\varepsilon^2 \tanh(x) + \varepsilon((\mathrm{sech}^2(x))$ (dashed) and asymptotic limit $2.06\varepsilon^2 \tanh(-2.06x + 2.1)$ (solid line).

More examples of decay: a frozen oscillation

In some cases the evolution of the initial profile results early and clearly *not* in an invariant solution from the list above; see next figure, obtained with IVBP $\{u(x,0) = -\alpha\varepsilon^2 \tanh(\alpha)(2x^4 - x^2), u(0,t) = 0, u(1,t) = -\alpha\varepsilon^2 \tanh(\alpha)\}$, $\varepsilon = 0.05$; $\alpha = 50$.

Note that the invariant solution with the same boundary values is $\mu(x) = -\alpha\varepsilon^2 \tanh(\alpha x)$.

Compare initial, asymptotic and invariant profiles

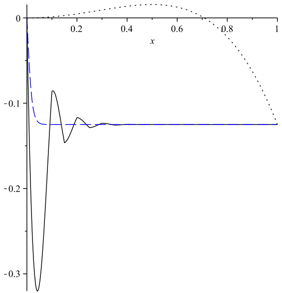
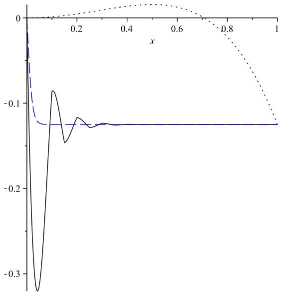


Figure : Initial profile $-\varepsilon^2 \tanh(x) + 1.6\varepsilon \sin(2\pi x)$, $u(0,t) = 0$, $u(1,t) = -\varepsilon^2 \tanh(1)$. Asymptotic limit (dash line) is the invariant solution $-\varepsilon^2 \tanh(x)$; n=1

Compare initial, asymptotic and invariant profiles



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$\mathcal{L}^2$-convergence

The stabilization may be rather quick. The graph of $\mathcal{L}^2$-estimate for the difference ν, $\int_0^1 (u(s,t) - \mu(s))^2\, ds$ is presented in

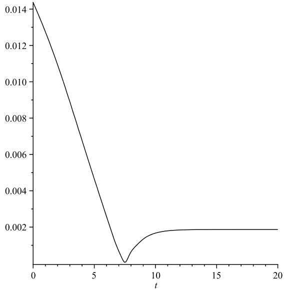

Figure : The graph of the $\langle \nu \rangle|_{\mathcal{L}^2}$, at $0 \leqslant t \leqslant 20$.

Stability

The effect is stable, as the asymptotic profile in this example do not to depend on perturbations of the initial one, provided boundary data is the same: identical asymptotics are obtained for
$u(x,0) = -\alpha\varepsilon^2 \tanh(\alpha)x$
or $-\alpha\varepsilon^2 \tanh(\alpha)x^2$, etc,:
see

ANIMATION, Click here.

$\mathcal{L}^2$-extremals

A stationary point may be an extremal of the $\mathcal{L}^2$-estimate functional,

$$\frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0} \int_a^b \left((\nu + \varepsilon h)_x^2 - \mu_x(\nu + \varepsilon h)^2\right) dx = 0. \tag{15}$$

It follows

$$\nu_{xx} + \mu_x \nu = 0. \tag{16}$$

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Decreasing solutions of the Burgers ($n=1$)-equation are of the form $\mu(x)=-a\tanh(ax+b)$ and the potential μ_x is "numerically finite". Some of solutions of (16) are discontinuous (eg, the real part of its complex solution is both discontinuous and multi-oscillating in some cases).
This discontinuity can generate a chaos on the numeric graph and be a possible reason of a failed smoothness of $\nu(x)$. Details will be published elsewhere.

Piecewise-smoothness?

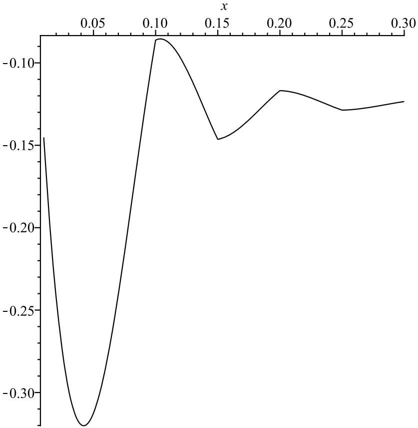

Figure : A part of the general view: piecewise-smooth difference; $t = 20$. The graph is composed with parts of different invariant solutions.

Breaks of the derivative

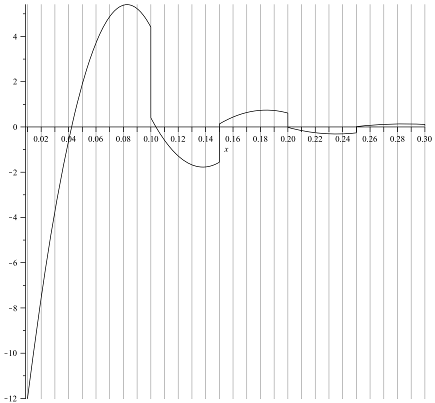

Figure : The graph of the derivative, $t = 20$.

The equation for the derivative $v = u'$ is
$v_t = \varepsilon^2 v'' - 2v^2 - 2v' D^{-1}(v)$. Breaks form in early stage of evolution.

References

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THANK YOU FOR YOUR ATTENTION