On natural invariants and equivalence of planar webs

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In memory of my friends and colleagues:
Maks Akvis and Vadim Goldberg

Intro

- Nomography is the graphical representation of mathematical relationships for purposes of calculation. Invented in 1880 by Maurice d'Ocagne (1862–1938), nomograms were used extensively well into the 1970s (and occasionally today) to provide engineers with fast graphical calculations of complicated formulas to a practical precision.

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- In other words, we are interested in whether an equation in the form $F(x,y,z)=0$ can be rewritten in the form $Z(z)=X(x)+Y(y)$. The Saint-Robert criterion says this is possible if

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- Is there a process, whereby a function of several variables is constructed using functions of two variables?
- The variant of the problem for continuous functions was resolved in 1957 by Vladimir Arnold (the Kolmogorov–Arnold representation theorem), but the variant for algebraic functions remains unresolved.

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- A set of vector fields :$\{X_1, ..., X_d\}, X_i \wedge X_j \neq 0, i \neq j \Longrightarrow$d-web with foliations that are integral curves of X_i.
- They satisfy the following relations: $\omega_i \wedge df_i = 0, \omega_i(X_i) = 0, X_i(f_i) = 0.$

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- The symmetric group $\mathbf{S}_d$:
$\mathbf{S}_d \ni \sigma : \{f_1, .., f_d\} \rightarrow \left\{f_{\sigma(1)}, ..., f_{\sigma(d)}\right\}$.

Ordered and non-ordered webs

- The webs $W(f_1, .., f_d)$ we call *ordered*, and the sets $\bigcup_{\sigma e \mathbf{S}_d} W\left(f_{\sigma(1)}, ..., f_{\sigma(d)}\right)$ we'll call *non-ordered* webs

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- Remark that the action of $\mathcal{D}iffeo_2$ is trivial if we fix any two components, say $\{f_i, f_j\}$.

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- Denote by $\pi_{i,,j} : \mathbb{R}^d \rightarrow \mathbb{R}^2$ the natural projections

$$\pi_{i,,j} : \left(u^1, ..., u^d\right) \rightarrow \left(u^i, u^j\right).$$

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- These two actions commute, and, at first, we will consider the action of the gauge group.

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- Let $\pi_k : J^k(\pi) \to \mathbb{R}^2$ be bundles of k-jets of the sections of the bundle π, i.e. k-jets of planar d-webs, and let $u^i_{\alpha,\beta}, \alpha, \beta \in \mathbb{N}, \alpha + \beta \leq k$, be the standard coordinates in the jet space $J^k(\pi)$.

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- Denote by $j_k(s)$ the k-jet of the section s, or the k-jet of corresponding web.

Web and Jet Geometry

- Web, that correspondens to a section s of the bundle π,is given by the restriction on $j_1(s)$

  functions $(x, y, u^2, ..., u^d)$,horisontal differential 1-forms: $\left(dx, dy, \widehat{d}u^3, ..., \widehat{d}u^d\right)$, where

$$\widehat{d}u^i = u^i_{10}dx + u^i_{01}dy$$

  are the total differentials.

Web and Jet Geometry-2

- In order to find vector fields that describe the web, we take the 2-form

$$\Omega = dx \wedge dy,$$

and define "Hamiltonian" vector fields X_i as horisontal vector fields

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Model

- Consider 1-dimensional bundle $\pi : \mathbb{R}^3\left(x, y, u\right) \to \mathbb{R}^2\left(x, y\right)$ with gauge action of Diffeo_1 :

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Theorem

The field of rational differential invariants of the gauge group Diffeo_1*-action is generated by the following invariant of the first order* $U = \frac{u_{0,1}}{u_{1,0}}$ *and its total derivatives* $U_{i,j} = \frac{d^{i+j} U}{dx^i dy^j}$.

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- Thus, we have another representation of ordered $d-$ webs as a collection of functions

$$\left(x, y, \frac{f_{3,y}}{f_{3,x}}, \ldots, \frac{f_{d,y}}{f_{d,x}}\right).$$

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where $\Delta_{pq} = p_{01}q_{10} - p_{10}q_{01}$.

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where $\Delta_{pq} = p_{01}q_{10} - p_{10}q_{01}$.
- In particular, we have

$$\frac{d}{dx} = \frac{d}{du^i}, \frac{d}{dx} = \frac{d}{du^j}$$

in the above theorem.

- The last representation k-th components as ratios $\frac{du^k}{du^j} / \frac{du^k}{du^i}$ gives us the trivial Diffeo_1^{d-2} action on the last $(d-2)$ components and the only action, we have now, is the action Diffeo_1^2 on the first two components $(x, y) \longrightarrow (X(x), Y(y))$, and, the induced by it, the action on the last $(d-2)$ components.

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where $X(x)$ and $Y(y)$ are smooth functions.
- Let Z_1 is the first prolongation the vector field Z into the first jet space $J^1(\mathbb{R}^3 \to \mathbb{R}^2)$. We have

$$Z_1 = X(x)\,\partial_x + Y(y)\,\partial_y + (X' - Y')\left(u_{1,0}\partial_{u_{1,0}} + u_{1,0}\partial_{u_{1,0}}\right),$$

and

$$Z_1\left(\frac{u_{0,1}}{u_{1,0}}\right) = (X' - Y')\left(\frac{u_{0,1}}{u_{1,0}}\right).$$

Gauge group action

- We have the bundle $\pi : \mathbb{R}^d \longrightarrow \mathbb{R}^2, \pi\left(x, y, u^3, ..., u^{d-2}\right) \longrightarrow (x, y)$, and any d-web $w = W_d\left(x, y, f_3.., f_d\right)$ defines a section S_w of the bundle, where

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- As we have seen vectors fields, that generate the Lie algebra of the gauge pseudogroup, are point vector fields on $\mathbf{J}^0(\pi) = \mathbb{R}^d$, having the form:

$$\mathcal{Z}(X, Y) = X(x)\partial_x + Y(y)\partial_y + \left(X' - Y'\right)\sum_{i=3}^{d} u^i \partial_{u^i}. \qquad (3)$$

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- Take the fibre $\mathbf{J}_0^k = \pi_k^{-1}(0)$ an consider the gauge action on this fibre, that gives by vector fields $\mathcal{Z}_k(X, Y)$, where $X(0) = 0, Y(0) = 0$.

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- We represent such vector fields $\mathcal{Z}_k(X, Y)$ in the form

$$\mathcal{Z}_k(X, Y) = \mathcal{Z}_k(X) + \mathcal{Z}_k(Y),$$

where $\mathcal{Z}_k(X), \mathcal{Z}_k(Y)$ are k-th prolongations of the vector fields

$$\mathcal{Z}(X) = X(x)\,\partial_x + X' \sum_{i=3}^{d} \left(u^i \partial_{u^i}\right), \quad \mathcal{Z}(Y) = Y(y)\,\partial_y - Y' \sum_{i=3}^{d} \left(u^i \partial_{u^i}\right).$$

Gauge Lie algebra

- To descrive the gauge actions on the fibres $\mathbf{J}_0^k$, we remark that they generate two nilpotent Lie algebras of vector fields on $\mathbf{J}_0^k$:

$$\mathcal{L}_1^k = \left\langle \xi_1 = \mathcal{Z}_k\left(x\right), ..., \xi_{k+1} = \mathcal{Z}_k\left(x^{k+1}\right)\right\rangle,$$
$$\mathcal{L}_2^k = \left\langle \theta_1 = \mathcal{Z}_k\left(y\right), ..., \theta_{k+1} = \mathcal{Z}_k\left(y^{k+1}\right)\right\rangle,$$

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- with the following commutation relations

$$\begin{aligned} [\xi_i, \xi_j] &= (j-i)\,\xi_{i+j-1}, \text{if } i+j-1 \leq k+1, \text{ and 0 otherwise}, \\ [\theta_i, \theta_j] &= (j-i)\,\theta_{i+j-1}, \text{if } i+j-1 \leq k+1, \text{ and 0 otherwise}, \\ [\xi_i, \theta_j] &= 0, \text{for all } i, j. \end{aligned}$$

Theorem

The k-th prolongations $\mathcal{Z}_k(X, Y)$ of the vector fields $\mathcal{Z}(X, Y)$ have the following form:

$$\mathcal{Z}_k(X, Y) = X(x)\,\partial_x + Y(y)\,\partial_y + \sum_{i=1}^{k+1} X^{(i)}\frac{\xi_i}{i!} + \sum_{j=1}^{k+1} Y^{(j)}\frac{\theta_j}{j!}. \qquad (4)$$

Theorem

There are $\nu_0 = d - 3$ *independent natural differential invariants of zero order for ordered planar d-webs , and*

$$\nu_k = (d-2)\,\frac{(k+1)\,(k+2)}{2} - 2\,(k+1) \qquad (5)$$

independent natural differential invariants of orders less or equal k*, when* $k \geq 1$.

Corollary

The Hilbert function HF for the field of rational invariants of planar d-webs equals $HF(k) = N_k - N_{k-1}$, *that is, the number of independent invariants of exact order* k,

$$\begin{aligned} HF(k) &= (k+1)(d-2) - 2, \text{ for } k > 1, \\ HF(1) &= 2d - 7, \; HF(0) = d - 3. \end{aligned} \tag{6}$$

Important remark

The Lie-Tresse theorem states that the field of all rational differential invariants of the planar webs is an *algebraic diffiety* with charts that organized as algebraic differential equations.
That is, they do have two independent ("basic") invariants, say b_1, b_2, some number of ("dependent") invariants, say $a_1, ..., a_m$, and all other invariants are just rational functions of these invariants and theirTresse derivatives $\frac{d^k a_i}{d^j b_1 d^{k-j} b_{2.}}$.
There are also relations (syzygies) between them. Thus, taking their restriction on a planar web, we'll get some algebraic differential equations.

Planar 3-webs

The above theorems show that, in the case $d = 3$, we have the first non-trivial invariants in the order $k = 3$.
Namely, we have

1. Two independent invariants, say q, p, in the order $k = 3$, that generate all invariants in order ≤ 3.
2. Three independent invariants, say w_1, w_2, w_3, in the order $k = 4$, and these 5 invariants generate all invariants in order ≤ 4.
3. Four independent invariants in the exact order $k = 5$, and these 9 invariants generate all invariants in order ≤ 5.

Taking Tresse derivatives

$$\frac{dw_i}{dq}, \frac{dw_i}{dp}, i = 1, 2, 3;$$

we get 6 invariants in order $k = 5$.
Therefore, there are two relations of the form:

$$\sum_{i=1}^{3} \left(A_i \left(w_1, w_2, w_3, q, p \right) \tfrac{dw_i}{dq} + B_i \left(w_1, w_2, w_3, q, p \right) \tfrac{dw_i}{dp} \right)$$

$$+C \left(w_1, w_2, w_3, q, p \right) = 0$$

with rational coefficients A_i, B_i, C.

$$q = \frac{\left(u_{0,0}^2 u_{1,2} - 3 u_{0,0} u_{0,1} u_{1,1} - u_{0,0} u_{0,2} u_{1,0} + 3 u_{0,1}^2 u_{1,0}\right)^2}{\left(u_{0,0} u_{1,1} - u_{0,1} u_{1,0}\right)^3 u_{0,0}}, \tag{7}$$

$$p = \frac{\left(u_{0,0}^2 u_{2,1} - u_{0,0} u_{0,1} u_{2,0} - u_{0,0} u_{1,0} u_{1,1} + u_{0,1} u_{1,0}^2\right)^2 u_{0,0}}{\left(u_{0,0} u_{1,1} - u_{0,1} u_{1,0}\right)^3},$$

$$w_1 = \frac{\left(u_{3,1}u_{0,0}^3 + \left(-u_{0,1}u_{3,0} - 2u_{1,1}u_{2,0}\right)u_{0,0}^2 + \left(2u_{0,1}u_{1,0}u_{2,0} + u_{1,0}^2u_{1,1}\right) u \right.}{\left(u_{0,0}u_{1,1} - u_{0,1}u_{1,0}\right)^2}$$

$$w_2 = \frac{u_{2,2}u_{0,0}^3 + \left(-3u_{0,1}u_{2,1} - u_{0,2}u_{2,0} - u_{1,0}u_{1,2}\right)u_{0,0}^2 + \left(3u_{0,1}^2u_{2,0} + 3u_{0,1}\right.}{\left(u_{0,0}u_{1,1} - u_{0,1}u_{1,0}\right)^2}$$

$$-3\frac{u_{1,0}^2u_{0,1}^2}{\left(u_{0,0}u_{1,1} - u_{0,1}u_{1,0}\right)^2},$$

$$w_3 = \frac{u_{0,0}^2u_{1,3} + \left(-6u_{0,1}u_{1,2} - 4u_{0,2}u_{1,1} - u_{0,3}u_{1,0}\right)u_{0,0} + \left(15u_{0,1}^2u_{1,1} + 10\right.}{\left(u_{0,0}u_{1,1} - u_{0,1}u_{1,0}\right)^2}$$

$$-15\frac{u_{1,0}u_{0,1}^3}{u_{0,0}\left(u_{0,0}u_{1,1} - u_{0,1}u_{1,0}\right)^2},$$

Theorem

The field of rational differential invariants of ordered planar three webs is generated by two of the 3-rd order invariants q, p*, three of the 4-rd order invariants* w_1, w_2, w_3 *and their Tresse derivatives*

$$\frac{d^{k+l} w_i}{dq^k \; dp^l}.$$

We say that a planar 3-web is *regular* if invariants q, p are in general position, or if the restriction 2-form $\widehat{d}q \wedge \widehat{d}p$ on the web surface is a volume form in the domain of consideration.

Example

The following 3-web $W_3 = \{x, y, xy\}$ is singular.

3-web syzygy equation

- The Tresse derivatives $w_{i,j}\frac{dw_i}{dq_j}$ are linear in the fifth order derivatives $u_{i,5-i}$ and do not contain $u_{0,5}, u_{5,0}$. Thus we look for the syzygies in the form:

$$\sum_{i=1}^{3}\sum_{j=1}^{2} A_{ij} w_{i,j} + A_0 = 0. \tag{9}$$

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- We get

$$A_{11}^{1} = -\frac{\left(3q_1^2 - 2w_3\right)\left(3q_2^2 - 2w_1\right)}{9q_2^4 - 12q_2^2 w_1 + 4w_1^2}, \tag{10}$$

$$A_{12}^{1} = -\frac{9q_2^3 q_1 - 6q_2^2 w_2 - 6q_2 q_1 w_1 + 6q_2^2 + 4w_1 w_2 - 4w_1}{9q_2^4 - 12q_2^2 w_1 + 4w_1^2}, \tag{11}$$

$$A_{21}^{1} = \frac{9q_2^3 q_1 - 6q_2^2 w_2 - 6q_2 q_1 w_1 + 18q_2^2 + 4w_1 w_2 - 12w_1}{9q_2^4 - 12q_2^2 w_1 + 4w_1^2}, \tag{12}$$

$$A_{31}^{1} = 0, A_{32}^{1} = 0, A_{22}^{1} = 1 \tag{13}$$

Two differential syzygies

$$\sum_{i=1}^{3}\sum_{j=1}^{2} A_{ij}^{1} w_{i,j} = \frac{(4w_2+2)\,q_2 - 4q_1 w_1}{3q_2^2 - 2w_1}, \tag{17}$$

$$\sum_{i=1}^{3}\sum_{j=1}^{2} A_{ij}^{2} w_{i,j} =$$

$$\frac{12q_2^3 w_3 + 60q_1\, q_2^2 + \left(\left(-12q_1^2 - 8w_3\right) w_1 - 8w_2^2 + 4w_2 + \right.}{\left(3q_2^2 - 2w_1\right)^2}$$

Theorem

The field of rational differential invariants of regular ordered planar three webs is generated by two of the 3-rd order invariants q, p, *three of the 4-rd order invariants* w_1, w_2, w_3 *and their Tresse derivatives* $\frac{d^{k+l} w_i}{dq^k \ dp^l}$. *There are two differential syzygies and all other syzygies are the Tresse derivatives of them.*

4-webs

- In the case of 4-webs, we have $d = 4$, and

$$\begin{aligned} HF(0) &= 1, HF(1) = 1, \\ HF(k) &= 2k, \ \ k \geq 2. \end{aligned} \tag{19}$$

Thus, we have

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- Four independent invariants, say w_1, w_2, w_3, w_4 in the pure order $k = 2$, and all these 6 invariants together with Tresse derivatives $\frac{d^k w_i}{dq^j dp^{k-j}}$ generate all invariants in the domains, where $\widehat{d}p \wedge \widehat{d}q \neq 0$.

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- In order $k = 3$, we have $HF(3) = 6$ independent invariants of pure order 3 and eight Tresse derivatives $\frac{dw_i}{dq}, \frac{dw_i}{dp}$. Therefore, in this case, we have two syzygies of the first order and they generate all others.

Summarizing, we get the following

Theorem

The field of rational differential invariants of ordered planar four webs is generated by two invariants q, p, of order zero and one respectively, by four invariants of the 2-rd order w_1, w_2, w_3, w_4 and their Tresse derivatives $\frac{d^{k+l} w_i}{dq_1^k \, dq_2^l}$. There are two differential syzygies of the first order, and all other syzygies are the Tresse derivatives of these.

Similar to the case of 3 webs we have

Theorem

The field of rational differential invariants of ordered planar four webs is generated by invariants:

$$
\begin{aligned}
q &= \frac{v_{0,0}}{u_{0,0}},\ p = \frac{u_{0,0}\,v_{0,1} - u_{0,1}\,v_{0,0}}{u_{0,0}\left(u_{0,0}\,v_{1,0} - u_{1,0}\,v_{0,0}\right)}, \qquad (20)\\
w_1 &= \frac{\left(u_{0,0}\,u_{1,1} - u_{0,1}\,u_{1,0}\right)u_{0,0}}{\left(u_{0,0}\,v_{1,0} - u_{1,0}\,v_{0,0}\right)^2},\\
w_2 &= \frac{\left(u_{0,0}^2\,v_{2,0} - u_{0,0}\,u_{1,0}\,v_{1,0} - u_{0,0}\,u_{2,0}\,v_{0,0} + u_{1,0}^2\,v_{0,0}\right)u_{0,0}}{\left(u_{0,0}\,v_{1,0} - u_{1,0}\,v_{0,0}\right)^2},\\
w_3 &= \frac{u_{0,0}^2\,v_{1,1} - u_{0,0}\,u_{0,1}\,v_{1,0} - u_{1,0}\,v_{0,1}\,u_{0,0} + u_{0,1}\,u_{1,0}\,v_{0,0}}{\left(u_{0,0}\,v_{1,0} - u_{1,0}\,v_{0,0}\right)^2},\\
w_4 &= \frac{u_{0,0}^2\,v_{0,2} - 3u_{0,1}\,v_{0,1}\,u_{0,0} - u_{0,0}\,u_{0,2}\,v_{0,0} + 3v_{0,0}\,u_{0,1}^2}{u_{0,0}\left(u_{0,0}\,v_{1,0} - u_{1,0}\,v_{0,0}\right)^2},
\end{aligned}
$$

Theorem

The syzygies in the field of rational differential invariants of ordered planar four webs is generated by two syzygies

$$qw_{1.q} + pw_{2,q} - -((qw_1 - w_3)\,p + w_4)w_{2,p} + w_{3,q} - (w_{3,p} - qw_{1,p})\,(- \,(p$$

$$q\left(w_2\,p^2 + (2qw_1 - 2w_3)\,p + w_4\right)w_{1,q} + p\,((qw_1 - w_3)\,p + w_4)\,w_{2,1} + ((q$$

$$-\left(w_2\,p^2 + (2qw_1 - 2w_3)\,p + w_4\right)w_{3,q} + w_{4,q} - (pw_2 + qw_1 - w$$

$$(-2q^2w_1^2 + (4qw_3 + 2p)\,w_1 + 2w_2w_4 - 2w_3^2)(pw_2 + qw_1$$

of the first order and their Tresse derivatives.

5-webs

- We have

$$\begin{aligned} HF(0) &= 2, HF(1) = 3, \\ HF(k) &= 3k+1, \ \ k \geq 2. \end{aligned} \tag{23}$$

Thus, we expect

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- In order $k=3$, we have $HF(3)=10$ independent invariants of pure order 3. Nine of them we get as the Tresse derivatives $\frac{d^2s_i}{dq^2}, \frac{d^2s_i}{dp^2}, \frac{d^2s_i}{dqdp}$ and two invariants as derivatives $\frac{dr}{dq}, \frac{dr}{dp}$. Therefore, we get a syzygy in this order.

Theorem

The field of rational differential invariants of ordered planar 5-webs is generated by two invariants q, p, of order zero , by three invariants of the first order s_1, s_2, s_3, one invariant r of order two their their Tresse derivatives $\frac{d^{k+l}s_i}{dq_1^k\ dq_2^l}, \frac{d^{k+l}r}{dq_1^k\ dq_2^l}$. In order 3 there is one differential syzygy, and all other syzygies are the Tresse derivatives of it.

Theorem

The field of rational differential invariants of ordered planar 5-webs is generated by two invariants q, p, of order zero ,

$$q = \frac{v_{0,0}}{u_{0,0}}, p = \frac{w_{0,0}}{u_{0,0}} \tag{24}$$

by three invariants of the first order s_1, s_2, s_3,

$$\begin{aligned} s_1 &= \frac{u_{0,0}v_{0,1} - u_{0,1}v_{0,0}}{u_{0,0}\left(u_{0,0}v_{1,0} - v_{0,0}u_{1,0}\right)}, s_2 = \frac{u_{0,0}w_{1,0} - w_{0,0}u_{1,0}}{u_{0,0}v_{1,0} - v_{0,0}u_{1,0}}, \\ s_3 &= \frac{u_{0,0}w_{0,1} - u_{0,1}w_{0,0}}{u_{0,0}\left(u_{0,0}v_{1,0} - v_{0,0}u_{1,0}\right)} \end{aligned} \tag{25}$$

and one invariant r of order two

$$r = \frac{\left(u_{0,0}u_{1,1} - u_{0,1}u_{1,0}\right)u_{0,0}}{\left(u_{0,0}v_{1,0} - v_{0,0}u_{1,0}\right)^2}. \tag{26}$$

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Models

- The fields of rational differential and natural invariants of planar webs organized (due to Lie-Tresse theorem) in the following way: we have two basic invariants, say p, q , and some additional invariants, say $w_1, ..., w_m$, such that all other invariants are rational functions of $p, q, w_1, ..., w_m$, and the Tresse derivatives $\frac{dw_i}{dp,}, \frac{dw_i}{dq,}$, subjected to some relations.

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- We consider the space $\mathbb{R}^m \times \mathbb{R}^2$ with coordinates $(W_1, ..., W_m, P, Q)$. Then web $\mathcal{W}$ defines the following mapping: $\phi_{\mathcal{W}} : \mathbb{R}^2 \to \mathbb{R}^m \times \mathbb{R}^2, \phi_{\mathcal{W}}(x, y) =$ $(W_1 = w_1(\mathcal{W}), ..., W_m = w_m(\mathcal{W}), P = p(\mathcal{W}), Q = q(\mathcal{W}))$.

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- The mapping $\phi_{\mathcal{W}}$ is an embedding for regular webs and its image $\Sigma_{\mathcal{W}} \subset \mathbb{R}^m \times \mathbb{R}^2$ we call the normal form (or model) of the web $\mathcal{W}$.

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- We consider the space $\mathbb{R}^m \times \mathbb{R}^2$ with coordinates $(W_1, ..., W_m, P, Q)$. Then web $\mathcal{W}$ defines the following mapping: $\phi_{\mathcal{W}} : \mathbb{R}^2 \to \mathbb{R}^m \times \mathbb{R}^2, \phi_{\mathcal{W}}(x, y) =$ $(W_1 = w_1(\mathcal{W}), ..., W_m = w_m(\mathcal{W}), P = p(\mathcal{W}), Q = q(\mathcal{W}))$.
- We say that the web $\mathcal{W}$ is *regular* if $\widehat{d}p \wedge \widehat{d}p \neq 0$,on $\mathcal{W}$, or when $p(\mathcal{W})$ and $q(\mathcal{W})$ are coordinates in the domain under consideration.
- The mapping $\phi_{\mathcal{W}}$ is an embedding for regular webs and its image $\Sigma_{\mathcal{W}} \subset \mathbb{R}^m \times \mathbb{R}^2$ we call the normal form (or model) of the web $\mathcal{W}$.
- 3-webs.
- 4-webs.
- 5-webs

Webs equivalence

Theorem

Two regular ordered webs $\mathcal{W}_1$ and $\mathcal{W}_2$ are equivalent in the domains of consideration if and only if their models coincide.