Lie Pseudo-Groups and Coverings of Differential Equations

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Talk at the Workshop on
Geometry of PDEs and Integrability,
Teplice nad Bečvou, Czech Republic,
17 October 2013

Lie pseudo-groups

A pseudo-group $\mathfrak{G}$ on a manifold M is a collection of local diffeomorphisms $\Phi\colon \mathcal{U} \to \hat{\mathcal{U}}$, $\Phi\colon x \mapsto \hat{x}$ such that

1) if $\Phi \in \mathfrak{G}$, $\Psi \in \mathfrak{G}$, and their composition $\Psi \circ \Phi$ is defined, then $\Psi \circ \Phi \in \mathfrak{G}$;
2) $\Phi \in \mathfrak{G} \Rightarrow \Phi^{-1} \in \mathfrak{G}$;
3) $\mathrm{id}_M \in \mathfrak{G}$;
4) if $\Phi\colon \mathcal{U} \to \hat{\mathcal{U}}$ belongs to $\mathfrak{G}$, then for any open subset $\mathcal{V} \subset \mathcal{U}$ $\Phi|_{\mathcal{V}} \in \mathfrak{G}$;
5) $\Phi_1\colon \mathcal{U}_1 \to \hat{\mathcal{U}}_1$ $\Phi_2\colon \mathcal{U}_2 \to \hat{\mathcal{U}}_2$ belong to $\mathfrak{G}$, and $\Phi_1|_{\mathcal{U}_1\cap\mathcal{U}_2} = \Phi_2|_{\mathcal{U}_1\cap\mathcal{U}_2}$, then $\Phi\colon \mathcal{U}_1 \cup \mathcal{U}_2 \to \hat{\mathcal{U}}_1 \cup \hat{\mathcal{U}}_2$ defined by $\Phi|_{\mathcal{U}_1} = \Phi_1$ and $\Phi|_{\mathcal{U}_2} = \Phi_2$ belongs to $\mathfrak{G}$.

A pseudo-group $\mathfrak{G}$ is called a Lie pseudo-group, if

6) the functions $\hat{x} = \Phi(x)$ are local analytic solutions of a system of PDEs (Lie equations of the pseudo-group $\mathfrak{G}$)

$$R\left(x, \Phi(x), \frac{\partial \Phi(x)}{\partial x}, \dots, \frac{\partial^{\#I}\Phi(x)}{\partial x^I}\right) = 0.$$

Lie pseudo-groups

EXAMPLE: Lie groups = finite Lie pseudo-groups

EXAMPLE: $\mathrm{SDiff}(\mathbb{R}^2)$:

$\Phi\colon (x, y) \mapsto (\hat{x}, \hat{y}), \quad \Phi^*(d\hat{x} \wedge d\hat{y}) = dx \wedge dy,$

Lie equations: $$\frac{\partial \hat{x}}{\partial x}\frac{\partial \hat{y}}{\partial y} - \frac{\partial \hat{x}}{\partial y}\frac{\partial \hat{y}}{\partial x} = 1$$

Lie pseudo-groups

- Sophus Lie's infinitesimal method :
  - infinitesimal generators;
  - linearized Lie equations;
  - integration.
- Éli Cartan's method :
  - Maurer-Cartan forms;
  - structure equations;
  - algebraic operations, differentiation.

EXAMPLE (cont'd): Maurer–Cartan forms for $\mathrm{SDiff}(\mathbb{R}^2)$:

$\omega^1, \omega^2 \in \Omega^1(\mathbb{R}^2 \times \mathrm{SL}(\mathbb{R}, 2))$,

$$\begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} = Q \cdot \begin{pmatrix} dx \\ dy \end{pmatrix}, \qquad Q \in \mathrm{SL}(\mathbb{R}, 2).$$

$\Psi \colon \mathbb{R}^2 \times \mathrm{SL}(\mathbb{R}, 2) \to \mathbb{R}^2 \times \mathrm{SL}(\mathbb{R}, 2)$, $\quad \Psi \colon \ (x, y, Q) \ \mapsto \ (\hat{x}, \hat{y}, \hat{Q})$

$$\Psi^* \begin{pmatrix} \hat{\omega}^1 \\ \hat{\omega}^2 \end{pmatrix} = \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix}, \ \text{ where } \ \begin{pmatrix} \hat{\omega}^1 \\ \hat{\omega}^2 \end{pmatrix} = \hat{Q} \cdot \begin{pmatrix} d\hat{x} \\ d\hat{y} \end{pmatrix} \quad \Longleftrightarrow$$

$$(\hat{x}, \hat{y}, \hat{Q}) = \left(f(x, y), g(x, y), Q \cdot \begin{pmatrix} f_x(x, y) & f_y(x, y) \\ g_x(x, y) & g_y(x, y) \end{pmatrix}^{-1} \right),$$

$$\det \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = 1 \qquad (\ = \text{Lie equations})$$

Structure equations for $\mathrm{SDiff}(\mathbb{R}^2)$:

$$\begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} = Q \begin{pmatrix} dx \\ dy \end{pmatrix}$$

$$\Longrightarrow$$

$$d\begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} = dQ \wedge \begin{pmatrix} dx \\ dy \end{pmatrix} = dQ \cdot Q^{-1} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix}$$

$$= \begin{pmatrix} \pi^1 & \pi^2 \\ \pi^3 & -\pi^1 \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix}$$

$$\Longrightarrow$$

$$\left\{ \begin{array}{l} d\omega^1 = \pi^1 \wedge \omega^1 + \pi^2 \wedge \omega^2, \\ d\omega^2 = \pi^3 \wedge \omega^1 - \pi^1 \wedge \omega^2. \end{array} \right.$$

Cartan's method

Third fundamental Lie's theorem in Cartan's form: For a Lie pseudo-group there exists a collection of Maurer–Cartan forms with involutive and compatible structure equations

$$d\omega^i = A^i_{\varrho k}\pi^\varrho \wedge \omega^k + \tfrac{1}{2}\, B^i_{jk}\omega^j \wedge \omega^k, \quad B^i_{jk} = -B^i_{kj}.$$

The coefficients $A^i_{\varrho k}$, B^i_{jk} are either constants or functions of invariants U^1, ... , U^s of the pseudo-group. In the latter case

$$dU^s = C^s_k(U)\,\omega^k.$$

Third inverse fundamental Lie's theorem in Cartan's form: For a given involutive and compatible system of structure equations there exists a collection of 1-forms ω^1, ..., ω^m and functions U^1, ..., U^s satisfying this system. The forms ω^1, ..., ω^m are Maurer–Cartan forms of a Lie pseudo-group, and the functions U^1, ..., U^s are invariants of this pseudo-group.

Cartan's method

Details:

- Cartan É. Œuvres Complètes. Paris: Gauthier - Villars, 1953
- Vasil'eva M.V. Structure of Infinite Lie Groups of Transformations. Moscow: MSPI, 1972 (in Russian)
- Stormark O. Lie's Structural Approach to PDE Systems. Cambridge: CUP, 2000

EXAMPLE: contact transformations on $J^q(\mathbb{R}^n \times \mathbb{R}^m)$.

$q = 2$, $m = 1$, Maurer–Cartan forms:

$$\Theta_0 = a\,(du - u_i\,dx^i),$$
$$\Theta_i = a\,B_i^j\,(du_j - u_{jk}\,dx^k) + g_i\,\Theta_0,$$
$$\Theta_{ij} = a\,B_i^k\,B_j^l\,(du_{kl} - u_{klm}\,dx^m) + s_{ij}\,\Theta_0 + w_{ij}^k\,\Theta_k,$$
$$\Xi^i = b_j^i\,dx^j + c^i\,\Theta_0 + f^{ij}\,\Theta_j,$$

where $\quad b_k^i\,B_j^k = \delta_j^i, \quad f^{ik} = f^{ki},\ s_{ij} = s_{ji}, \quad w_{ij}^k = w_{ji}^k,$
$u_{klm} = u_{lkm} = u_{kml}.$

Structure equations:

$$d\Theta_0 = \Phi_0^0 \wedge \Theta_0 + \Xi^i \wedge \Theta_i,$$
$$d\Theta_i = \Phi_i^0 \wedge \Theta_0 + \Phi_i^k \wedge \Theta_k + \Xi^k \wedge \Theta_{ik},$$
$$d\Theta_{ij} = \Phi_i^k \wedge \Theta_{kj} - \Phi_0^0 \wedge \Theta_{ij} + \Upsilon_{ij}^0 \wedge \Theta_0 + \Upsilon_{ij}^k \wedge \Theta_k + \Xi^k \wedge \Theta_{ijk},$$
$$d\Xi^i = \Phi_0^0 \wedge \Xi^i - \Phi_k^i \wedge \Xi^k + \Psi^{i0} \wedge \Theta_0 + \Psi^{ik} \wedge \Theta_k$$

Symmetry pseudo-groups of differential equations

EXAMPLE: contact symmetries of a differential equation $\mathcal{E} \subset J^q(\mathbb{R}^n \times \mathbb{R}^m)$.

Details:

- Fels M., Olver P.J., Acta Appl. Math., 1998, 51, 161–213
- M., J. Phys. A: Math. Gen., 2002, 35, 2965–2977
- M., J. Math. Sci., 2006, 135, 2680–2694

Applications:

- equivalence problems for DEs;
- symmetry analysis of classes of DEs;
- differential invariants, automorphic systems of DEs, Vessiot group splittings of DEs;
- Darboux integrability;
- coverings of DEs ;
- recursion operators;
- ...

- $n = 2$, finite-dimensional coverings:

  Bryant R.L., Griffiths P.A., Duke Math. J., 1995, 78, 531–676

- $n > 2$, infinite-dimensional coverings:

  M., J. Geom. Phys., 2009, 59, 1461–1475

EXAMPLE: integrable extension of the structure equations for $\mathrm{SDiff}(\mathbb{R}^2)$.

The structure equations

$$\left\{ \begin{array}{l} d\omega^1 = \pi^1 \wedge \omega^1 + \pi^2 \wedge \omega^2, \\ d\omega^2 = \pi^3 \wedge \omega^1 - \pi^1 \wedge \omega^2 \end{array} \right.$$

We have: $d(\omega^1 \wedge \omega^2) = 0 \quad \Longrightarrow$

$$\left\{ \begin{array}{l} d\omega^1 = \pi^1 \wedge \omega^1 + \pi^2 \wedge \omega^2, \\ d\omega^2 = \pi^3 \wedge \omega^1 - \pi^1 \wedge \omega^2, \\ d\omega^3 = \omega^1 \wedge \omega^2. \end{array} \right.$$

The rdDym equation

$$u_{ty} = u_x\, u_{xy} - u_y\, u_{xx}$$

- Błaszak M., Phys. Lett. A, 2002, 297, 191–195
- Pavlov M.V., J. Math. Phys., 2003, 44, 4134–4156
- Ovsienko V., Adv. Pure Appl. Math., 2010, 1, 7–17

The 4-dimensional generalization of the rdDym equation

$$u_{ty} = u_z\, u_{xy} - u_y\, u_{xz}$$

- Martínez Alonso L., Shabat A.B., Theor. Math. Phys. 2004, 140, 1073–1085

4-dimensional generalization of rdDym equation

The structrure equations for the symmetry pseudo-group

$$
\begin{aligned}
d\theta_0 &= \eta_1 \wedge \theta_0 + \xi^1 \wedge \theta_1 + \xi^2 \wedge \theta_2 + \xi^3 \wedge \theta_3 + \xi^4 \wedge \theta_4,\\
d\theta_1 &= (\eta_1 - \eta_2) \wedge \theta_1 - \eta_3 \wedge \theta_2 + \theta_0 \wedge (\xi^4 - \theta_{24}) + \xi^1 \wedge \theta_{11} + \xi^2 \wedge \theta_{12} - \xi^3 \wedge \theta_{24} + \xi^4 \wedge \theta_{14},\\
d\theta_2 &= (\eta_1 - \eta_4) \wedge \theta_2 + \theta_0 \wedge (\xi^3 - \theta_{23}) + \xi^1 \wedge \theta_{12} + \xi^2 \wedge \theta_{22} + \xi^3 \wedge \theta_{23} + \xi^4 \wedge \theta_{24},\\
d\theta_3 &= -\xi^1 \wedge \theta_{24} + \xi^2 \wedge \theta_{23} + \xi^3 \wedge \theta_{33} + \xi^4 \wedge \theta_{34},\\
d\theta_4 &= (\eta_4 - \eta_2) \wedge \theta_4 + \eta_3 \wedge \theta_3 + \theta_3 \wedge \theta_4 + \xi^1 \wedge \theta_{14} + \xi^2 \wedge \theta_{24} + \xi^3 \wedge \theta_{34} + \xi^4 \wedge \theta_{44},\\
d\xi^1 &= \eta_2 \wedge \xi^1,\\
d\xi^2 &= \eta_3 \wedge \xi^1 + \eta_4 \wedge \xi^2,\\
d\xi^3 &= (\eta_1 + \theta_3) \wedge \xi^3 - (\eta_3 - \theta_4) \wedge \xi^4,\\
d\xi^4 &= (\eta_1 + \eta_2 - \eta_4) \wedge \xi^4,\\
d\theta_{11} &= (\eta_1 - 2\,\eta_2) \wedge \theta_{11} - 2\,\eta_3 \wedge \theta_{12} + \eta_9 \wedge \xi^2 - \eta_{10} \wedge (\theta_0 + \xi^3) + \eta_{11} \wedge \xi^1 + \eta_{12} \wedge \xi^4\\
&\quad - 2\,\theta_1 \wedge \theta_{24} + \theta_2 \wedge (\xi^3 + \theta_{14}),\\
d\theta_{12} &= (\eta_1 - \eta_2 - \eta_4) \wedge \theta_{12} - \eta_3 \wedge \theta_{22} + \eta_5 \wedge (\theta_0 + \xi^3) + \eta_7 \wedge \xi^2 + \eta_9 \wedge \xi^1 + \eta_{10} \wedge \xi^4\\
&\quad + \theta_1 \wedge (\theta_{23} - \xi^3),\\
d\theta_{14} &= (\eta_4 - 2\,\eta_2) \wedge (\xi^3 + \theta_{14}) - (\eta_1 + \theta_3) \wedge \xi^3 + \eta_3 \wedge (3\,\xi^4 - 2\,\theta_{24}) + \eta_{10} \wedge \xi^2\\
&\quad + (\eta_{12} - 2\,\theta_1) \wedge \xi^1 - \theta_4 \wedge \xi^4,\\
d\theta_{22} &= (\eta_1 - 2\,\eta_4) \wedge \theta_{22} - \eta_5 \wedge \xi^4 + \eta_6 \wedge (\theta_0 + \xi^3) + \eta_7 \wedge \xi^1 + \eta_8 \wedge \xi^2 + \theta_2 \wedge (\theta_{23} - \xi^3),\\
d\theta_{23} &= (\eta_1 + \theta_3) \wedge \xi^3 - (\eta_3 - \theta_4) \wedge \xi^4 + \eta_4 \wedge (\xi^3 - \theta_{23}) + \eta_5 \wedge \xi^1 + \eta_6 \wedge \xi^2,
\end{aligned}
$$

. . . .

4-dimensional generalization of rdDym equation

THEOREM 1. The above structure equations have no contact inte- grable extensions with constant coefficients. Each their contact integrable extension with one additional invariant is contact - equivalent to the following one

$$d\omega_0 = (\eta_1+\omega_2)\wedge\omega_0+W_1\,(\omega_2+\theta_4)\wedge\xi^1+W_1\,\omega_1\wedge\xi^2-(\omega_1+\theta_3)\wedge\xi^3 \\ +W_1\,\omega_2\wedge\xi^4,$$

$$dW_1 = W_1\,(\eta_1-\eta_4)+Z_1\,(\omega_0-W_1\,\xi^2+\xi^3),$$

where Z_1 is an arbitrary parameter.

REMARK. $Z_1 = 0 \quad \Rightarrow \quad$ a non-removable parameter in the corresponding covering

4-dimensional generalization of rdDym equation

COROLLARY 1. The 4D rdDym equation has two coverings defined by systems

$$\left\{ \begin{array}{lcl} q_t & = & u_z\, q_x - \lambda^{-1}\, q_z, \\ q_y & = & \lambda\, u_y\, q_x, \end{array} \right.$$

and

$$\left\{ \begin{array}{lcl} q_t & = & u_z\, q_x - q^{-1}\, q_z, \\ q_y & = & u_y\, q\, q_x. \end{array} \right.$$

The parameter λ in the first system is non-removable.

REMARK. Nonlocal symmetries, nonlocal conservation laws, Kähler structure.

Recursion operators

- Recursion operators as integro-differential operators
  - P.J. Olver, 1977
  - B. Fuchssteiner, 1979
  - V.E. Zakharov, B.G. Konopelchenko, 1984
  - A.S. Fokas. P.M. Santini, 1986
  - . . .
  - J.A. Sanders, J.P. Wang, 2001
- Recursion operators as Bäcklund autotransformations for tangent coverings
  - C.J. Papachristou, B.K. Harrison, 1988
  - C.J. Papachristou, 1990
  - G.A. Guthrie, 1994
  - I.S. Krasil′shchik, P.H.M. Kersten, 1994
  - M. Marvan, 1996
  - A. Sergyeyev, 2000
  - . . .
  - C.J. Papachristou, B.K. Harrison, 2010
  - I.S. Krasil′shchik, A.M. Verbovetsky, R. Vitolo, 2011
  - M. Marvan, A. Sergyeyev, 2012

Recursion operators

- A holonomic section $\varphi\colon \mathcal{E} \to \mathcal{T}(\mathcal{E})$ of the tangent covering is a symmetry of $\mathcal{E}$.
- A recursion operator for symmetries of $\mathcal{E}$ is a Bäcklund autotransformation of $\mathcal{T}(\mathcal{E})$

Details and examples:

- M. Marvan. Differential Geometry and Applications. Brno: Masaryk University, 1996, 393–402
- I.S. Krasil′shchik, A.M. Verbovetsky, R. Vitolo, Acta Appl. Math., 2012, 120, 199–218

The technique of the contact integrable extensions can be adapted to the problem of finding recursion operators for symmetries of PDEs.

EXAMPLE: The 4D rdDym equation

$$u_{ty} = u_z\, u_{xy} - u_y\, u_{xz} \tag{1}$$

The linearization:

$$v_{ty} = u_z\, v_{xy} - u_y\, v_{xz} + u_{xy}\, v_z - u_{xz}\, v_y. \tag{2}$$

The tangent covering: the infinite prolongation of system (1), (2) equipped with the restrictions of the extended total derivatives.

Recursion operators

The structure equations for the symmetry pseudo-group of the tangent covering is a CIE of the structure equations for the symmetry pseudo-group of the 4D rdDym equation:

$$
\begin{aligned}
d\zeta_0 &= (\eta_1 - \zeta_3 + X_3\,\xi^1 + X_2\,\xi^2 + X_1\,\xi^3 + X_4\,\xi^4) \wedge \zeta_0 - \zeta_1 \wedge \xi^1 - \zeta_2 \wedge \xi^2 - \zeta_3 \wedge \xi^3 \\
&\qquad -\zeta_4 \wedge \xi^4 + \theta_0 \wedge (\zeta_3 - X_1\,\xi^3 - X_4\,\xi^4), \\
d\zeta_1 &= -\theta_{24} \wedge \zeta_0 + (\eta_1 - \eta_2 - \zeta_3) \wedge \zeta_1 - \eta_3 \wedge \zeta_2 - \theta_1 \wedge \zeta_3 + (\eta_{34} + X_1\,(\zeta_1 + \theta_1)) \wedge \theta_0 \\
&\qquad +\eta_{32} \wedge \xi^1 + \eta_{33} \wedge \xi^2 + \eta_{34} \wedge \xi^3 + \eta_{35} \wedge \xi^4, \\
d\zeta_2 &= \zeta_0 \wedge (\theta_0 + \theta_{23}) + \zeta_1 \wedge (X_2\,\xi^1 - X_1\,\xi^4) + (\eta_1 - \eta_4 + X_3\,\xi^1 - X_1\,\theta_0 + X_4\,\xi^4) \wedge \zeta_2 \\
&\qquad +(\zeta_2 + \theta_2) \wedge \zeta_3 + (\eta_{24} - \eta_{34} - X_1\,\theta_1 - X_4\,\theta_2 - X_5\,(\theta_{23} + \theta_0) \wedge \xi^4 + \eta_{33} \wedge \xi^1 \\
&\qquad +\eta_{36} \wedge \xi^2 + \eta_{37} \wedge \xi^3 + (\eta_{37} + X_1\,\theta_2) \wedge \theta_0, \\
d\zeta_3 &= (\eta_{34} + X_1\,(\zeta_1 + \theta_1) - X_3\,\zeta_3) \wedge \xi^1 + (\eta_{37} + \zeta_0 + X_1\,(\zeta_2 + \theta_2) - X_3\,\zeta_3) \wedge \xi^2 \\
&\qquad +\eta_{38} \wedge \xi^3 + \eta_{39} \wedge \xi^4, \\
d\zeta_4 &= (\eta_3 - X_4\,\xi^3) \wedge \zeta_3 - (\zeta_3 + \eta_2 - \eta_4 - X_3\,\xi^1 - X_2\,\xi^2 - X_1\,\xi^3) \wedge \zeta_4 + \eta_{40} \wedge \xi^4 \\
&\qquad +(\eta_{35} - \zeta_0 + X_4\,(\zeta_1 + \theta_1)) \wedge \xi^1 - (\eta_{34} + X_1\,(\zeta_1 + \theta_1) + X_5\,\theta_{23} - \theta_{24}) \wedge \xi^2 \\
&\qquad +(\eta_{39} + X_4\,\theta_3 - X_1\,\theta_4) \wedge \xi^3.
\end{aligned}
$$

THEOREM 2. The structure equations for the symmetry pseudo-group of the tangent covering of the 4D rdDym equation have a CIE

$$d\zeta_5 = W_2\,(\xi^2 - X_5\,\xi^1)\wedge\zeta_5 + (\theta_{24} - W_2\,\zeta_4)\wedge\xi^1 + (W_2\,\zeta_3 - \theta_{23})\wedge\xi^2 + \zeta_6\wedge\xi^3 + \zeta_7\wedge\xi^4,$$

$$dW_2 = W_2\,(\zeta_3 - \zeta_5 - \eta_4 - (X_3 + X_5\,W_2)\,\xi^1 - (X_2 - W_2)\,\xi^2) + Y_2\,\xi^3 + Y_3\,\xi^4.$$

The inverse third fundamental Lie theorem $\Longrightarrow$

$$\zeta_5 = \frac{1}{w}\left(dw - \frac{u_y\,v_z - u_z\,v_y + (u_z\,u_{xy} - u_y\,u_{xz})\,w}{u_y}\,dt - \frac{u_{xy}\,w - v_y}{u_y}\,dx - w_y\,dy - w_z\,dz\right).$$

$\zeta_5 = 0 \quad \Longrightarrow$

$$\left\{ \begin{array}{rcl} w_t & = & u_y^{-1}\,(u_y\,v_z - u_z\,v_y + (u_z\,u_{xy} - u_y\,u_{xz})\,w)\,, \\ w_x & = & u_y^{-1}\,(u_{xy}\,w - v_y) \end{array} \right.$$

Exclude $v \quad \Longrightarrow \quad w$ is solution to the linearization $\quad \Longrightarrow$

COROLLARY 2. System

$$\left\{ \begin{array}{rcl} D_t(\psi) & = & u_y^{-1}\,(u_y\,D_z(\varphi) - u_z\,D_y(\varphi) + (u_z\,u_{xy} - u_y\,u_{xz})\,\psi)\,, \\ D_x(\psi) & = & u_y^{-1}\,(u_{xy}\,\psi - D_y(\varphi)) \end{array} \right.$$

defines a recursion operator for symmetries of 4D rdDym equation. The inverse operator is defined by

$$\left\{ \begin{array}{rcl} D_y(\varphi) & = & -u_y\,D_x(\psi) + u_{xy}\,\psi, \\ D_z(\varphi) & = & D_t(\psi) - u_z\,D_x(\psi) + u_{xz}\,\psi. \end{array} \right.$$

Ovsienko's system

$$u_{ty} = u_x\, u_{xy} - u_y\, u_{xx},$$

$$v_{ty} = u_x\, v_{xy} - u_y\, v_{xx} + 2\,(u_{xx}\, v_y - u_{xy}\, v_x) + 2\,\kappa\,(2\, u_x\, u_{xy} - u_y\, u_{xx})$$

- $\kappa = 1$: Ovsienko V. Adv. Pure Appl. Math. 2010, 1, 7–17: bi-Hamiltonian system on the dual space to the looped Virasoro algebra.
- $\kappa = 0$: the cotangent covering of the rdDym equation

THEOREM 3. The structure equations for the symmetry pseudo-group of the rdDym equations have a CIE defined by the forms

$$\omega_0 = p_{00}\,(dv - v_t\,dt - v_x\,dx - v_y\,dy) + \sum_{j=0}^{3} q_{0j}\,\theta_j,$$

$$\begin{aligned}\omega_i = {} & p_{i0}\,(dv - v_t\,dt - v_x\,dx - v_y\,dy) + p_{i1}\,(dv_t - v_{tt}\,dt - v_{tx}\,dx - R\,dy) \\ & + p_{i2}\,(dv_x - v_{tx}\,dt - v_{xx}\,dx - v_{xy}\,dy) + p_{i3}\,(dv_y - R\,dt - v_{xy}\,dx - v_{yy}\,dy) \\ & + \sum_{j=0}^{3} q_{ij}\,\theta_j + \sum{}^{*} s_{ijk}\,\theta_{jk}, \qquad i \in \{1,2,3\},\end{aligned}$$

where R is the r.h.s. of the second equation of the system, θ_j and θ_{jk} are the M-C forms of the symmetry pseudo-group of the first equations, while $\sum{}^{*}$ means summation over all j, k such that $1 \le j \le k \le 3$ and $(j,k) \neq (1,3)$. In these forms $p_{00} \neq 0$, $\det(p_{ij}) \neq 0$ for $1 \le i \le 3$, $1 \le j \le 3$, and $q_{01}^2 + q_{02}^2 + q_{03}^2 \neq 0$.

Ovsienko's system

$$
\begin{aligned}
d\omega_0 &= \eta_{12}\wedge\omega_0+\xi^1\wedge\omega_1+\xi^2\wedge\omega_2+\xi^3\wedge\omega_3+\eta_{13}\wedge\theta_0+\eta_{14}\wedge\theta_1+\eta_{15}\wedge\theta_2+\eta_{16}\wedge\theta_3,\\
d\omega_1 &= (2\,\theta_{22}+3\,X_1\,\xi^3)\wedge\omega_0+(\eta_1+\eta_{12}+(X_1+2)\,\theta_3-2\,\theta_{23}+2\,X_2\,\xi^3)\wedge\omega_1\\
&\quad +3\,(\theta_2+\tfrac{1}{2}\,(X_1+2)\,\xi^3)\wedge\omega_2+3\,\eta_{13}\wedge\theta_1+\eta_{14}\wedge(2\,\theta_1+\theta_{11})+\eta_{15}\wedge(X_1\,\theta_1+\theta_{12})\\
&\quad +\eta_{16}\wedge(\theta_{22}-3\,(\theta_2+\xi^3))+\eta_{17}\wedge(\theta_0+\xi^3)+\eta_{18}\wedge\xi^1+\eta_{19}\wedge\xi^2,\\
d\omega_2 &= \omega_0\wedge(2\,\xi^1-X_1\,\xi^2)+\tfrac{1}{2}\,(4-3\,X_1)\,\omega_1\wedge\xi^1+2\,\omega_3\wedge\xi^3+\eta_{17}\wedge\xi^2+\eta_{19}\wedge\xi^1\\
&\quad +\omega_2\wedge(\theta_{23}-\eta_{12}+\xi^1-\tfrac{1}{2}\,(X_1+2)\,(\theta_3+\xi^2)-X_2\,\xi^3)+\eta_{13}\wedge(3\,(\theta_2-\tfrac{1}{2}\,X_1\,\theta_0)+5\,\xi^3)\\
&\quad +\eta_{14}\wedge(\theta_0+\theta_{12}-\tfrac{1}{2}\,X_1\,\theta_1+\xi^3)+\eta_{16}\wedge(\theta_{23}-\tfrac{1}{2}\,(X_1+6)\,\theta_3-3\,\xi^2-X_2\,\xi^3)\\
&\quad +\eta_{15}\wedge(\theta_{22}-\tfrac{1}{4}\,X_1\,(3\,X_1+2)\,\theta_0+\tfrac{3}{2}\,X_1\,\theta_2+\tfrac{1}{2}\,(7\,X_1+4)\,\xi^3),\\
d\omega_3 &= -(X_1\,\omega_0+\tfrac{1}{2}\,(3\,X_1+2)\,\omega_2)\wedge\xi^1+\omega_3\wedge(\eta_1-\eta_{12}+3\,\theta_3+\tfrac{1}{2}\,(X_1+6)\,\xi^2)\\
&\quad +\eta_{13}\wedge(3\,\theta_3+5\,\xi^2)+\eta_{14}\wedge(2\,\theta_2-\tfrac{3}{2}\,X_1\,\theta_0+\theta_{22}+\xi^2)+\eta_{17}\wedge\xi^1+\eta_{20}\wedge\xi^3\\
&\quad +\eta_{15}\wedge((X_1+2)\,\theta_3+\theta_{23}+\tfrac{1}{2}\,(7\,X_1+4)\,\xi^2)+\eta_{16}\wedge(\theta_{33}-3\,\xi^1-X_2\,\xi^2).
\end{aligned}
$$

$$
X_1 = u_y\,u_{xxy}\,u_{xy}^{-2}, \qquad X_2 = u_y\,u_{xxx}\,(u_{xy}\,u_{yy}-u_y\,u_{xyy})\,u_{xy}^{-4}
$$

Ovsienko's system

The coefficients of the CIE do not depend on $\kappa \implies$

COROLLARY 3. There exists a Bäcklund transformation of the form

$$\begin{cases} \tilde{t} = h_1(t,x,y), \\ \tilde{x} = h_2(t,x,y), \\ \tilde{y} = h_3(t,x,y), \\ \tilde{u} = h_4(t,x,y,u), \\ \tilde{v} = h_5(t,x,y,u,u_t,u_x,u_y,v), \\ \tilde{v}_{\tilde{t}} = h_6(t,x,y,u,u_t,u_x,u_y,u_{tt},u_{tx},u_{xx},u_{xy},u_{yy},v,v_t,v_x,v_y), \\ \tilde{v}_{\tilde{x}} = h_7(t,x,y,u,u_t,u_x,u_y,u_{tt},u_{tx},u_{xx},u_{xy},u_{yy},v,v_t,v_x,v_y), \\ \tilde{v}_{\tilde{y}} = h_8(t,x,y,u,u_t,u_x,u_y,u_{tt},u_{tx},u_{xx},u_{xy},u_{yy},v,v_t,v_x,v_y) \end{cases}$$

mapping Ovsienko's system to the cotangent covering of the rdDym equation.