On integrable classes of surfaces in Euclidean space

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Surfaces in Euclidean space

The first and the second fundamental form associated with a surface $\mathbf{r}(x^1, x^2)$ in Euclidean space are

$$g_{ij} = \mathbf{r}_{x^i} \cdot \mathbf{r}_{x^j}, \quad h_{ij} = \mathbf{r}_{x^i x^j} \cdot \mathbf{n},$$

where $\mathbf{n}$ is the unit normal. The vectors $\mathbf{r}_{x^1}, \mathbf{r}_{x^2}, \mathbf{n}$ satisfy the *Gauss–Weingarten* equations

$$\begin{pmatrix} \mathbf{r}_{x^1} \\ \mathbf{r}_{x^2} \\ \mathbf{n} \end{pmatrix}_{x^k} = A_k \begin{pmatrix} \mathbf{r}_{x^1} \\ \mathbf{r}_{x^2} \\ \mathbf{n} \end{pmatrix}, \quad k = 1, 2,$$

where A_1, A_2 are certain matrices constructed from the components g_{ij}, h_{ij} and their derivatives. The compatibility conditions

$$D_{x^2} A_1 - D_{x_1} A_2 + [A_1, A_2] = 0$$

of the Gauss–Weingarten equations are equivalent to the *Gauss–Mainardi–Codazzi–Peterson* system. The latter system consists of three equations in six unknowns g_{ij}, h_{ij}. In curvature coordinates x, y we have $g = u^2 \,\mathrm{d}x^2 + v^2 \,\mathrm{d}y^2$, $h = pu^2 \,\mathrm{d}x^2 + qv^2 \,\mathrm{d}y^2$, where p, q are principal curvatures; then the Gauss–Mainardi–Codazzi–Peterson system assumes the form

$$\begin{aligned}
&(p - q)uu_y + u^2 p_y = 0, \\
&(p - q)vv_x - v^2 q_x = 0, \\
&uu_{yy} + vv_{xx} - \frac{vu_x v_x}{u} - \frac{uu_y v_y}{v} + u^2 v^2 pq = 0.
\end{aligned}$$

These are 3 equations in 4 unknowns. We have one condition at our disposal to permit integrability. To be geometric, the extra condition must be written in terms of invariants of surfaces with respect to rigid movements and reparameterizations. The simplest such invariants are the principal curvatures p, q; others can be obtained by invariant differentiations $(1/u)D_x$, $(1/v)D_y$ (i.e., differentiation along the lines of curvature parameterized by the arc length).

Criterion of integrability

In the case of two independent variables x, y and a matrix Lie algebra $\mathfrak{g}$, a $\mathfrak{g}$-valued *zero curvature representation* (ZCR) for a system of PDE $\mathcal{E}$ is a form $\alpha = A\,dx + B\,dy$ with $A, B \in \mathfrak{g}$ such that

$$D_y A - D_x B + [A, B] = 0 \quad \operatorname{mod} \mathcal{E}.$$

A *gauge transformation* with respect to a gauge matrix S

$$A \longmapsto (D_x S)S^{-1} + SAS^{-1}, \quad B \longmapsto (D_y S)S^{-1} + SBS^{-1}.$$

Such a ZCR is the compatibility condition for the linear system $D_x\Psi = A\Psi$, $D_y\Psi = B\Psi$; the gauge transformation corresponds to $\Psi \longmapsto S\Psi$.

To be indicative of integrability, the ZCR should depend on a parameter that is not removable by the gauge transformation.

Example. Gauss–Weingarten equations = a parameterless zero curvature representation of the Gauss–Mainardi–Codazzi equations.

The spectral parameter problem

Given a parameterless ZCR, when a parameter can be incorporated?

A cohomological solution

To solve the spectral parameter problem in a given Lie algebra:

1. Consider the conditions, resulting from expanding the ZCR in terms of the (prospective) spectral parameter $A = \sum_i A_i\lambda^i$, $B = \sum_i B_i\lambda^i$

$$\begin{aligned}
&D_y A_0 - D_x B_0 + [A_0, B_0] = 0 \quad \text{(the seed ZCR)},\\
&D_y A_1 - D_x B_1 + [A_1, B_0] + [A_0, B_1] = 0,\\
&D_y A_2 - D_x B_2 + [A_2, B_0] + [A_1, B_1] + [A_0, B_2] = 0,\\
&\quad\vdots
\end{aligned}$$

2. The second equation can be rewritten as $(D_x - \mathrm{ad}_{A_0})B_1 = (D_y - \mathrm{ad}_{B_0})A_1$, meaning that $A_1\,\mathrm{d}x + B_1\,\mathrm{d}y$ is a cocycle. Coboundaries $A_1 = (D_x - \mathrm{ad}_{A_0})S$, $B_1 = (D_y - \mathrm{ad}_{B_0})S$ correspond to removable λ. Denote by H^1 the cohomology group $\{\text{cocycles}\}/\{\text{coboundaries}\}$. Then H^1 contains obstructions to expandability of $A_0\,\mathrm{d}x + B_0\,\mathrm{d}y$ to $A(\lambda)\,\mathrm{d}x + B(\lambda)\,\mathrm{d}y$: If $H^1 = 0$, then no non-removable λ can be incorporated.

3. For an arbitrary generator $A_1\,\mathrm{d}x + B_1\,\mathrm{d}y$ of H^1, attempt to compute A_i, B_i in successive steps, $i \ge 2$. If any of these steps fails, then no non-removable λ can be incorporated.

4. For the remaining candidates compute the full zero curvature representation $A(\lambda)\,\mathrm{d}x + B(\lambda)\,\mathrm{d}y$. Information obtained in the previous steps helps to cut off branches.

M.M., On the spectral parameter problem, *Acta Appl. Math.* **109** (2010) 239–255.

M.M., A direct procedure to compute zero-curvature representations. The case $\mathfrak{sl}_2$, in: *Secondary Calculus and Cohomological Physics,* Proc. Conf. Moscow, 1997 (ELibEMS, 1998) pp. 10.

The classification project for surfaces

We depart from the always-existing zero curvature representation equivalent to the Gauss–Weingarten system, and solve the spectral parameter problem as explained above. Computations are conveniently performed in curvature coordinates.

Weingarten surfaces

H. Baran and M. Marvan, Classification of integrable Weingarten surfaces possessing an sl(2)-valued zero curvature representation, *Nonlinearity* **23** (2010) 2577–2597.

By definition, *Weingarten surfaces* satisfy a functional relation between the principal curvatures p, q. We ask which functional relations $f(p,q) = 0$ determine an integrable class of Weingarten surfaces.

Let ρ, σ denote the principal radii of curvature, $\rho = 1/p$, $\sigma = 1/q$. In the Weingarten case, $\rho = \rho(\sigma)$, the Mainardi–Codazzi subsystem can be explicitly solved, while the Gauss equation assumes the form

$$R_{xx} + S_{yy} + T = 0,$$

where R, S, T are some functions of σ.

The result

Weingarten surfaces determined by an explicit dependence $\rho(\sigma)$ possess a one-parametric zero curvature representation if the *determining equation*

$$\rho''' = \frac{3}{2\rho'}\rho''^2 + \frac{\rho' - 1}{\rho - \sigma}\rho'' + 2\frac{(\rho' - 1)\rho'(\rho' + 1)}{(\rho - \sigma)^2}$$

holds (the prime denotes $\mathrm{d}/\mathrm{d}\sigma$).

The general solution $\rho(\sigma)$ is given by the elliptic integral

$$\rho + \sigma = \frac{1}{m}\int^{m(\rho-\sigma)} \frac{1 \pm s^2}{\sqrt{1 + 2cs^2 + s^4}}\,\mathrm{d}s.$$

Here m is a scaling parameter, the integration constant is an offsetting parameter, and c is a "true" parameter. There exists a link to deformations of quadrics of revolution.

Summary of special cases

All the special cases when the above elliptic integral reduces to elementary functions were known in the XIX century. Except tubular surfaces ($\rho = \text{const}$ or $\sigma = \text{const}$), they are

No.	relation	integrable equation
1.	$\rho + \sigma = 0$	$z_{xx} + z_{yy} + \mathrm{e}^z = 0$
2.	$\rho\sigma = 1$	$z_{xx} + z_{yy} - \sinh z = 0$
3.	$\rho\sigma = -1$	$z_{xx} - z_{yy} + \sin z = 0$
4.	$\rho - \sigma = \sinh(\rho + \sigma)$	$(\tanh z - z)_{xx} + (\coth z - z)_{yy} + \operatorname{csch} 2z = 0$
5.	$\rho - \sigma = \sin(\rho + \sigma)$	$(\tan z - z)_{xx} + (\cot z + z)_{yy} + \csc 2z = 0$
6.	$\rho - \sigma = 1$	$z_{xx} + (1/z)_{yy} + 2 = 0$
7.	$\rho - \sigma = \tanh \rho$	$\frac14 (\sinh z - z)_{xx} + (\coth \frac12 z)_{yy} + \coth \frac12 z = 0$
8.	$\rho - \sigma = \tan \rho$	$\frac14 (\sin z - z)_{xx} + (\cot \frac12 z)_{yy} + \cot \frac12 z = 0$
9.	$\rho - \sigma = \coth \rho$	$\frac14 (\sinh z + z)_{xx} - (\tanh \frac12 z)_{yy} + \tanh \frac12 z = 0$
10.	$\rho - \sigma = -\cot \rho$	$\frac14 (\sin z + z)_{xx} + (\tan \frac12 z)_{yy} + \tan \frac12 z = 0$

up to scaling and offsetting. The first three integrable equations are well known, the fourth and fifth have been solved by Darboux, the others are new.

The constant astigmatism equation

H. Baran and M. Marvan, On integrability of Weingarten surfaces: a forgotten class, *J. Phys. A: Math. Theor.* **42** (2009) 404007.

The sixth equation

$$z_{xx} + \left(\frac{1}{z}\right)_{yy} + 2 = 0 \tag{1}$$

is called the *constant astigmatism equation*. The ZCR with a non-removable parameter λ is given by

$$A = \begin{pmatrix} \frac{1}{2}\frac{\lambda^2 z_x}{(\lambda^2-1)z} + \frac{1}{2}\frac{\lambda z_y}{\lambda^2-1} & \frac{\lambda^2 z}{\lambda^2-1} \\ \frac{1}{\lambda^2-1} & -\frac{1}{2}\frac{\lambda^2 z_x}{(\lambda^2-1)z} - \frac{1}{2}\frac{\lambda z_y}{\lambda^2-1} \end{pmatrix},$$

$$B = \begin{pmatrix} \frac{1}{2}\frac{\lambda z_x}{(\lambda^2-1)z^2} + \frac{1}{2}\frac{\lambda^2 z_y}{(\lambda^2-1)z} & \frac{\lambda}{\lambda^2-1} \\ \frac{\lambda}{(\lambda^2-1)z} & -\frac{1}{2}\frac{\lambda z_x}{(\lambda^2-1)z^2} - \frac{1}{2}\frac{\lambda^2 z_y}{(\lambda^2-1)z} \end{pmatrix}.$$

Open problem. Solve the constant astigmatism equation by spectral methods.

A link to equiareal patterns on the sphere

A. Hlaváč and M.M., Another integrable case in two-dimensional plasticity, *J. Phys. A: Math. Theor.*, to appear.

The geometric meaning of the variable z can be seen from the third fundamental form, which turns out to be

$$\mathrm{d}\mathbf{n}\cdot\mathrm{d}\mathbf{n} = z\,\mathrm{d}x^2 + \frac{1}{z}\,\mathrm{d}y^2. \tag{2}$$

Since $\mathrm{d}\mathbf{n}\cdot\mathrm{d}\mathbf{n}$ coincides with the first fundamental form of the Gaussian sphere $\mathbf{n}(x, y)$, it follows that one obtains a parameterization of the unit sphere with the following properties:

1. the coordinate lines are orthogonal;
2. the parameterization is area preserving.

These properties characterize *equiareal patterns* on the sphere. Evenly distributed coordinate lines cover the surface with curvilinear rectangles of equal area.

Example. The well-known Archimedean projection of the cylinder $(\cos y, \sin y, x)$ onto an inscribed sphere (see Fig. 1) provides an example of an orthogonal equiareal pattern. We have

$$g_{\text{Arch}} = \frac{\mathrm{d}x^2}{1-x^2} + (1-x^2)\,\mathrm{d}y^2,$$

where $z = 1/(1-x^2)$ is a solution of the constant astigmatism equation which corresponds to von Lilienthal surfaces.

A converse statement says that if (2) is an orthogonal equiareal parameterization of the unit sphere, then z is a solution of the the constant astigmatism equation (1). The proof is by computation of the Gaussian curvature of the sphere through the Brioschi formula.

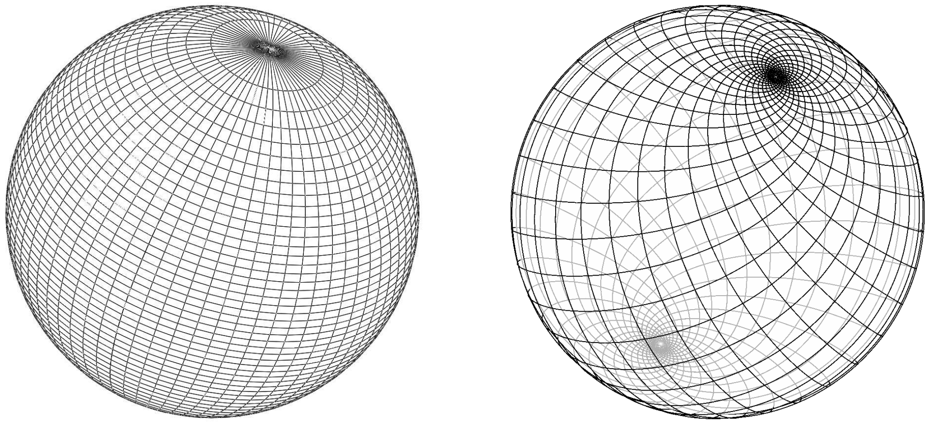

Figure 1: The Archimedean equiareal parameterisation of the sphere (left) and a slip line field composed of loxodromes (right)

A link to two-dimensional plasticity

The orthogonal equiareal patterns were first observed in connection with plane plasticity (Boussinesq 1872, Sadowsky 1941) under Tresca yield condition. Conversely, given an orthogonal equiareal pattern (2), then the two-dimensional tensor σ with components

$$\sigma_1^1 = \tfrac{1}{2} \ln z, \quad \sigma_2^1 = \sigma_1^2 = 0, \quad \sigma_2^2 = \tfrac{1}{2} (\ln z - 2). \tag{3}$$

has the necessary properties of the stress tensor for ductile materials in the absence of "body" forces:

1. the symmetry, $\sigma_{ij} = \sigma_{ji}$,
2. the equilibrium equation $\sigma^{ij}_{;j} = 0$,
3. the Tresca yield condition $\sigma_1^1 - \sigma_2^2 = \text{const}$.

The pattern itself is then composed of principal stress lines.

The physical meaning is that a plastic material *yields* under stress; yielding occurs along slip lines, which are positioned at the angle of $\frac{1}{4}\pi$ to the principal stress lines.

Therefore, by a *slip line field* associated with the orthogonal equiareal pattern (2) on a surface S we shall mean a parameterization ξ, η such that the angle between ∂_x and ∂_ξ as well as the angle between ∂_y and ∂_η is equal to $\frac{1}{4}\pi$.

Continuing the example of Archimedean parameterization, we easily see that the corresponding orthogonal net of slip lines is formed by the $\pm 45°$ loxodromes (lines of constant bearing); see Fig. 1(b) or model No. 249 in the Göttingen collection of mathematical models `http://www.uni-math.gwdg.de/modellsammlung/`.

A link to the sine-Gordon equation

To obtain solutions of the constant astigmatism equation depending on an arbitrary number of parameters we can extend the well-known Bäcklund transformation of the sine-Gordon equation

$$\omega_{\xi\eta} = \tfrac{1}{2} \sin 2\omega.$$

The Bäcklund transform of a surface $\mathbf{r}(\xi,\eta)$ is related to another sine-Gordon solution $\omega^{(\lambda)}$, satisfying compatible first-order equations

$$\omega^{(\lambda)}_\xi = \omega_\xi + \lambda \sin(\omega^{(\lambda)} + \omega), \quad \omega^{(\lambda)}_\eta = -\omega_\eta + \frac{1}{\lambda}\sin(\omega^{(\lambda)} - \omega). \tag{4}$$

Here λ is a constant called the Bäcklund parameter. Particularly useful is Bianchi's superposition principle

$$\tan\frac{\omega^{(\lambda_1\lambda_2)} - \omega}{2} = \frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}\tan\frac{\omega^{(\lambda_1)} - \omega^{(\lambda_2)}}{2} \tag{5}$$

If a general solution of system (4) is known for every value of the Bäcklund parameter λ, solutions $\omega^{(\lambda_1\lambda_2\cdots\lambda_s)}$ depending on any finite number of Bäcklund parameters and integration constants can be obtained step by step, by purely algebraic manipulations, via (5).

In the particular case of $\lambda = \pm 1$ the Bäcklund transformation coincides with Bianchi's complementarity relation. Consequently, the superposition formula (5) yields a method to obtain abundant pairs of complementary sine-Gordon solutions $\omega^{(\lambda_1\lambda_2\cdots\lambda_s)}$ and $\omega^{(\lambda_1\lambda_2\cdots\lambda_s 1)}$.

Surfaces of constant astigmatism are easy to obtain from a pair of complementary pseudospherical surfaces $\mathbf{r}$ and $\mathbf{r}^{(1)}$. Denote

$$\tilde{\mathbf{n}} = \mathbf{r}^{(1)} - \mathbf{r} = \frac{\sin(\omega-\tilde\omega)}{\sin(2\omega)}\mathbf{r}_\xi + \frac{\sin(\omega+\tilde\omega)}{\sin(2\omega)}\mathbf{r}_\eta. \tag{6}$$

Then $\tilde{\mathbf{n}}$ is a unit vector tangent to both surfaces $\mathbf{r}$ and $\mathbf{r}^{(1)}$ and determines what is called a pseudospherical congruence. Normal surfaces of this congruence are the constant astigmatism surfaces sought. The following proposition is essentially due to Bianchi.

Proposition 1. *Let $\omega^{(1)}(\xi,\eta,c)$ be a general solution of system (4), where we set $\lambda = 1$ and c denotes the integration constant. Then $\tilde{\mathbf{r}} = \mathbf{r} - f\tilde{\mathbf{n}}$, where $f = \ln(\mathrm{d}\omega^{(1)}/\mathrm{d}c)$ and $\tilde{\mathbf{n}}$ is the unit vector given by formula (6), is a surface of constant astigmatism having surfaces $\mathbf{r}$ and $\mathbf{r}^{(1)}$ as evolutes.*

Proposition 1 implies that the constant astigmatism surfaces $\tilde{\mathbf{r}} = \mathbf{r} - f\tilde{\mathbf{n}}$ can be found by purely algebraic manipulations and differentiation once a one-parameter family of pseudopotentials $\omega^{(1)}$ is known. The coordinates ξ,η have a geometric meaning of a slip line field.

Proposition 2. *If S is a constant astigmatism surface, then the asymptotic coordinates on the focal surfaces of S correspond to slip line fields on the Gaussian image of S.*

Proposition 1 as such yields neither a solution of the constant astigmatism equation nor an orthogonal equiareal pattern on the sphere $\tilde{\mathbf{n}}$.

Proposition 3. *Let $\omega^{(1)}(\xi,\eta,c)$ be a general solution of system (4), where $\lambda = 1$ and c denotes the integration constant, let $f = \ln(\mathrm{d}\omega^{(1)}/\mathrm{d}c)$ and $x = \mathrm{d}f/\mathrm{d}c$. Let $y(\xi,\eta)$ be a solution of the system*

$$y_\xi = \mathrm{e}^{-f}\sin(\omega + \omega^{(1)}), \quad y_\eta = \mathrm{e}^{-f}\sin(\omega - \omega^{(1)}). \tag{7}$$

Then x,y are adapted curvature coordinates on the surface $\tilde{\mathbf{r}}$. Moreover, if $z = \mathrm{e}^{-2f}$, then $z(x,y)$ is a solution of the constant astigmatism equation (1). Finally, $z\,\mathrm{d}x^2 + \mathrm{d}y^2/z$ is an orthogonal equiareal pattern on the unit sphere $\tilde{\mathbf{n}}$, while ξ,η is the associated slip line field (see above).

One can also obtain superposition formulas for f, x, y similar to formula (5). Given two sine-Gordon solutions ω and $\omega^{(\lambda)}$ related by the Bäcklund transformation $\mathcal{B}^{(\lambda)}$, let $f^{(\lambda)}$, $x^{(\lambda)}$, $y^{(\lambda)}$ denote functions, called *associated potentials*, satisfying the compatible equations

$$\begin{aligned}
f_\xi^{(\lambda)} &= \lambda \cos(\omega^{(\lambda)} + \omega), & f_\eta^{(\lambda)} &= \frac{1}{\lambda}\cos(\omega^{(\lambda)} - \omega), \\
x_\xi^{(\lambda)} &= \lambda \mathrm{e}^{f^{(\lambda)}} \sin(\omega^{(\lambda)} + \omega), & x_\eta^{(\lambda)} &= \frac{1}{\lambda}\mathrm{e}^{f^{(\lambda)}} \sin(\omega^{(\lambda)} - \omega), \\
y_\xi^{(\lambda)} &= \lambda \mathrm{e}^{-f^{(\lambda)}} \sin(\omega^{(\lambda)} + \omega), & y_\eta^{(\lambda)} &= -\frac{1}{\lambda}\mathrm{e}^{-f^{(\lambda)}} \sin(\omega^{(\lambda)} - \omega).
\end{aligned} \tag{8}$$

Proposition 4. *Let $\omega, \omega^{(\lambda_1)}, \omega^{(\lambda_2)}, \omega^{(\lambda_1\lambda_2)}$ be four sine-Gordon solutions related by the Bianchi superposition principle* (5)*. Then the associated potentials $f^{(\lambda_1\lambda_2)}$, $x^{(\lambda_1\lambda_2)}$, $y^{(\lambda_1\lambda_2)}$ corresponding to the pair $\omega^{(\lambda_1)}, \omega^{(\lambda_1\lambda_2)}$ are related to the associated potentials $f^{(\lambda_2)}$, $x^{(\lambda_2)}$, $y^{(\lambda_2)}$ corresponding to the pair $\omega, \omega^{(\lambda_2)}$ by formulas*

$$\begin{aligned}
f^{(\lambda_1\lambda_2)} &= f^{(\lambda_2)} - \ln\Big(2\cos(\omega^{(\lambda_1)} - \omega^{(\lambda_2)}) - \frac{\lambda_1}{\lambda_2} - \frac{\lambda_2}{\lambda_1}\Big), \\
x^{(\lambda_1\lambda_2)} &= \frac{\lambda_1\lambda_2}{\lambda_1^2 - \lambda_2^2}\left(x^{(\lambda_2)} - \frac{2\lambda_1\lambda_2 \sin(\omega^{(\lambda_1)} - \omega^{(\lambda_2)})}{\lambda_1^2 - 2\lambda_1\lambda_2\cos(\omega^{(\lambda_1)} - \omega^{(\lambda_2)}) + \lambda_2^2}\mathrm{e}^{f^{(\lambda_2)}}\right), \\
y^{(\lambda_1\lambda_2)} &= \Big(\frac{\lambda_1}{\lambda_2} - \frac{\lambda_2}{\lambda_1}\Big) y^{(\lambda_2)} - 2\mathrm{e}^{-f^{(\lambda_2)}} \sin(\omega^{(\lambda_1)} - \omega^{(\lambda_2)}),
\end{aligned} \tag{9}$$

up to an additive constant.

Example.

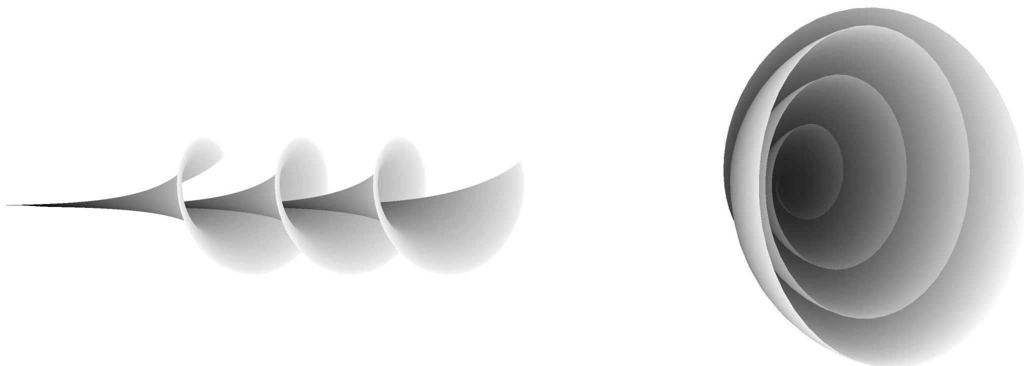

Figure 2: Dini's pseudospherical surface (left) and its constant astigmatism involute (right).

Lipschitz surfaces in principal coordinates

In 1887 Lipschitz presented a class of surfaces of constant astigmatism in terms of spherical coordinates related to the Gaussian image. Consider the unit sphere

$$\mathbf{n} = (\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta)$$

parameterised by the latitude θ and longitude ϕ, yet unknown functions of parameters x, y. The Lipschitz class is specified by allowing the *position angle* ω between $\mathbf{n}_\theta$ and $\mathbf{n}_x = \phi_x \mathbf{n}_\phi + \theta_x \mathbf{n}_\theta$ to depend solely on the latitude θ.

Proposition 5. *The general Lipschitz solution of the constant astigmatism equation* (1) *depends on four constants $h_{11}, h_{10}, h_{01}, h_{00}$ and consists of functions*

$$z = \frac{1 - h^2 + \sqrt{(1 - h^2)^2 - 4(H_1 h - H_2)^2}}{2(h_{11}x + h_{01})^2}, \tag{10}$$

where $h = h_{11}xy + h_{10}x + h_{01}y + h_{00}$, $H_1 = h_{11}$, $H_2 = h_{11}h_{00} - h_{10}h_{01}$. Formula (10) *covers all Lipshitz solutions except a particular solution*

$$z = \frac{1}{c_1 - (x - c_0)^2},$$

c_1, c_0 being arbitrary constants.

Actually, (10) is a symmetry-invariant solution of the constant astigmatism equation.

Proposition 6. *The general Lipschitz solution* (10) *satisfies*

$$h_{11}\mathfrak{s} + h_{01}\mathfrak{t}^x - h_{10}\mathfrak{t}^y = 0,$$

where $\mathfrak{t}^x = z_x$, $\mathfrak{t}^y = z_y$, $\mathfrak{s} = xz_x - yz_y + 2z$ are generators of the Lie symmetries of the constant astigmatism equation.

The orthogonal equiareal pattern corresponding to the general Lipschitz solution is given by

$$\theta = \arccos h,$$

$$\phi = -\frac{\ln(h_{11}x + h_{01})}{h_{11}} + \int \frac{1 - h^2 + \sqrt{(1 - h^2)^2 - 4(H_1 h - H_2)^2}}{2(H_1 h - H_2)(1 - h^2)} \, \mathrm{d}h.$$

The function ϕ can be expressed in terms of elementary functions if and only if

$$h_{00} = \pm 1 \quad \text{or} \quad h_{00} = \pm\frac{1 + h_{11}^2}{2h_{11}}.$$

For $h_{00} = 1$ we obtain the pattern on Fig. 3. In the other case ϕ and θ cannot be simultaneously real.

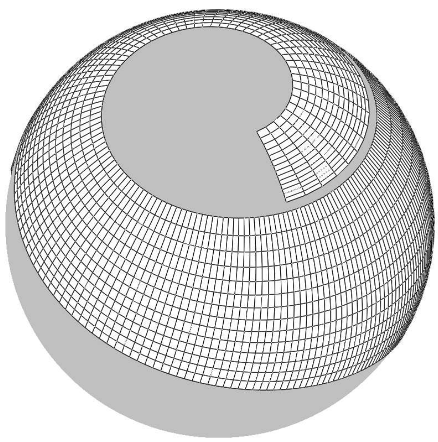

Figure 3: An orthogonal equiareal pattern on the sphere corresponding to the Lipschitz solution