Classification of integrable PDE in the differential geometry of surfaces

Michal Marvan

Silesian University in Opava, Czech Republic

A joint project with Hynek Baran and Artur Sergyeyev.

Introduction

Around 1970, soliton theory started, bringing new powerful integration methods for nonlinear PDE.

Open question. Which equations are integrable in the sense of soliton theory?

Up to now, indirect approaches have been the most successful

– singularity analysis;

– symmetry analysis.

However, the majority of classification problems in differential geometry appear to be beyond the scope of these methods.

Integrability criterion

Existence of a zero curvature representation depending on a nonremovable (spectral) parameter.

Given a system $\mathcal{E}$ of PDE in independent variables x, y, a Lie algebra $\mathfrak{g}$, a $\mathfrak{g}$-valued zero curvature representation for $\mathcal{E}$ is a form $\alpha = A\,dx + B\,dy$ with $A, B \in \mathfrak{g}$ such that

$$D_y A - D_x B + [A, B] = 0$$

as a consequence of the system $\mathcal{E}$.

Applications

– Zakharov–Shabat formulation of the inverse spectral transform,

– starting point to obtain explicit solutions,

– Bäcklund/Darboux transformations,

– nonlocal symmetries,

– recursion operators and hierarchies of symmetries.

Example

The mKdV equation $u_t + u_{xxx} - 6u^2 u_x = 0$ has an $\mathfrak{sl}_2$-valued zero curvature representation $A\,dx + B\,dt$ with

$$A = \begin{pmatrix} u & \lambda \\ 1 & -u \end{pmatrix},$$

$$B = \begin{pmatrix} -u_{xx} + 2u^3 - 4\lambda u & 2\lambda u_x + 2\lambda u^2 - 4\lambda^2 \\ -2u_x + 2u^2 - 4\lambda & u_{xx} - 2u^3 + 4\lambda u \end{pmatrix}.$$

Indeed, $D_t(A) - D_x(B) + [A, B] = (u_t + u_{xxx} - 6u^2 u_x) \cdot C$, where

$$C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Here λ is a parameter (the spectral parameter).

The problem

How to tell whether a given nonlinear system has a zero curvature representation?

The famous **Wahlquist–Estabrook** method

- algorithmizable under favourable conditions,
- usually not the case with serious classification problems.

The method used

Resources:

M.M., On zero curvature representations of partial differential equations, in: Differential Geometry and Its Applications, Proc. Conf. Opava, Czechoslovakia, Aug. 2428, 1992 (Silesian University, Opava, (1993) 103122.

M.M., A direct procedure to compute zero-curvature representations. The case $\mathfrak{sl}_2$, in: *Secondary Calculus and Cohomological Physics*, Proc. Conf. Moscow, 1997 (ELibEMS, 1998) pp. 10.

P. Sebestyén, Normal forms of irreducible $\mathfrak{sl}_3$-valued zero curvature representations, *Rep. Math. Phys* 55 (2005) No. 3, 435–445.

P. Sebestyén, On normal forms of irreducible $\mathfrak{sl}_n$-valued zero curvature representations, *Rep. Math. Phys* 62 (2008) No. 1.

Overview

Supposing A, B, C_l to be in a *normal form*, the determining system

$$(D_y A - D_x B + [A, B])|_{\mathcal{E}} = 0,$$

$$\sum_{I,l} (-\widehat{D})_I \left(\frac{\partial F^l}{\partial u_I^k} C_l \right) \Big|_{\mathcal{E}} = 0$$

has the following properties:

- is a system of differential equations in total derivatives;
- has the same number of unknowns as equations;
- is quasilinear in A, B and linear in C_l;
- impossible to solve without computer algebra;
- solution algorithms are resource demanding;
- tractable if A, B, C_l in a semisimple Lie algebra.

One successful example

M.M., Scalar second order evolution equations possessing an irreducible sl_2-valued zero curvature representation, *J. Phys. A: Math. Gen.* 35 (2002) 9431–9439.

Negatives of the method

- the calculations tend to be prohibitively resource-demanding;
- one-parametric families of zero curvature representations, which are characteristic of integrability, have to be selected from the vast corpus of calculation results.

Fortunately, helps to solve an important subproblem.

The spectral parameter problem

M.M., On the spectral parameter problem, *Acta Appl. Math.*, DOI 10.1007/s10440-009-9450-4.

Question. When a given zero curvature representation can be incorporated into a one-parameter family?

Warning. The family can exist only in a larger Lie algebra.

The method to solve the problem in a given Lie algebra:

1) compute cohomological obstructions, obtained when expanding the zero curvature representation in terms of the (prospective) spectral parameter;

2) use the information obtained in the first step to cut off branches when computing the full zero curvature representation.

Cutting off branches Image courtesy of Jiří Škarda

The integrable surfaces problem

Surfaces correspond to solutions of the Gauss–Mainardi–Codazzi equations (up to rigid motions).

Example. Pseudospherical surfaces $\leftrightarrow$ sine-Gordon equation.

A. Sym, Soliton surfaces and their applications. Soliton geometry from spectral problems, in: R. Martini, ed., *Geometric Aspects of the Einstein Equations and Integrable Systems,* Lecture Notes in Physics 239 (Springer, Berlin, 1985) 154–231.

To start with, we focus on Weingarten surfaces, i.e., immersed surfaces in $\mathbf{E}^3$ with a functional relation between the principal curvatures k_1, k_2.

Example. All rotation surfaces; constant Gaussian curvature surfaces; constant mean curvature surfaces.

Problem. Which functional relations $f(k_1, k_2) = 0$ determine an integrable class of Weingarten surfaces?

The Finkel–Wu conjecture

A well-known answer: Any linear relation between the mean curvature $\frac{1}{2}(k_1 + k_2)$ and the Gauss curvature $k_1 k_2$:

$$ak_1k_2 + b(k_1 + k_2) + c = 0$$

determines an integrable class (linear Weingarten surfaces).

Conjecture. The only class of integrable Weingarten surfaces are the linear Weingarten surfaces.

F. Finkel, On the integrability of Weingarten surfaces, in: A. Coley et al., ed., *Bäcklund and Darboux Transformations. The Geometry of Solitons*, AARMS-CRM Workshop, June 4-9, 1999, Halifax, N.S., Canada, (Amer. Math. Soc., Providence, 2001) 199–205.

Hongyou Wu, Weingarten surfaces and nonlinear partial differential equations, *Ann. Global Anal. Geom.* **11** (1993) 49–64.

Preliminaries

Parameterized by the lines of curvature, surfaces $\mathbf{r}(x, y)$ have the fundamental forms

$$\mathrm{I} = u^2 \, \mathrm{d}x^2 + v^2 \, \mathrm{d}y^2, \qquad \mathrm{II} = \frac{u^2}{\rho} \, \mathrm{d}x^2 + \frac{v^2}{\sigma} \, \mathrm{d}y^2.$$

where ρ, σ are the principal radii of curvature.

In the Weingarten case, $\rho = \rho(\sigma)$, the Mainardi–Codazzi subsystem can be explicitly solved. The full GMC system then reduces to the Gauss equation alone.

Moreover, the Gauss equation can be written in the form

$$R_{xx} + S_{yy} + T = 0,$$

where R, S, T are functions of σ.

A non-parametric zero curvature representation

The Gauss–Mainardi–Codazzi equations always posses a non-parametric zero curvature representation

$$A_0 = \begin{pmatrix} \dfrac{\mathrm{i}u_y}{2v} & -\dfrac{u}{2\rho} \\ \dfrac{u}{2\rho} & -\dfrac{\mathrm{i}u_y}{2v} \end{pmatrix}, \qquad B_0 = \begin{pmatrix} -\dfrac{\mathrm{i}v_x}{2u} & -\dfrac{\mathrm{i}v}{2\sigma} \\ -\dfrac{\mathrm{i}v}{2\sigma} & \dfrac{\mathrm{i}v_x}{2u} \end{pmatrix}$$

(x, y label the lines of curvature).

Question. Can we incorporate a parameter?

Answer. No, unless we impose a suitable additional condition.

Problem. Which *geometric* conditions imply integrability?

Results of the computation

Weingarten surfaces determined by an explicit dependence $\rho(\sigma)$ possess a one-parametric zero curvature representation if and only if the **determining equation**

$$\rho''' = \frac{3}{2\rho'}\rho''^2 + \frac{\rho' - 1}{\rho - \sigma}\rho'' + 2\frac{(\rho' - 1)\rho'(\rho' + 1)}{(\rho - \sigma)^2}$$

holds (the prime denotes $\mathrm{d}/\mathrm{d}\sigma$).

This equation has

– a general solution in terms of elliptic integrals;

– a number of special solutions when the relation between ρ and σ can be expressed in terms of elementary functions.

Surprise. All the special cases were known in the XIX century.

Corollary. The Finkel–Wu conjecture is false.

Summary of special cases

up to scaling and offsetting; ρ, σ are the principal radii of curvature.

relation	integrable equation
$\rho + \sigma = 0$	$z_{xx} + z_{yy} + \mathrm{e}^z = 0$
$\rho\sigma = 1$	$z_{xx} + z_{yy} - \sinh z = 0$
$\rho\sigma = -1$	$z_{xx} - z_{yy} + \sin z = 0$
$\rho - \sigma = \sinh(\rho + \sigma)$	$(\tanh z - z)_{xx} + (\coth z - z)_{yy} + \operatorname{csch} 2z = 0$
$\rho - \sigma = \sin(\rho + \sigma)$	$(\tan z - z)_{xx} + (\cot z + z)_{yy} + \csc 2z = 0$
$\rho - \sigma = 1$	$z_{xx} + (1/z)_{yy} + 2 = 0$
$\rho - \sigma = \tanh \rho$	$\frac{1}{4}(\sinh z - z)_{xx} + (\coth \frac{1}{2} z)_{yy} + \coth \frac{1}{2} z = 0$
$\rho - \sigma = \tan \rho$	$\frac{1}{4}(\sin z - z)_{xx} + (\cot \frac{1}{2} z)_{yy} + \cot \frac{1}{2} z = 0$
$\rho - \sigma = \coth \rho$	$\frac{1}{4}(\sinh z + z)_{xx} - (\tanh \frac{1}{2} z)_{yy} + \tanh \frac{1}{2} z = 0$
$\rho - \sigma = -\cot \rho$	$\frac{1}{4}(\sin z + z)_{xx} + (\tan \frac{1}{2} z)_{yy} + \tan \frac{1}{2} z = 0$

Surfaces of constant astigmatism

The relation $\rho - \sigma = \text{const}$ was among the special solutions.

H. Baran and M.M., On integrability of Weingarten surfaces: a forgotten class, *J. Phys. A: Math. Theor.* **42** (2009) 404007.

Popular among nineteenth-century geometers:

A. Ribaucour, Note sur les développées des surfaces, *C. R. Acad. Sci. Paris* 74 (1872) 1399–1403.

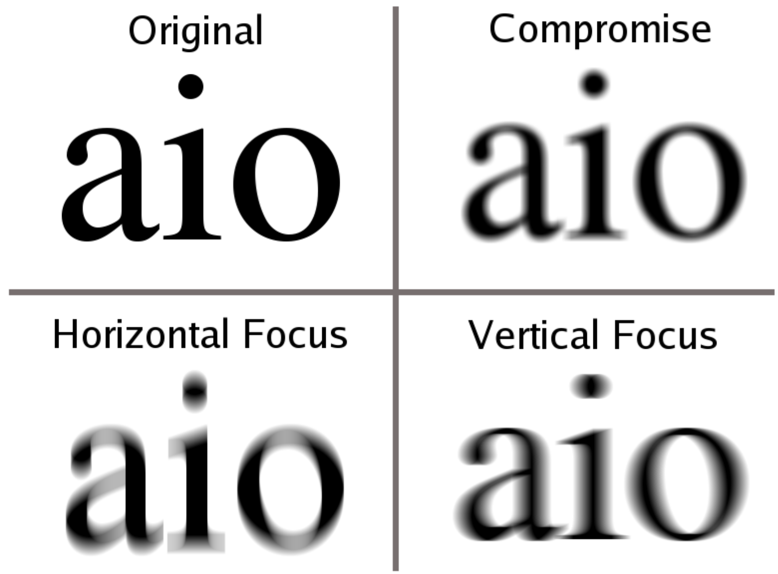
A. Mannheim, Sur les surfaces dont les rayons de courbure principaux sont fonctions l'un de l'autre, *Bull. S.M.F.* 5 (1877) 163–166.

R. Lipschitz, Zur Theorie der krummen Oberflächen, *Acta Math.* 10 (1887) 131–136.

R. von Lilienthal, Bemerkung über diejenigen Flächen bei denen die Differenz der Hauptkrümmungsradien constant ist, *Acta Math.* 11 (1887) 391–394.

Astigmatism

A general reflecting or refracting surface exhibits two focuses in perpendicular directions at distances equal to ρ and σ.



Tallfred, `http://en.wikipedia.org/wiki/Astigmatism_(eye)`

The difference $\rho - \sigma$ is known as the *interval of Sturm* or the *astigmatic interval* or the *amplitude of astigmatism* or the *astigmatism*.

The constant astigmatism equation

The astigmatic interval can be always reduced to 1 by rescaling the ambient metric. In the case of $\rho - \sigma = 1$, the Gauss equation can be put in the form

$$z_{yy} + \left(\frac{1}{z}\right)_{xx} + 2 = 0,$$

which we propose to call the constant astigmatism equation.

The equation has obvious translational symmetries (reparameterization) ∂_x, ∂_y, the scaling symmetry (offsetting)

$$2z\frac{\partial}{\partial_z} - x\frac{\partial}{\partial_x} + y\frac{\partial}{\partial_y},$$

and a discrete symmetry (swapping the orientation & taking the parallel surface at the unit distance)

$$x \longrightarrow y, \quad y \longrightarrow x, \quad z \longrightarrow \frac{1}{z}.$$

Two third-order symmetries

One of them has the generator

$$\frac{z^3}{K^3}(z_{xxx} - zz_{xxy})$$
$$- \frac{3}{K^5} z^3 (z_x - zz_y)(z_{xx} - zz_{xy})^2 - \frac{2}{K^5} z^5 (9z_x - zz_y) z_{xx}$$
$$+ \frac{1}{2K^5} z^2 (9z_x^2 + 4zz_x z_y - z^2 z_y^2)(z_x - zz_y) z_{xx}$$
$$- \frac{2}{K^5} z^3 z_x (z_x - zz_y)(4z_x - zz_y) z_{xy} + \frac{4}{K^5} z^6 z_x z_{xy}$$
$$+ \frac{3}{K^5} z^4 (5z_x - zz_y) z_x^2 - \frac{3}{K^5} z (z_x - zz_y) z_x^4,$$

where $K = \sqrt{(z_x - zz_y)^2 + 4z^3}$.

The other symmetry is obtained by conjugation with the discrete symmetry above.

A recursion operator

due to A. Sergyeyev (private communication).

If Z is a generating function of a symmetry, then so is

$$Z' = -z_y U + z_x V + 2zW,$$

where U, V, W satisfy

$$D_x U = Z, \qquad D_x V = W, \qquad D_x W = D_y Z,$$
$$D_y U = W, \qquad D_y V = \frac{Z}{z^2}, \qquad D_y W = D_x \frac{Z}{z^2}.$$

In the pseudodifferential form:

$$Z' = -z_y D_x^{-1} + z_x D_x^{-2} D_y + 2z D_x^{-1} D_y.$$

Takes local symmetries to nonlocal ones.

Relation to the sine–Gordon equation

A. Ribaucour, Note sur les développées des surfaces, *C. R. Acad. Sci. Paris* 74 (1872) 1399–1403.

The focal surfaces of surfaces satisfying $\rho - \sigma = \text{const}$ are pseudospherical. Hence a relation to the sine-Gordon equation.

Let $w = \frac{1}{2}\ln z$. Determine function ϕ' and coordinates ξ, η from

$$\cos\phi' = \frac{w_x^2 - \mathrm{e}^{2w} - \mathrm{e}^{4w} w_y^2}{\sqrt{(w_x + \mathrm{e}^{2w} w_y)^2 + \mathrm{e}^{2w}}\,\sqrt{(w_x - \mathrm{e}^{2w} w_y)^2 + \mathrm{e}^{2w}}},$$

$$\sin\phi' = -\frac{2\mathrm{e}^{w} w_x}{\sqrt{(w_x + \mathrm{e}^{2w} w_y)^2 + \mathrm{e}^{2w}}\,\sqrt{(w_x - \mathrm{e}^{2w} w_y)^2 + \mathrm{e}^{2w}}},$$

$$d\xi = \tfrac{1}{2}\sqrt{(w_x + \mathrm{e}^{2w} w_y)^2 + \mathrm{e}^{2w}}\; dx + \tfrac{1}{2}\sqrt{(\mathrm{e}^{-2w} w_x + w_y)^2 + \mathrm{e}^{-2w}}\; dy,$$

$$d\eta = \tfrac{1}{2}\sqrt{(w_x - \mathrm{e}^{2w} w_y)^2 + \mathrm{e}^{2w}}\; dx - \tfrac{1}{2}\sqrt{(\mathrm{e}^{-2w} w_x - w_y)^2 + \mathrm{e}^{-2w}}\; dy.$$

Then $\phi'(\xi, \eta)$ is a solution to the sine-Gordon equation $\phi_{\xi\eta} = \sin\phi$.

The Bianchi transformation

Another solution of the sine-Gordon equation can be obtained from the other focal surface.

The two focal surfaces are related by the classical Bianchi transformation:

– Corresponding points have a constant distance equal to $\rho - \sigma$;

– Corresponding normals are orthogonal;

– The line joining the corresponding points is tangent to both focal surfaces.

The Bianchi transformation is, however, superseded by the classical Bäcklund transformation, where the condition on the angle between the normals is relaxed from being right to being constant.

This probably explains why surfaces of constant curvature fell into oblivion.

Inverse relation to the sine–Gordon equation

An arbitrary pseudospherical surface can be equipped with a parabolic geodesic net. Involutes of the geodesics along the same starting line form a surface of constant astigmatism.

Let $\phi(\xi,\eta)$ be a solution of the sine-Gordon equation $\phi_{\xi\eta} = \sin\phi$. Let α,β be solutions of the compatible equations

$$\beta_\xi = -\sin\alpha, \qquad \alpha_\eta = -\sin\beta, \qquad \alpha - \beta = \phi.$$

Compute functions X, x, y from

$$dX = \cos\alpha\, d\xi + \cos\beta\, d\eta,$$

$$dx = \mathrm{e}^{-X}(\sin\alpha\, d\xi + \sin\beta\, d\eta),$$

$$dy = \mathrm{e}^{X}(\sin\alpha\, d\xi + \sin\beta\, d\eta).$$

Then $\mathrm{e}^{-2X(x,y)}$ is a solution of the constant astigmatism equation.

Von Lilienthal surfaces

R. von Lilienthal, Bemerkung über diejenigen Flächen bei denen die Differenz der Hauptkrümmungsradien constant ist, *Acta Math.* 11 (1887) 391–394.

A special case of the Lipschitz solution

R. Lipschitz, Zur Theorie der krummen Oberflächen, *Acta Math.* 10 (1887) 131–136.

Von Lilienthal surfaces are (made of) involutes (of meridians) of the pseudosphere (starting at the same 'parallel').

The pseudosphere itself is the involute of the catenoid.

All they are rotation surfaces:

- Catenoid = rotation of the catenary.
- Pseudosphere = rotation of the tractrix.
- Von Lilienthal surfaces = see the picture.

Weingarten's 'new class of surfaces'

Surfaces satisfying relation $\rho - \sigma = \sin(\rho + \sigma)$.

J. Weingarten, Über die Oberflächen, für welche einer der beiden Hauptkrümmungshalbmesser eine function des anderen ist, *J. Reine Angew. Math.* **62** (1863) 160–173.

Covered in §§ 745, 746, 766, 769, 770 of

G. Darboux, *"Leçons sur la théorie générale des surface et les applications géométriques du calcul infinitésimal,"* Vol. I–IV.

and §§ 135, 245, 246 of

L. Bianchi, *"Lezioni di Geometria Differenziale,"* Vol. I, II.

Darboux gave a general solution of the associated equation $(\tan z - z)_{xx} + (\cot z + z)_{yy} + \csc 2z = 0$. He also gave a remarkable geometric construction, further developed by Bianchi.

Darboux correspondence

Darboux discovered a relationship with translation surfaces.

A *translation surface* is a surface that admits a parameterization $\tilde{\mathbf{r}}(\xi, \eta)$ such that

$$\tilde{\mathbf{r}}_{\xi\eta} = 0.$$

Equivalently, $\tilde{\mathbf{r}}(\xi, \eta) = \tilde{\mathbf{r}}_1(\xi) + \tilde{\mathbf{r}}_2(\eta)$. The curves $\tilde{\mathbf{r}}_1(\xi)$ and $\tilde{\mathbf{r}}_2(\eta)$ are called the *generating curves*.

Otherwise said, a translation surface is obtained when translating a curve along another curve. Translation surfaces are manifestly integrable.

Bianchi observed that the translation surface in question is the *middle evolute*, which consists of mid-points between the two focal surfaces.

Darboux–Bianchi theorem I

Proposition. Let $\mathbf{r}$ be a Weingarten surface, let ξ, η be the common asymptotic coordinates of its focal surfaces. Then

(i) the coordinates ξ, η render the middle evolute $\tilde{\mathbf{r}}$ as a translation surface, i.e., $\tilde{\mathbf{r}}(\xi, \eta) = \tilde{\mathbf{r}}_1(\xi) + \tilde{\mathbf{r}}_2(\eta)$;

(ii) the generating curves $\tilde{\mathbf{r}}_1$, $\tilde{\mathbf{r}}_2$ have opposite nonzero constant torsion;

(iii) the normal vector $\mathbf{n}$ to the surface $\mathbf{r}$ at a point belongs to the intersection of the osculating planes of the generating curves $\tilde{\mathbf{r}}_1$, $\tilde{\mathbf{r}}_2$ through the corresponding point.

Darboux–Bianchi theorem II

Proposition. Let $\mathbf{s}(\xi,\eta)=\mathbf{s}_1(\xi)+\mathbf{s}_2(\eta)$ be a nonplanar translation surface. Assume that the generating curves $\mathbf{s}_1(\xi)$ and $\mathbf{s}_2(\eta)$ are of opposite nonzero constant torsion τ and $-\tau$, respectively. Denote by $\mathbf{b}_1$ and $\mathbf{b}_2$ the respective binormal vectors of the generating curves $\mathbf{s}_1(\xi)$ and $\mathbf{s}_2(\eta)$ and by $\Theta=\arccos(\mathbf{b}_1,\mathbf{b}_2)$ the angle between them, $0<\Theta<\pi$. Then the surface

$$\mathbf{r}=\mathbf{s}+\frac{\Theta+c_0}{\tau\sin\Theta}\mathbf{b}_1\times\mathbf{b}_2$$

satisfies Weingarten's relation

$$\frac{\rho-\sigma}{c_1}=\sin\left(\frac{\rho+\sigma}{c_1}-c_0\right). \tag{1}$$

with $c_1=2/\tau$.

Invariant characterization

Proposition. Consider a Weingarten surface which is not a sphere. Hence, the focal surfaces $\mathbf{r}^{(1)}, \mathbf{r}^{(2)}$ satisfy $\det \mathrm{II}^{(1)} \neq 0$, $\det \mathrm{II}^{(2)} \neq 0$. Let $\widetilde{\mathrm{II}}$ denote the second fundamental form of the middle evolute $\tilde{\mathbf{r}}$. Then the following statements are equivalent:

(i) surface belongs to the 'new Weingarten class';

(ii) $\operatorname{tr}(\mathrm{II}^{(1)-1}\,\widetilde{\mathrm{II}}) = 0$;

(iii) $\operatorname{tr}(\mathrm{II}^{(2)-1}\,\widetilde{\mathrm{II}}) = 0$;

(iv) $\mathbf{n}^{(2)} \cdot \Delta^{(2)}_{\mathrm{II}} \mathbf{r}^{(1)} = \mathbf{n}^{(1)} \cdot \Delta^{(1)}_{\mathrm{II}} \mathbf{r}^{(2)}$, where $\Delta^{(1)}_{\mathrm{II}}$ and $\Delta^{(2)}_{\mathrm{II}}$ are the Laplace–Beltrami operators with respect to $\mathrm{II}^{(1)}$ and $\mathrm{II}^{(2)}$.

Remark. The normal components $\mathbf{n}^{(1)} \cdot \Delta^{(1)}_{\mathrm{II}} \mathbf{r}^{(1)}$, $\mathbf{n}^{(2)} \cdot \Delta^{(2)}_{\mathrm{II}} \mathbf{r}^{(2)}$ are constant and equal to 2.

S. Haesen, S. Verpoort and L. Verstraelen, The mean curvature of the second fundamental form, *Houston J. Math.*