Diffeological differential geometry.

Thesis for the Master degree in Mathematics.

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May 28, 2008

Abstract

The main objective for this thesis is the construction of a tensor bundle on a diffeological space X. Thereby getting access to the exterior bundle of anti-symmetric tensors on X, and smooth sections here on i.e. differential forms. We shall list certain requirements that any reasonable tensor bundle on a diffeological space should fulfil. And show that the given construction fulfil these requirements. The main idea of the approach taken in this thesis is to associate to each smooth curve $\alpha : \mathbb{R} \to X$ a map $d\alpha : C^\infty(X) \to \mathbb{R}$ defined by $d\alpha(f) := d_0(f \circ \alpha)$ (where d_0 denotes differentiation at 0). This leads to reasonable tangent spaces, tangent bundles, tensor bundles and finally differential forms. These differential forms will in a natural way be $\mathcal{D}$-forms. In order to archive the main objective we shall also need to develop some theory concerning diffeological bundles, and vector bundles.

Abstract

Det primær mål i dette speciale er, at konstruer et tensor bundt på et diffeologisk rum X. Og derigemmen at studere differentialeformer på X. Vi sætter en række krav op et rimeligt defineret tensor bundt bør opfylde, og viser at det konstruerede tensor bundt opfylder disse krav. Den grundlæggende ide i konstruktion er, at associer til en hver glat kurve $\alpha : \mathbb{R} \to X$ en afbildning $d\alpha : C^\infty(X) \to \mathbb{R}$ defineret ved $d\alpha(f) := d_0(f \circ \alpha)$ (hvor d_0 betegner differentiation i 0). Dette leder til rimelige tangent rum, tangent bundter, tensor bundter og differentialeformer. Disse differentialeformer er på en naturlig måde $\mathcal{D}$- differentialeformer. For at nå det primær mål er det nødvendigt også at studer diffeologiske bundter og vektor bundter.

Contents

Introduction

The theory of diffeological spaces tries to capture the essence of smooth spaces, it generalizes smooth manifolds. It is fair to say, that the theory of diffeological spaces, is an attempt at construeting a theory in which all kind of smooth surfaces can be studied, including singular surfaces, infinite dimensional smooth spaces and functional spaces.

The main objective for this thesis is the construction of a k-tensor bundle $T^k(X)$ on a diffeological space X. Thereby getting access to the exterior bundle $\Lambda^k(X)$ of antisymmetric k-tensors on X, and smooth sections here on i.e. differential forms. We shall list certain requirements that any reasonable tensor bundle on a diffeological space should fulfil. And show that our construction fulfil these requirements. We shall however not discuss, in detail, the uniqueness of the constructions. Although it will be clear that, many of, the constructions involved have certain uniqueness properties in relation to these requirements.

It is worth noticing that there is a short cut to defining differential forms on diffeological spaces, these forms are called $\mathcal{D}$-forms. $\mathcal{D}$-forms can be defined without first defining a tensor bundle, a tangent bundle and a tangent space. The traditional approach in diffeology is to define tangent spaces as induced by $\mathcal{D}$-forms. The approach we shall take in this thesis is different, it is inspired by a more traditional approach, as taken in finite dimensional locally euclidean differential geometry . Hence our line of construction will be

$$\text{tangent vector} \rightarrow \text{tangent space} \rightarrow \text{tangent bundle} \rightarrow \text{tensor bundle.}$$

Furthermore smooth sections of $\Lambda^k(X)$ will in a natural way be $\mathcal{D}$-forms. In fact any reasonable construction of a tensor bundle will admit differential forms which are $\mathcal{D}$-forms, in a natural way.

Due to the richness and complex nature of diffeological spaces the problem of constructing a suitable tensor bundle on a diffeological space is not a trivial one. Before we go into more details on the objectives and approaches to the problems taken in this thesis, we shall introduce the category of diffeological spaces.

The category

An early version of diffeological spaces, now usually called Chen spaces, was introduced by Chen in Chen [1977] under the name *differentiable spaces*. We shall in this thesis use a slightly different notion, namely that of diffeological space as introduced in Souriau [1980]. Other more or less similar approaches includes Frölicher spaces and Sikorski's differential spaces (among physicists generally called d-spaces). For a short introduction to Frölicher spaces see chapter V of Kriegl and Michor [1997], an introduction to Sikorski's differential spaces can for example be found in Gruszczak et al. [1988]. The category of Frölicher space is a full subcategory of the category of diffeological spaces, in Vincent [2008] we discuss this relation further.

Iglesias-Zemmour has done a great deal of work relating to the theory of diffeology, a review of diffeology and an introduction to $\mathcal{D}$-forms may be found in Iglesias-Zemmour [2007c]. There are also some interesting unpublished work, in particular the unfinished book Iglesias-Zemmour [2007b].

Applications of diffeology. In Souriau [1980] the axioms of diffeology is formulated, as a part of an attempt at formalizing quantum mechanics. Furthermore physicists have used, the related, Sikorski's differential spaces in attempts to model the physical space time. But there are today, it seems, still no successful and important application of diffeology, or related theories. However a well established theory of smooth spaces able to handle singularities and functional spaces would be of interest, and could have many applications.

We shall not discuss applications of diffeology, and related theories, any further. We shall however, throughout the thesis, illustrate the theory with examples of the studied constructions.

The idea of diffeological spaces is to define the structure on a space by specifying the collection of, what will be, diffeologically smooth maps into the space and having domains in $\mathrm{O}\mathbb{R}^\infty$. By $\mathrm{O}\mathbb{R}^\infty$ we mean the category with objects open subsets of euclidean spaces, and morphism usual smooth maps. Let us consider a collection Ω of maps into a set X, having there domain in $\mathrm{O}\mathbb{R}^\infty$. What should we require for such a collection in order to justify calling the structure defined by Ω, a smooth structure. The following three axioms are reasonable to require (we shall state these precisely in section 1.1.1)

(1) Every constant map is smooth, i.e. is an element of Ω.

(2) If $\alpha \in \Omega$ and h is a usual smooth map composable with α then $\alpha \circ h \in \Omega$.

(3) Smoothness is a local property.

Given a collection $\mathcal{D}_X$ fulfilling the three axioms above we say that $\mathcal{D}_X$ is a diffeology on X, and call the elements of $\mathcal{D}_X$ plots. A set X equipped with a diffeology $\mathcal{D}_X$ is then said to be a diffeological space. The notation of smooth maps between diffeological spaces X and Y is natural, as we simply require that composition of smooth maps must be smooth. Hence we say that a map $f : Y \to X$ is smooth if for every plot α on Y the composition $f \circ \alpha$ is a plot for X, see fig. 1. Notice that this implies that plots are exactly smooth maps with domain in $\mathrm{O}\mathbb{R}^\infty$.

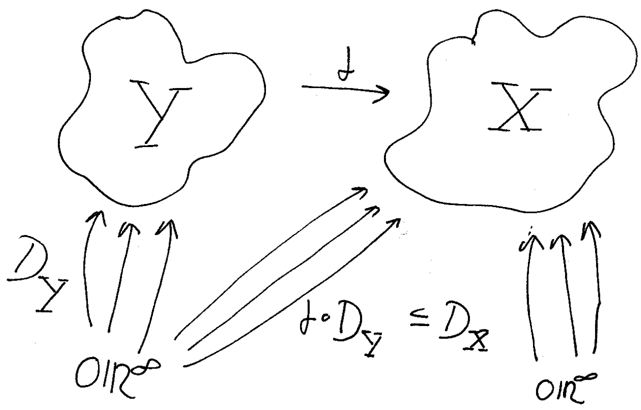


Figure 1: A smooth map $f : Y \to X$.

Categorical properties. The collection of diffeological spaces form a category, with morphisms smooth maps. The category of diffeological spaces is a nice category, below is listed some of its properties;

- It is complete and cocomplete, i.e. it has all limits and colimits. In particular we may form products, coproducts (disjoint union), equalizers and coequalizers, pushouts and pullbacks. Furthermore these limits behave nicely with respect to the category of sets (Set), as the forgetfull functor Dif $\longrightarrow$ Set preserves limits and colimits.
- The collection of smooth maps $C^\infty(X, Y)$ between diffeological spaces have a natural smooth structure as a diffeological space, *the functional diffeology*. And the category of diffeological spaces is cartesian closed, that is there is a natural diffeomorphism

$$C^\infty(X \times Y, Z) \simeq C^\infty(X, C^\infty(Y, Z)).$$

- Every subset of a diffeological space has a natural subspace diffeology. Every quotient (given by a equivalence relation on the set) of a diffeological space has a natural quotient diffeology.
- Every diffeological space can be given a natural topology, given this topology all smooth maps are continuous.
- Finite dimensional manifolds (with or without boundary) are diffeological spaces, and smooth maps between these are precisely those that are smooth in the usual sense. In other words the category of finite dimensional manifolds with boundary is a full subcategory of Dif.

It has be said that the philosophy is that, it is better to have a nice category containing some pathological spaces, than a ugly category of nice spaces.

Objectives and organization of the thesis

We shall in this thesis mainly study the differential geometric properties of diffeological spaces. We shall in particular construct tangent spaces, tangent bundles and tensor bundles on diffeological spaces. The final goal is to construct a sensible tensor bundle, whereby we shall mean a tensor bundle fulfilling the requirements listed below. Note that we do not claim that this list is complete, in any way, the investigation of this would require further study. See also chapter VII of Kriegl and Michor [1997] in which different approaches in defining differential forms on infinite dimensional smooth space are carefully and detailed studied. However in a different setting than diffeology, namely infinite dimensional manifolds modelled on locally convex vector spaces. In fact the approach we shall take in defining tensor bundles is partly inspired by their conclusions.

The constructed tensor bundle $T^k(X)$ should fulfil the following requirements:

(a) The fiber over $x \in X$ should be diffeomorphic with the vector space consisting of smooth multilinear maps

$$\mathrm{T}_x X \times \cdots \times \mathrm{T}_x X \to \mathbb{R}.$$

(b) If $U \in \mathrm{O}\mathbb{R}^\infty$ then the set of smooth sections $\Gamma(T^k(U))$ on $T^k(U)$ (tensor fields) should consist of all smooth functions

$$\phi : U \times \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{\text{k copies}} \to \mathbb{R}$$

with $\phi(u)$ multilinear for each $u \in U$.

(c) Each smooth map $\phi : X \to Y$ should induce a pullback

$$\phi^* : \Gamma(T^k(Y)) \to \Gamma(T^k(X))$$

of tensor fields. And for euclidean spaces this pullback should be the usual one.

(d) Given k vector fields $V_1, \ldots, V_k$ on X and a tensor field $\mathcal{T}$, the map

$$x \to \mathcal{T}(x)(V_1(x), \ldots, V_k(x))$$

should be smooth.

In order to reach this objective we shall need to construct tangent spaces and tangent bundles. Furthermore we will need to study diffeological vector spaces, diffeological bundles and vector bundles.

As we shall discuss a bit further in chapter 6 (b) and (c) above implies that differential forms, related to the tensor bundle, are in a natural way $\mathcal{D}$-forms.

In chapter 1 we introduce the category of diffeological spaces. The chapter contains several examples of diffeological spaces, and discuss smooth maps here on. We shall, in this chapter, omit most of the proofs as they can be found other places for example Iglesias-Zemmour [2007b] or Vincent [2008]. Note however that some might only be found in Vincent [2008], and proof for all statements without proof may be found herein. Note also that examples 9 to 11 and 13 are new.

In section 1.4 we state some few simple observations concerning diffeological products. Most of these might be found other places as well, we shall, however, include the proofs.

In section 1.7 we introduce the functional diffeology, and prove the cartesian closure of the category of diffeological spaces. This will be important for the constructions dealt with in this thesis. Except for the examples we shall follow more or less Laubinger [2006].

In chapter 2 we introduce the notion of diffeological vector spaces, as introduced in Iglesias-Zemmour [2007c]. In section 2.1 we follow Iglesias-Zemmour [2007c], in section 2.2 we generalize *the fine diffeology* for vector spaces, as found in Iglesias-Zemmour [2007c]. Section 2.3 is new, and my own work. Section 2.4 only contains simple observations.

In chapter 3 we construct tangent spaces for a diffeological space. This has been done before, but the approach we shall taken in this thesis is different. In Laubinger [2006] tangent spaces are constructed as certain co-limits in the category of vector spaces. Some short notes on how to construct tangent spaces induced by $\mathcal{D}$-forms can be found in the, still unfinished, book Iglesias-Zemmour [2007b]. The main idea of the approach taken in this thesis is to associate to each smooth curve $\alpha : \mathbb{R} \to X$ a map $d\alpha : C^\infty(X) \to \mathbb{R}$ defined by

$$d\alpha(f) := d_0(f \circ \alpha),$$

(where d_0 denotes differentiation at 0). As wee shall see in chapter 3 this leads to reasonable tangent spaces, and furthermore the idea is relatively simple and intuitive, having in mind the usual definition of tangent vectors, on a finite dimensional manifolds M, as derivations on $C^\infty(M)$. In fact the maps $d\alpha$ are smooth derivations.

In chapters 4 and 5 we introduce the notion of diffeological bundles and diffeological vector bundles. We need to study bundle structures in order to reach our objective of constructing a tangent bundle and a tensor bundle. The richness of the category of diffeological spaces, hence the possible high complexity of a diffeological space, implies high complexity of the tangent bundle. We therefore need a fairly broad notion of bundles. In fact fairly simple examples will show us that we need to deal with bundles which are not locally trivial.

In chapter 4 we introduce general diffeological bundles, which is a generalization of the concept found in Iglesias-Zemmour [2007c]. In chapter 5 we introduce diffeological vector bundles, as well as construction involving these.

Finally in chapter 6 we utilize the theory developed in chapters 4 and 5 in order to construct, and study, our tangent bundle, and tensor bundle. We shall in particular show that the constructed tensor bundle fulfills the regiments (a)-(d) as listed above.

Chapters 3 to 6 are new, and my own work. The ideas for the constructions involved are mainly inspired by, but not equal to, constructions found Kriegl and Michor [1997], Iglesias-Zemmour [2007c], Warner [1971] among others.

Chapter 1

A review of the theory of diffeological spaces

This chapter contains a review, mainly without proofs, of the theory of diffeological spaces. Proofs for the statements may for example be found in Iglesias-Zemmour [2007b] or Vincent [2008], or in any other text on diffeology.

In section 1.1 we introduce the category of diffeological spaces, and give some examples. In section 1.2 we introduce the concepts of generating diffeology. The lattice structure of lattice consisting of diffeologies on a set is studied in section 1.3. Product and co-products in section 1.4. In section 1.5 we collect a few result relating to diffeomorphisms between diffeological spaces. Each diffeological space carries a natural topology, the $\mathcal{D}$-topology, we introduce it in section 1.6. The functional diffeology will be important for the work done in this thesis, we shall therefore give the full proof of all statements relating to this. The functional diffeology will be introduced in section 1.7. Finally in section 1.8 we study a few examples.

1.1 The category of diffeological spaces

In this section, the axioms of diffeology will be given, thereby defining diffeological spaces. The concept of smooth maps between diffeological spaces, will be defined. And we shall see that the collection of diffeological spaces forms a category with morphisms smooth maps. The definitions will be followed up by examples, many of which we shall use later to illustrate points of the theory.

1.1.1 The axioms of diffeology

Let in the following X denote a set. A diffeology on X is a collection of certain maps into X, in order for this collection to be a diffeology 3 axioms must hold, covering, smoothness and locality. We may think of a diffeology on X as the collection of *all* smooth maps into X, with domains objects of $\mathrm{O}\mathbb{R}^\infty$. Before we give the exact statement of these axioms, we will need to define which types of maps the diffeology is a collection of. We will therefore start by making the following definitions;

Definition 1.1.1 By $\mathrm{O}\mathbb{R}^\infty$ we will denote the category with objects all open sets of any finite dimensional euclidean space, and morphisms smooth maps between these open sets.

Note that we will write $U \in \mathrm{O}\mathbb{R}^\infty$, if U is a object of $\mathrm{O}\mathbb{R}^\infty$. And that we will use this notation for any category.

Note that $\mathrm{O}\mathbb{R}^\infty$ is a small category.

Definition 1.1.2 A *parametrization* of X is any map into X with domain an object of $\mathrm{O}\mathbb{R}^\infty$.

We will by $\mathrm{Par}(X)$ denote the set consisting of all parametrizations of X. If a parametrization is a constant map, we will say that is a *constant parametrization.* Consider a subset Ω of $\mathrm{Par}(X)$. We shall say that Ω is a *covering* of X if axiom 1 below holds.

Note that $\mathrm{Par}(X)$ is indeed a set.

Axiom 1 (Covering)
All constant parametrizations belongs to Ω.

In other words axiom 1 express that all constant maps into a diffeological space are considered smooth. Before we state axiom 2 we need one further definition;

Definition 1.1.3 Let α be any parametrization, and h any morphism in $\mathrm{O}\mathbb{R}^\infty$, then h is said to be *composable* with α if $\mathrm{Im}(h) \subseteq \mathrm{Dom}(\alpha)$.

We will use the following notation;

$$\mathcal{G}(\alpha) := \{\alpha \circ h \mid h \text{ any morphism in } \mathrm{O}\mathbb{R}^\infty \text{ composable with } \alpha\}$$

Our second axiom stats that the composition of a diffeologically smooth map with a usual smooth map (i.e a morphism in $\mathrm{O}\mathbb{R}^\infty$) is again a diffeologically smooth map, to be exact we state the following;

Axiom 2 (Smoothness)
For any parametrization α in Ω it holds that $\mathcal{G}(\alpha) \subseteq \Omega$.

A collection Ω of parametrization of X is said to be a *smooth collection* if axiom 2 holds for all parametrizations of Ω. If we let $\mathcal{G}(\Omega) := \cup_{\alpha\in\Omega}\mathcal{G}(\alpha)$, then a collection is smooth if and only if $\mathcal{G}(\Omega) = \Omega$. Note also that $\mathcal{G}(\Omega)$ is a smooth collection. And if $\alpha \in \mathcal{G}(\Omega)$, then $\alpha|_U \in \mathcal{G}(\Omega)$ for any open $U \subseteq \mathrm{Dom}(\alpha)$.

Our last axiom states that diffeologically smoothness is a local property, in order to state it we need the notion of *locally belonging to* as defined below.

Definition 1.1.4 We say that a parametrization $U \xrightarrow{\alpha} X$ *locally belongs to* Ω if there exist a open covering $\{U_i\}_{i\in\mathcal{I}}$ of U such that $\alpha|_{U_i} \in \Omega$ for all $i \in \mathcal{I}$.

We shall write $\alpha \overset{\mathrm{loc}}{\in} \Omega$ if α locally belongs to Ω, and write $\Omega \overset{loc}{\subseteq} \Omega'$ if $\alpha \in \Omega \Rightarrow \alpha \overset{\mathrm{loc}}{\in} \Omega'$. And say that a parametrization α is *locally constant* if it locally belongs to the collection of constant parametrizations. As simple consequences of definition 1.1.4 we get the following propositions;

Lemma 1.1.5 *Let Ω and Ω' be subsets of* $\mathrm{Par}(X)$ *then*

(i) If $\Omega \subseteq \Omega'$ then $\alpha \overset{\mathrm{loc}}{\in} \Omega \Rightarrow \alpha \overset{\mathrm{loc}}{\in} \Omega'$

(ii) $\Omega \overset{loc}{\subseteq} \Omega$

and for a family $\{\Omega_i\}_{i\in\mathcal{I}}$ *of subsets of* $\mathrm{Par}(X)$

(iii) $\alpha \overset{\text{loc}}{\in} \cap_{i\in\mathcal{I}}\Omega_i$ *implies that* $\alpha \overset{\text{loc}}{\in} \Omega_i$ *for all* $i \in \mathcal{I}$*.*

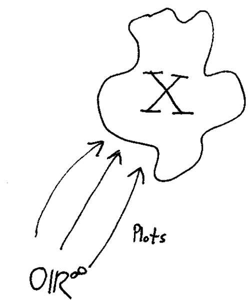
Proof: The propositions are trivial consequences of the definition 1.1.4. ▪

Axiom 3 is as follows;

Axiom 3 (Locality)
It holds that $\alpha \overset{\text{loc}}{\in} \Omega \Rightarrow \alpha \in \Omega$.

A collection Ω of parametrization of X is said to be a *local collection* if axiom 3 holds. We are now ready to give the definition of a diffeological space. Our central definition, of this thesis, are as follows

Definition 1.1.6 A *diffeology* on a set X is a collection $\mathcal{D}_X$ of parametrizations of X (called plots) for which axioms 1 to 3 holds.



A diffeological space.

For a set X with a diffeology $\mathcal{D}_X$ on it, we will use the notation $(X, \mathcal{D}_X)$ or just X or $\mathcal{D}_X$ if the meaning is clear from the context. A set together with a diffeology is called a diffeological space. Given two diffeologies D_1 and D_2 on X with $\mathcal{D}_1 \subseteq \mathcal{D}_2$, we shall say that $\mathcal{D}_1$ is *weaker* than $\mathcal{D}_2$, or equivalent that D_2 is *stronger* than $\mathcal{D}_1$.

Next, we will need a concept of smooth maps between diffeological spaces. but before we move on to this, we will look at some simple examples of diffeological spaces. Let us start with the simplest possible examples of diffeological spaces on X;

Examples of diffeological spaces

Example 1 (The discrete and indiscrete diffeology)
The collection of all parametrizations of X which are locally constant is a diffeology on X, it will be denote by $\mathcal{D}_X^\circ$, this diffeology is called the *discrete diffeology*. The collection of all parametrizations of X is also a diffeology, it will be denoted by $\mathcal{D}_X^\bullet$ and called the *indiscrete diffeology*. For any diffeology $\mathcal{D}_X$ on X evidently

$$\mathcal{D}_X^\circ \subseteq \mathcal{D}_X \subseteq \mathcal{D}_X^\bullet.$$ ▪

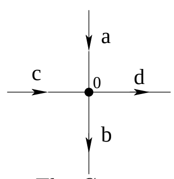
Example 2 (The canonical diffeology on $\mathbb{R}$)
Let

$$\mathcal{D}_\mathbb{R} := \bigcup_{U\in \mathrm{O}\mathbb{R}^\infty} C^\infty(U,\mathbb{R})$$

clearly $\mathcal{D}_\mathbb{R}$ is a diffeology on $\mathbb{R}$, it's called the canonical diffeology on $\mathbb{R}$. Now for any object V of $\mathrm{O}\mathbb{R}^\infty$ we may define, as above, the canonical diffeology on V, i.e we define

$$\mathcal{D}_\mathbb{V} := \bigcup_{U\in \mathrm{O}\mathbb{R}^\infty} C^\infty(U,V).$$ ▪

Next let us look at the simplest singularity we can think of (beside an isolated point);



The Cross

Example 3 (A diffeology on the cross)
Consider the set

$$\mathcal{X} := \frac{\mathbb{R}_v \coprod \mathbb{R}_h}{0_v \sim 0_h}$$

this set will be called *the cross* (here $\coprod$ denotes disjoint union, for convenience we have labeled the copies of $\mathbb{R}$). We shall here give two different diffeologies on $\mathcal{X}$, the line diffeology and the subspace diffeology. Later, in example 6, we will see that the line diffeology is the natural diffeology on $\mathcal{X}$ if we think of $\mathcal{X}$ as constructed by products and quotients. If we on the other hand consider $\mathcal{X}$ as a subspace of the plan the subspace diffeology is the natural one to consider. And there is a difference, as we shall see below, but first we define the diffeologies;

> *The Line diffeology*, $\mathcal{D}_{\text{line}}$ consist of all parametrizations locally having there image entirely in one of the subspaces $\mathbb{R}_v$ or $\mathbb{R}_h$, and which are smooth in the ordinary sense, i.e. as maps into $\mathbb{R}$.
>
> *The Subspace diffeology.:* Let $\iota : \mathcal{X} \hookrightarrow \mathbb{R}^2$ be the inclusion where $\mathbb{R}_v$ is mapped to the vertical axis, and $\mathbb{R}_h$ to the horizontal axis, then the subspace diffeology is
>
> $$\mathcal{D}_{\text{sub}} := \{\alpha \in \text{Par}(X) \mid \iota \circ \alpha \text{ is smooth}\}$$

In is not difficult to verify that the above defines diffeologies, nor is it hard to see that $\mathcal{D}_{\text{line}} \subseteq \mathcal{D}_{\text{sub}}$. To see that they are not equal, consider the smooth map $h : \mathbb{R} \to \mathbb{R}$ given by

$$h(x) := \begin{cases} \exp(-1/x) & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

then define the parametrization $\lambda : \mathbb{R} \to \mathcal{X}$ by

$$\lambda(x) := \begin{cases} \iota_h \circ h(x) & x > 0 \\ 0 & x = 0 \\ \iota_v \circ h(-x) & x < 0 \end{cases}$$

■

where ι_h and ι_v are the obvious inclusions, see fig. 1.1. Note that $\iota \circ \lambda(x) = (h(x), h(-x))$ is smooth hence $\alpha \in \mathcal{D}_{\text{sub}}$. On the other hand for any open neighbourhood U of $0 \in \mathbb{R}$. the image of $\lambda|_U$ is not entirely in either of the subspace $\mathbb{R}_h$ or $\mathbb{R}_v$, hence $\alpha \notin \mathcal{D}_{\text{line}}$. To sum up the subspace diffeology contains singular curves (as for example $\lambda(x)$), the line diffeology do not.

1.1.2 Smooth maps

Having defined a smooth structure on sets, diffeologies, the next natural question is, *when is a map between diffeological spaces smooth?* The answer follows almost naturally from the definition of diffeologies, as it seems very reasonable to require that the composition of two smooth maps is a smooth map, we therefore make the following;

Definition 1.1.7 A map $f : X \to Y$ between diffeological spaces is said to be *smooth*, or C^∞, if for any plot α for X the composition $f \circ \alpha$ is a plot for Y.

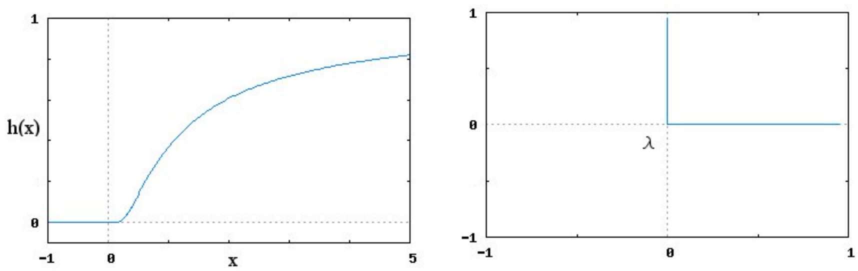


Figure 1.1: On the left a plot of $h(x)$, on the right a plot of $\lambda(x)$, just stopping at 0 (i.e. the derivations $\lambda^{(n)}(0) = 0$ for all $n \in \mathbb{N}$), but not staying. Had λ "stayed" at 0 for any amount of "time" it would had been a plot for the line diffeology.

See also fig. 1. The set of all smooth maps from $(X, \mathcal{D}_X)$ to $(Y, \mathcal{D}_Y)$ will be denoted by $C^\infty(D_X, D_Y)$. Using the notation $f \circ \mathcal{D}_X := \{f \circ \alpha \mid \alpha \in \mathcal{D}_X\}$, we may write the set of smooth maps from $(X, \mathcal{D}_X)$ to $(Y, \mathcal{D}_Y)$ simply as

$$C^\infty(D_X, D_Y) = \{f \in \mathrm{Maps}(X, Y) \mid f \circ \mathcal{D}_X \subseteq \mathcal{D}_Y\}.$$

Notation as $C^\infty(X, Y)$ or mixtures as $C^\infty(\mathcal{D}_X, Y)$, if the diffeologies are clear from the context, will also be used. Furthermore we may also say that a map $f : X \to Y$ is $\mathcal{D}_X$-smooth if $f \in C^\infty(\mathcal{D}_X, Y)$.

It is worth noting that the plots for a diffeological space $(X, \mathcal{D}_X)$ are exactly the collection of smooth maps into X with there domain in $\mathrm{O}\mathbb{R}^\infty$, in other words

$$\mathcal{D}_X = \bigcup_{U \in \mathrm{O}\mathbb{R}^\infty} C^\infty(U, X).$$

Definition 1.1.7 has the following simple, but desireable consequences;

Proposition 1.1.8 *The following holds, for diffeological spaces* $(X, \mathcal{D}_X), (Y, \mathcal{D}_Y)$ *and* $(Z, \mathcal{D}_Z)$.

(i) If $f \in C^\infty(X, Y)$ *and* $g \in C^\infty(Y, Z)$ *then* $g \circ f \in C^\infty(X, Z)$.

(ii) $\mathrm{id}_X \in C^\infty(X, X)$.

Proof:

$$(g \circ f) \circ \mathcal{D}_X \subseteq g \circ \mathcal{D}_Y \subseteq D_Z \qquad \blacksquare$$

The above proposition says that the collection of diffeological spaces, with morphisms smooth maps, is a category. We shall call this category the *category of diffeological spaces*, and denote it by Dif.

1.2 Generating diffeologies

Given any collection Ω of parametrization of X there exist a weakest diffeology on X containing Ω, namely the intersection of all diffeologies containing Ω, which is by lemma 1.2.2, given below, a diffeology. We will say that this diffeology is generated by Ω, hence we make the following definition;

Definition 1.2.1 The diffeology generated by Ω is the intersection of all diffeologies on X containing Ω.

The diffeology generated by Ω is denoted by $\langle\Omega\rangle$. And if given a diffeology $\mathcal{D}$ and a subset $\Omega \subseteq \mathcal{D}$ such that $\langle\Omega\rangle = \mathcal{D}$, then we shall say that Ω is a *generating family* of $\mathcal{D}$, and call the plots in Ω for generating plots. We shall also say that $\mathcal{D}$ is the *weak diffeology* generated by Ω, the motivation for this name will, if not already, become clear in section 1.3. Note also that if we have a canonical injection $X \hookrightarrow Y$ then we may view Ω as a collection of parametrizations on Y. And that the weak diffeology on X generated by Ω is in general different from the weak diffeology on Y generated by Ω. It will therefore sometimes be convenient to use the notation $\langle\Omega\rangle_Y$ meaning the weak diffeology on Y generated by Ω. We still need to verify that the intersection of a collection of diffeologies on a common set is infact a diffeology on that set. It is however not hard, a proof may be found in Vincent [2008], Iglesias-Zemmour [2007b] or indeed in any introductory text on diffeology.

Lemma 1.2.2 *Given a family $\mathbb{D}$ of diffeologies on a set X,the intersection $\cap_{\mathcal{D}\in\mathbb{D}}\mathcal{D}$ is a diffeology on X.*

Let us collect some imitate properties of definition 1.2.1;

Lemma 1.2.3 *Given collections Ω and Ω' of parametrization of X the following holds*

(i) $\Omega \subseteq \langle\Omega\rangle$

(ii) $\Omega \subseteq \Omega' \Rightarrow \langle\Omega\rangle \subseteq \langle\Omega'\rangle$

(iii) $\langle\langle\Omega\rangle\rangle = \langle\Omega\rangle$

Proof: Clearly we have the following inclusions for any diffeology $\mathcal{D}$ containing Ω

$$\Omega \subseteq \langle\Omega\rangle \subseteq \mathcal{D}$$

this implies (i) and (iii). (ii) is also a simple consequences of the definition. ▪

1.2.1 Plots of generated diffeologies

Given a collection Ω of parametrizations of X, what do the plots of $\langle\Omega\rangle$ look like? The answer to this question is the content of theorem 1.2.4, a proof can be found in Vincent [2008] or Iglesias-Zemmour [2007b].

Recall that $\mathcal{D}^\circ_X$ denotes the discrete diffeology on X, consiting of all locally constant parametrizations

Theorem 1.2.4
The following equality holds

$$\langle\Omega\rangle = \left\{\alpha \in \mathrm{Par}(X) \mid \alpha \overset{\mathrm{loc}}{\in} \mathcal{D}^\circ_X \cup \mathcal{G}(\Omega)\right\}$$

Remark 1.2.5 In other words theorem 1.2.4 says that a plot $\alpha : U \to X$ is in $\langle\Omega\rangle$ if and only if for all $x \in U$ there exist an open neighbourhood $V \subseteq U$ of x such that $\alpha|_V$ is constant or $\alpha|_V = \beta \circ h$ for a parametrization $\beta \in \Omega$ and a smooth map $h : V \to \mathrm{Dom}(\beta)$

Corollary *If Ω is a covering of X then $\alpha \in \langle\Omega\rangle$ if and only if $\alpha \overset{\mathrm{loc}}{\in} \mathcal{G}(\Omega)$.*

Corollary *If Ω is a smooth covering of X then $\alpha \in \langle\Omega\rangle$ if and only if $\alpha \overset{\mathrm{loc}}{\in} \Omega$.*

1.2.2 Smooth maps and generated diffeologies

If a diffeological space X has a generating family Ω, only the plots in Ω need to be checked when determining if a map $X \to Z$, into any space Z, is smooth. To be precise we have the following;

Lemma 1.2.6 *Let Ω be a collection of parametrizations on X, a map*

$$f : (X, \langle\Omega\rangle) \to (Y, \mathcal{D}_Y)$$

is smooth if and only if $f \circ \Omega \subseteq \mathcal{D}_Y$.

1.3 Lattice structure and constructions

We shall in this section discuss the lattice structure of the collection of diffeologies on a fixed set. A more detailed discussion, and proof of the results cited in this section may be found in Vincent [2008].

We shall by $\mathcal{P}(X)$ denote the complete lattice consisting of subsets of parametrizations of X, ordered by set inclusion. Recall that, in the complete lattice $\mathcal{P}(X)$, taking supremum corresponds to taking union, and infimum to intersection. By $\mathbb{D}(X)$ we will denote the partially ordered set of diffeologies on X, with the order induced by the inclusion $\mathbb{D}(X) \hookrightarrow \mathcal{P}(X)$. Furthermore we will call the map

$$\mathcal{P}(X) \xrightarrow{\Omega \to \langle\Omega\rangle} \mathbb{D}(X)$$

the *generating map* and denote it by $\langle\cdot\rangle$ (here $\langle\Omega\rangle$ denotes the diffeology generated by the collection Ω of parametrizations on X). The following observation is an important property of the genrating map.

Theorem 1.3.1
The genrating map is a closure operator on $\mathcal{P}(X)$ and the closed elements are exactly the diffeologies on X.

Hence the collection of diffeologies on X, $\mathbb{D}(X)$, is a complete lattice. As a side remark we may note that in example 1 we saw that the discrete diffeology on X, $\mathcal{D}_X^\circ$, is the bottom of $\mathbb{D}(X)$, and the indiscrete diffeology, $\mathcal{D}_X^\bullet$, the top. We have the following corollary to theorem 1.3.1

Corollary *Consider a collection $\{\mathcal{D}_i\}_{i\in\mathcal{I}}$ (for some index set $\mathcal{I}$) of diffeologies on X. Then the infimum and supremum in $\mathbb{D}(X)$ of $\{\mathcal{D}_i\}_{i\in\mathcal{I}}$ is respectively*

$$\inf_{i\in\mathcal{I}} \mathcal{D}_i = \bigcap_{i\in\mathcal{I}} \mathcal{D}_i \quad \text{and} \quad \sup_{i\in\mathcal{I}} \mathcal{D}_i = \left\langle \bigcup_{i\in\mathcal{I}} \mathcal{D}_i \right\rangle$$

Remark 1.3.2 Note that by the definition of the generating map

$$\sup_{i\in\mathcal{I}} \mathcal{D}_i = \inf \{\mathcal{D} \in \mathbb{D}(X) \mid \mathcal{D}_i \subseteq \mathcal{D} \text{ for all } i \in \mathcal{I}\}$$

It is, by the above, evident that, the infimum $\inf_{i\in\mathcal{I}} \mathcal{D}_i$ is the *strongest diffeology* on X contained in $\mathcal{D}_i$ for all $i \in \mathcal{I}$. And that the supremum $\sup_{i\in\mathcal{I}} \mathcal{D}_i$ is the *weakest diffeology* on X containing $\mathcal{D}_i$ for all $i \in \mathcal{I}$.

1.3.1 Pushforward and pullback of diffeologies

Consider a map $f : X \to Y$, and assume that a diffeology D_X is given for X, we will be interested in finding the weakest diffeology on Y such that f is smooth, this diffeology will be called the *pushforward* of $\mathcal{D}_X$. Note that the strongest diffeology on Y such that f is smooth is uninteresting, why, it is just the top diffeology $\mathcal{D}^\bullet_X$. Dually, given a diffeology $\mathcal{D}_Y$ on Y we have the strongest diffeology on X such that f is smooth, called the *pullback* of $\mathcal{D}_Y$.

Definition 1.3.3 The *pullback* of the diffeology $\mathcal{D}_Y$ along a smooth map $f : X \to Y$ is

$$\overleftarrow{f}(\mathcal{D}_Y) := \{p \in \mathrm{Par}(X) \mid f \circ p \in \mathcal{D}_Y\}$$

which is a diffeology on X.

Definition 1.3.4 The *pushforward* of the diffeology $\mathcal{D}_X$ along a smooth map $f : X \to Y$ is

$$\overrightarrow{f}(D_X) := \langle f \circ \mathcal{D}_X \rangle = \left\{ p \in \mathrm{Par}(Y) \mid p \overset{\text{loc}}{\in} \mathcal{D}^\circ_Y \cup f \circ \mathcal{D}_X \right\}$$

which is a diffeology on Y.

Remark 1.3.5 We may note that if $f = \mathrm{id}_X$ then $\overrightarrow{f} = \overleftarrow{f} = \mathrm{id}_{\mathbb{D}(X)}$. And if $U \in \mathrm{O}\mathbb{R}^\infty$ with the standard diffeology, as defined in example 2, and $\alpha : U \to X$ is any map then $\alpha^*(\mathcal{D}_U) = \mathcal{G}(\alpha)$, hence $\overrightarrow{\alpha}(\mathcal{D}_U) = \langle \alpha \rangle$.

Lemma 1.3.6 *For a map $f : (X, \mathcal{D}_X) \to (Y, \mathcal{D}_Y)$ the following is equivalent*

(i) f is smooth.

(ii) $\overrightarrow{f}(\mathcal{D}_X) \subseteq \mathcal{D}_Y$

(iii) $\mathcal{D}_X \subseteq \overleftarrow{f}(\mathcal{D}_Y)$

Theorem 1.3.7
Let $f : X \to Y$ be a map, and let there be given a diffeology D_X on X, then the pushforward of D_X is the the weakest diffeology on Y such that f is smooth. Dually, let there be given a diffeology D_Y on Y, then the pullback of D_Y is the strongest diffeology on X such that f is smooth.

Lemma 1.3.8 *Consider two maps, a map f with range X and a map g with domain X then*

$$\overrightarrow{(g \circ f)} = \overrightarrow{g} \circ \overrightarrow{f} \quad \text{and} \quad \overleftarrow{(g \circ f)} = \overleftarrow{f} \circ \overleftarrow{g}$$

Subduction and inductions

Definition 1.3.9 A map between diffeological spaces is said to be a *pre-induction* if the diffeology of the domain space equals the pullback of the diffeology of the range space, and an *induction* if in addition it is injective. The map is said to be a *pre-subduction* if the pushforward of the diffeology of the domain space equals the diffeology of the range space, and a *subduction* if in addition it is surjective.

Remark 1.3.10 So a map $f : (X, \mathcal{D}_X) \to (Y, \mathcal{D}_Y)$ is a pre-induction if $\overleftarrow{f}(\mathcal{D}_Y) = \mathcal{D}_X$, and a pre-subduction if $\overrightarrow{f}(\mathcal{D}_X) = \mathcal{D}_Y$.

Remark 1.3.11 Notice that a smooth map $\pi : X \to Y$ is a pre-subduction if and only if for each plot $\beta : U \to Y$ and each $u_0 \in U$ there exist an open $U_0 \subseteq U$ with $u_0 \in U_0$ and a plot $\alpha : U_0 \to X$ such that $\pi \circ \alpha = \beta|_{U_0}$.

Example 4 (Subspaces)
Consider any subspace $A \subseteq \mathbb{R}^n$, we may then equip A with the subspace diffeology. Let $\iota : A \hookrightarrow \mathbb{R}^n$ be the canonical inclusion, then

$$\mathcal{D}_A := \{\alpha \in \mathrm{Par}(A) \mid \iota \circ \alpha \text{ is smooth (in the usual sens)}\} = \overleftarrow{\iota}(\mathcal{D}_{\mathbb{R}^n})$$

is a diffeology on A, it is called the subspace diffeology. Examples 2 and 14 as well as the subspace diffeology of example 3 are all examples of subspace diffeologies.

We may generalize this construction; consider a subset A of a diffeological space X the subspace diffeology on A is simply $\overleftarrow{\iota}(\mathcal{D}_X)$, where ι is the canonical inclusion. Note that, by theorem 1.3.7, the *subspace diffeology* is the strongest diffeology such that ι is smooth. Furthermore it is evident that ι becomes a induction. ▪

Example 5 (Quotients)
Let $(X, \mathcal{D}_X)$ be a diffeological space, and $\sim$ an equivalence relation on the set X. Let $\pi : X \to X/\sim$ denote the quotient map, i.e the surjective map taking $x \in X$ to its equivalence class. The *quotient diffeology* on the set $X/\sim$ is $\overrightarrow{\pi}(\mathcal{D}_X)$. Note that π becomes a subduction, when $X/\sim$ is given the quotient diffeology. ▪

1.3.2 The weak and the strong diffeology

We shall in this section introduce the notation of covers and co-covers, and the weak and strong diffeology. We shall later in the thesis find much use for the two lemmas 1.3.14 and 1.3.15. Although not important in this thesis, we shall however, for the sake of completeness, shortly discuss how the weak and strong diffeology relates to universal constructions in the category of diffeologies.

Covers and co-covers

Consider a diffeological space $(Y, \mathcal{D}_Y)$, and a collection of diffeologies on the set Y say $\{\mathcal{D}_i\}_{i\in\mathcal{I}}$. If, for any space Z, a map $f : (X, \mathcal{D}_Y) \to Z$ is smooth if and only if $f : (Y, \mathcal{D}_i) \to Z$ is smooth for all $i \in \mathcal{I}$, it would seems plausible to say that the collection $\{\mathcal{D}_i\}_{i\in\mathcal{I}}$ covers D_Y. In fact, as stated in theorem 1.3.13, any collection $\{\mathcal{D}_i\}_{i\in\mathcal{I}}$ of diffeologies on Y covers $\sup_{i\in\mathcal{I}} \mathcal{D}_i$, this characterizes supremum. We therefore make the following;

Definition 1.3.12 A family of diffeologies on a set X is said to be a *cover* of the space $(X, \mathcal{D}_X)$ if the supremum equals $\mathcal{D}_X$. And said to be a *co-cover* of $(X, \mathcal{D}_X)$ if the infimum equals $\mathcal{D}_X$.

Our main observation, in this section, is the content of the following theorem.

Theorem 1.3.13
Let $\{D_i\}_{i\in\mathcal{I}}$ be a family of diffeologies on Y. Then $\{D_i\}_{i\in\mathcal{I}}$ is a cover of $(Y,\mathcal{D}_Y)$ if and only if for any space $(Z,\mathcal{D}_Z)$

$$C^\infty(\mathcal{D}_Y,\mathcal{D}_Z)=\bigcap_{i\in\mathcal{I}}C^\infty(\mathcal{D}_i,\mathcal{D}_Z),$$

and a co-cover if and only if for any space $(X,\mathcal{D}_X)$

$$C^\infty(\mathcal{D}_X,\mathcal{D}_Y)=\bigcap_{i\in\mathcal{I}}C^\infty(\mathcal{D}_X,\mathcal{D}_i).$$

Next we will look at, what could be described as, a multi-map generalization of inductions and subductions. The setup is as follows, consider a collection of spaces $\{(X_j,\mathcal{D}_{X_j})\}_{j\in J}$, with any index set J, together with a collection of maps $f_j : X_j \to Y$. The following lemmas 1.3.14 and 1.3.15 are central to the constructions of limits and colimit in Dif, as we shall see below.

Lemma 1.3.14 *The following is equivalent*

(i) *$\mathcal{D}_Y$ is the weakest diffeology on Y such that for any $j\in J$ the map f_j is smooth.*

(ii) *$\{\overrightarrow{f_j}(\mathcal{D}_{X_j})\}_{j\in J}$ cover $(Y,\mathcal{D}_Y)$.*

(iii) *For any space Z, any map $g : Y \to Z$ is smooth if and only if $g\circ f_j$ is smooth for all $j\in J$.*

(iv) $\mathcal{D}_Y=\left\langle\bigcup_{j\in J}f_j\circ\mathcal{D}_{X_j}\right\rangle=\left\{\alpha\in\mathrm{Par}(Y)\mid\exists j\in J:\alpha\overset{\mathrm{loc}}{\in}\mathcal{D}_Y^\circ\cup f_j\circ\mathcal{D}_{X_j}\right\}$

Lemma 1.3.14 has a dual, to state this consider a collection of spaces $\{(Z_j,\mathcal{D}_{Z_j})\}_{j\in J}$, together with a collection of maps $g_j : Y\to Z_j$.

Lemma 1.3.15 *The following is equivalent*

(i) *$\mathcal{D}_Y$ is the strongest diffeology on Y such that for any $j\in J$ the map g_j is smooth.*

(ii) *$\{\overleftarrow{g_j}(\mathcal{D}_{Z_j})\}_{j\in J}$ co-cover $(Y,\mathcal{D}_Y)$.*

(iii) *For any space X, any map $f : X\to Y$, f is smooth if and only if $g_j\circ f$ is smooth for all $j\in J$.*

(iv) $\mathcal{D}_Y=\left\{\alpha\in\mathrm{Par}(Y)\mid\forall j\in J:g_j\circ\alpha\in\mathcal{D}_{Z_j}\right\}$

Definition 1.3.16 Consider a collection of maps $\{X_j\xrightarrow{f_j}Y\}_{j\in J}$. We shall say that the diffeology $\sup_{j\in J}\overrightarrow{f_j}(\mathcal{D}_{X_j})$ is the *weak diffeology* induced by the collection $\{f_j\}_{j\in J}$. And given a collection of maps $\{Y\xrightarrow{g_j}Z_j\}_{j\in J}$. We shall say that the diffeology $\inf_{j\in J}\overleftarrow{g_j}(\mathcal{D}_{Z_j})$ is the *strong diffeology* induced by by the collection $\{g_j\}_{j\in J}$.

Universal constructions

The category of diffeological space is complete and co-complete, i.e. all small limits and small colimits exist. Well known examples of limits are products, equalizers and pullbacks, there dual co-limits are coproducts, coequalizers, pushouts. An introduction to categorical limits and co-limits can, for example, be found in MacLane [1971].

As we shall see the completeness and cocompleteness follows more or less directly from lemma 1.3.14 and its dual.Theorem 1.3.17 below, shows the existence of limits, and tells us how to find them, its dual theorem 1.3.18 does the same for colimits.

Consider a small scheme (or index category) J and a diagram of type J, i.e a functor $\mathbf{D} : \mathrm{J} \to \mathrm{Dif}$, we wish to determine the limit and colimit of $\mathbf{D}$. Note that since Dif is a concrete category (denote by $|\cdot| : \mathrm{Dif} \to \mathrm{Set}$ the forgetfull functor), we may consider any diagram $\mathbf{D}$ in Dif as a diagram $|\mathbf{D}|$ in Set. The following theorem tells us how to lift limits in Set to limits in Dif.

Theorem 1.3.17
Let $\mathbf{D} : \mathrm{J} \to \mathrm{Dif}$ *be a small diagram, and* $|\mathbf{D}| : \mathrm{J} \to \mathrm{Set}$ *its associated diagram in* Set. *Let the cone* $\{X \xrightarrow{\eta_j} |\mathbf{D}(j)|\}_{j\in J}$ *be the limit of* $|\mathbf{D}|$. *Then the limit of* $\mathbf{D}$ *is*

$$\left\{\left(X, \inf_{j\in J} \overleftarrow{\eta_j}(\mathcal{D}_{\mathbf{D}(j)})\right) \xrightarrow{\eta_j} \mathbf{D}(j)\right\}_{j\in J}$$

Theorem 1.3.17 has a dual;

Theorem 1.3.18
Let $\mathbf{D} : \mathrm{J} \to \mathrm{Dif}$ *be a small diagram, and* $|\mathbf{D}| : \mathrm{J} \to \mathrm{Set}$ *its associated diagram in* Set. *Let the co-cone* $\{|\mathbf{D}(j)| \xrightarrow{\eta_j} X\}_{j\in J}$ *be the co-limit of* $|\mathbf{D}|$. *Then the co-limit of* $\mathbf{D}$ *is*

$$\left\{\mathbf{D}(j) \xrightarrow{\eta_j} \left(X, \sup_{j\in J} \overrightarrow{\eta_j}(\mathcal{D}_{\mathbf{D}(j)})\right)\right\}_{j\in J}$$

1.4 Products and co-products

We shall, several times, use constructions involving products. It is therefore convenient to collect some basic observations about diffeological products.

1.4.1 The product

Let $\{(X_i, \mathcal{D}_i\}_{i\in\mathcal{I}}$ be any family of diffeological spaces. The direct product of the sets is denoted by

$$\prod_{i\in\mathcal{I}} X_i$$

and the canonical projections by $\pi_i : \Pi_{i\in\mathcal{I}} X_i \to X_i$. In accordance with theorem 1.3.17, we define;

Definition 1.4.1 The *product diffeology* on $\Pi_{i\in\mathcal{I}} X_i$ is the co-cover diffeology $\inf_{i\in\mathcal{I}} \overleftarrow{\pi_i}(\mathcal{D}_i)$.

We have, as usual, that a map into a product is smooth if and only if all the coordinate maps are smooth, that is;

Proposition 1.4.2 *Let Y be any diffeological space, then a map $f : Y \to \Pi_{i\in\mathcal{I}} X_i$ is smooth if and only if $\pi_i \circ f$ is smooth for all $i \in \mathcal{I}$.*

Proof: By lemma 1.3.15. ▪

Furthermore,

Lemma 1.4.3 *The projections π_i are subductions.*

Proof: Let $j \in \mathcal{I}$, by lemma 1.3.15, the projections π_j are smooth. Let $\alpha : U \to X_j$ be a plot for X_j, and choose for each $i \in I$ an element $x_i \in X_i$, then define $\beta : U \to \Pi_{i\in I} X_i$ by

$$\beta(u) := \begin{cases} x_i & u \in X_i, \text{ for } i \neq j \\ \alpha(u) & u \in X_j \end{cases}$$

clearly β is smooth, i.e. a plot. Furthermore $\pi_j \circ \beta = \alpha$, it follows (by definition 1.3.4) that π_j is a subduction. ▪

Products and subspaces

Lemma 1.4.4 *Let $\{X_i\}_{i\in\mathcal{I}}$ and $\{A_i\}_{i\in\mathcal{I}}$ be collections of diffeological spaces, such that A_i is a subspace of X_i. Then the subspace diffeology on $\prod_{i\in\mathcal{I}} A_i \subseteq \prod_{i\in\mathcal{I}} X_i$ is the product diffeology.*

Proof: Let $\alpha : U \to \prod_{i\in\mathcal{I}} A_i$ be a plot for the subspace diffeology, that is α may be considered as a plot for the product diffeology on $\prod_{i\in\mathcal{I}} X_i$. It follows that $\pi_i \circ \alpha$ is smooth, hence a plot for the subspace diffeology on A_i. This implies that α is a plot for the product diffeology on $\prod_{i\in\mathcal{I}} A_i$.

Now if α is a plot for the product diffeology on $\prod_{i\in\mathcal{I}} A_i$, then since $\pi_i \circ \alpha$ is smooth for each $i \in \mathcal{I}$ it is a plot for the product diffeology on $\prod_{i\in\mathcal{I}} X_i$. Hence α is a plot for the subspace diffeology. ▪

Products of generated diffeologies

Definition 1.4.5 Let X_1 and X_2 be diffeological space, $\alpha_1 : U_1 \to X_1$ and $\alpha_2 : U_2 \to X_2$ parametrizations. Then define the *product parametrization* $\alpha_1 \times \alpha_2 : U_1 \times U_2 \to X_1 \times X_2$ by

$$\alpha_1 \times \alpha_2(u_1, u_2) := (\alpha_1(u_1), \alpha_2(u_2))$$

Note that if α_1 and α_2 are plots then $\alpha_1 \times \alpha_2$ is a plot for the product diffeology.

Definition 1.4.6 Let X_1 and X_2 be diffeological space and Ω_1 and Ω_2 be collections of parametrizations of respectively X_1 and X_2. Then let

$$\Omega_1 \times \Omega_2 := \{\alpha_1 \times \alpha_2 \mid \alpha_1 \in \Omega_1 \quad \text{and} \quad \alpha_2 \in \Omega_2\}.$$

Lemma 1.4.7 *Let X_1 and X_2 be diffeological spaces, with there diffeology generated by respectively the coverings Ω_1 and Ω_2. Then the product diffeology on $X_1 \times X_2$ is generated by $\Omega_1 \times \Omega_2$.*

Proof: We claim that $\langle \Omega_1 \times \Omega_2 \rangle = \mathcal{D}_{X_1 \times X_2}$. First of all the inclusion $\subseteq$ is obvious. For the other, let $\alpha : U \to X_1 \times X_2$ be a plot, and let $\alpha_i := \pi_i \circ \alpha$ for $i = 1, 2$. Then the α_i maps are, by proposition 1.4.2, plots for X_i, that is $\alpha_i \in \langle \Omega_i \rangle$. It follows, by a simple argument, that there exist an open cover $\{U_j\}_{j \in J}$ such that $\alpha_i|_{U_j} = \beta_{i,j} \circ h_{i,j}$ for $i = 1, 2$ with $\beta_{i,j} \in \Omega_i$ and $h_{i,j} : U_j \to \mathrm{Dom}(\beta_{i,j})$ a smooth map. Note that the map $h'_j : U_j \to \mathrm{Dom}(\beta_{1,j}) \times \mathrm{Dom}(\beta_{2,j})$ given by $h'_j(u) = (h_{1,j}(u), h_{2,j}(u))$ is smooth, therefore

$$\alpha|_{U_j}(u) = \big(\alpha_1|_{U_j}(u), \alpha_2|_{U_j}(u)\big) = (\beta_{1,j} \times \beta_{2,j}) \circ h'_j(u) \in \langle \Omega_1 \times \Omega_2 \rangle \,.$$

That is $\alpha \overset{\text{loc}}{\in} \langle \Omega_1 \times \Omega_2 \rangle$, hence $\alpha \in \langle \Omega_1 \times \Omega_2 \rangle$. ▪

1.4.2 The co-product

Let $\{(X_i, \mathcal{D}_i\}_{i \in \mathcal{I}}$ be any family of diffeological spaces. The coproduct (disjoint union) of the sets is denoted by

$$\coprod_{i \in \mathcal{I}} X_i$$

and the canonical injections by $\iota_i : X_i \to \coprod_{i \in \mathcal{I}} X_i$. And in accordance with the dual of theorem 1.3.17, we define

Definition 1.4.8 The *coproduct diffeology* on $\coprod_{i \in \mathcal{I}} X_i$ is the cover diffeology $\sup_{i \in \mathcal{I}} \overrightarrow{\iota_i}(\mathcal{D}_i)$

Lemma 1.4.9 *Let Z be any diffeological space, then a map $f : \coprod_{i \in \mathcal{I}} X_i \to Z$ is smooth if and only if $f \circ \iota_i$ is smooth for all $i \in \mathcal{I}$.*

Proof: By lemma 1.3.14. ▪

Note that if $X_i \in \mathrm{O}\mathbb{R}^\infty$ for all $i \in I$ then, by lemma 1.3.14 (iv) and since $\iota_i \circ \mathcal{D}_{X_i} = \mathcal{G}(\iota_i)$, the diffeology of the coproduct $\coprod_{i \in \mathcal{I}} X_i$ is just $\langle \{\iota_i\}_{i \in I} \rangle$.

Example 6 (Constructing the Line diffeology)
Recall examples 3 and 12 then consider the diffeology

$$\mathcal{X} := \frac{\mathbb{R}_v \coprod \mathbb{R}_h}{0_v \sim 0_h}$$

and let $\pi : \mathbb{R}_v \coprod \mathbb{R}_h \longrightarrow \mathcal{X}$ be the quotient map, and denote by ι_v and ι_h the canonical injections $\mathbb{R} \hookrightarrow \mathcal{X}$. Then by the above the diffeology of $\mathcal{X}$ is

$$\begin{aligned}
\mathcal{D}_{\mathcal{X}} &= \overrightarrow{\pi} \left(\mathcal{D}_{\mathbb{R}_v \coprod \mathbb{R}_h}\right) \\
&= \overrightarrow{\pi} \left\langle \iota_v, \iota_h \right\rangle \\
&= \left\langle \pi \circ \iota_v, \pi \circ \iota_h \right\rangle \\
&= \mathcal{D}_{line}.
\end{aligned}$$

The third equality requires a little work see Vincent [2008] for details. ▪

1.5 Diffeomorphism

A large part of the work done in this thesis will be concerned with proving that certain maps are diffeomorphism. We shall therefore, here, collect some simple observation concerning diffeomorphism. By diffeomorphism we mean a isomorphism in the category of diffeological spaces, i.e a smooth bijective map with smooth inverse.

Proposition 1.5.1 *For a bijective map f the following is equivalent*

(i) f is a diffeomorphism.

(ii) f is a subduction.

(iii) f is a induction.

Lemma 1.5.2 *A bijective map $\varphi : X \to Y$, between diffeological spaces, is a diffeomorphism if and only if the following holds*

(i) φ is smooth, and

(ii) there exist a genrating family Ω for $\mathcal{D}_Y$ such that $\Omega \overset{loc}{\subseteq} \varphi \circ \mathcal{D}_X$.

Proof: Let $\alpha \in \Omega$ then there exist an open cover $\{U_i\}_{i\in\mathcal{I}}$ of $\mathrm{Dom}(\alpha)$. By assumption there exist, for each $i \in \mathcal{I}$, a plot β_i for X such that

$$\phi^{-1} \circ \alpha|_{U_i} = \beta_i,$$

that is $\phi^{-1} \circ \alpha \overset{\text{loc}}{\in} \mathcal{D}_X$, hence $\phi^{-1} \circ \alpha \in \mathcal{D}_X$. By lemma 1.2.6 ϕ^{-1} is smooth. ▪

Lemma 1.5.3 *Let $\phi : X \to Y$ be a diffeomorphism and let A be a subspace of X. Then the restriction $\phi|_A$ is a diffeomorphism onto the subspace $\phi(A)$.*

Proof: By lemma 1.5.2. ▪

Local diffeomorphisms

Definition 1.5.4 Let $\varphi : X \to Y$ be a map, and let $x \in X$. Then φ is said to be *local diffeomorphism* at x if there exist a $\mathcal{D}$-open neighborhood A of x such that $\varphi(A)$ is $\mathcal{D}$-open and $\varphi : A \to \varphi(A)$ is a diffeomorphism (where the subsets are given the subspace diffeology).

Lemma 1.5.5 *A bijective map $\phi : X \to Y$ which is a local diffeomorphism at every point $x \in X$ is a diffeomorphism.*

Proof: By lemma 1.5.2. ▪

1.6 The $\mathcal{D}$-topology

Any diffeological space has an associated natural topology, called the $\mathcal{D}$-topology. The $\mathcal{D}$-topology will not play an important role in this thesis, except in section 3.5. We shall not go into a detailed study of the $\mathcal{D}$-topology. It is important to note that the $\mathcal{D}$-topology of a diffeological space is not an extra structure, but simply a topology naturally carried by a diffeological space.

Definition 1.6.1 Let $\mathcal{D}_X$ be a diffeology on the set X. The *D-topology* of D_X is the final topology with respect to the plots.

Remark 1.6.2 By definition 1.6.1 a subset A of X is $\mathcal{D}_X$-open if and only if for all $\alpha \in \mathcal{D}_X$ the preimage $\alpha^{-1}(A)$ is open.

Smooth maps are $\mathcal{D}$-continuous, to see this let $f : (X, \mathcal{D}_X) \to (Y, \mathcal{D}_Y)$ be a smooth map, and let $A \subseteq Y$ be a $\mathcal{D}_Y$-open subset, then $\alpha^{-1}(f^{-1}(A)) = (f \circ \alpha)^{-1}(A)$ for any plot $\alpha \in \mathcal{D}_X$ and since f is smooth $f \circ \alpha \in \mathcal{D}_Y$, hence $f^{-1}(A)$ is $\mathcal{D}_X$-open.

If a diffeology is generated the final topology with respect to the generating plots is the $\mathcal{D}$-topology, as we have the following;

Lemma 1.6.3 *Let Ω be a collection of parametrizations on X, then a subset $A \subseteq X$ is $\langle\Omega\rangle$-open if and only if $\alpha^{-1}(A)$ is open for all $\alpha \in \Omega$.*

Example 7 ($\mathcal{D}$-topology of the discrete diffeology)
The $\mathcal{D}$-topology of the discrete diffeology is just the discrete topology. ∎

Lemma 1.6.4 *Let $A \subseteq X$ be a subspace (i.e A is equipped with the subspace diffeology) then every set $\mathcal{U} \subseteq A$ open in the supspace topology is $\mathcal{D}_A$-open.*

Proof: Trivial. ∎

Our next example shows that the $\mathcal{D}$-topology of a subspace A of X is not always the subspace topology. When the $\mathcal{D}$-topology equals the subspace topology we say that $A \hookrightarrow X$ is an *embedding*.

Example 8 ($\mathbb{Q} \hookrightarrow \mathbb{R}$ is not an embedding)
Consider the standard diffeology $\mathcal{D}_{\mathbb{R}}$ on $\mathbb{R}$. Then the $\mathcal{D}$-topology of $(\mathbb{R}, \mathcal{D}_{\mathbb{R}})$ is the usual topology on $\mathbb{R}$. To see this let U be an open subset of the real line, then $\alpha^{-1}(U)$ is open for any plot α, since the plots, in this case, are smooth (in the usual sense) and therefore continues. And if A is a non open subset of $\mathbb{R}$ then $\beta^{-1}(A)$ is not open if β is the plot $\mathrm{id}_{\mathbb{R}}$.

Now consider $\mathbb{Q}$ as a subspace of $\mathbb{R}$, the subspace diffeology $\mathcal{D}_{\mathbb{Q}}$ is the discrete diffeology, as any plot in $\mathcal{D}_{\mathbb{R}}$ is continues hence only the constant plots can have there image contained in $\mathbb{Q}$. By example 7 the $\mathcal{D}$-topology of $(\mathbb{Q}, \mathcal{D}_{\mathbb{Q}})$ is the discrete topology. On the other hand the subspace topology of $\mathbb{Q}$ as a subspace of $\mathbb{R}$ with the usual topology, is not the discrete topology (consider the one point sets, they are not open, as every open set of $\mathbb{R}$ contains infinitely many rationals). Hence the $\mathcal{D}$-topology of $\mathbb{Q}$ as a subspace of $\mathbb{R}$ is not the subspace topology. ∎

We note that the $\mathcal{D}$-topology of the canonical diffeology on any open subset of any finite dimensional euclidean space is the standard topology, this follows by the first part of the argument in example 8.

Example 9 (The subspace cross is an embeding into $\mathbb{R}^2$.)
The cross equipped with the subspace diffeology is an embedding into $\mathbb{R}^2$. In order to justify this claim we only need to consider connected neighborhoods of 0. Since the canonical inclusions ι_v and ι_h (see example 3) are smooth it follows that every connected neighbourhood of 0 is similar to the first one shown in the figure. It follows that the $\mathcal{D}$-topology equals the subspace topology.

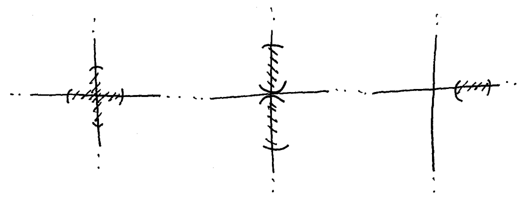


Figure 1.2: Examples of $\mathcal{D}$-open set on the cross.

1.7 The functional diffeology

The category of diffeological spaces is cartesian closed, as we shall show in this section. The canonical diffeology on function spaces is called the functional diffeology, it is introduced below. The functional diffeology will play a central role in many of the constructions we shall deal with in this thesis.

Definition 1.7.1 The *evaluation map* is the map $\text{eval} : X \times C^\infty(X,Y) \to Y$ defined by

$$\text{eval}(x, f) := f(x).$$

Definition 1.7.2 Let $M \subseteq C^\infty(X,Y)$, and let $\gamma : U \to M$ be any parametrization of M, and $\alpha : V \to X$ a parametrization of X. Then the map $\gamma \cdot \alpha : U \times V \to Y$ is the parametrization of Y given by

$$\gamma \cdot \alpha(u, v) := \gamma(u)\,[\alpha(v)]$$

Theorem 1.7.3
Let $M \subseteq C^\infty(X,Y)$. Then there exist a strongest diffeology on M such that the evaluation map is smooth. This diffeology is

$$\mathcal{D}_M := \{\gamma \in \text{Par}(M) \mid \forall \alpha \in \mathcal{D}_X : \gamma \cdot \alpha \in \mathcal{D}_Y\}$$

Proof: We must show (1) that $\mathcal{D}_M$ is a diffeology on M. And (2) that for any other diffeology $\mathcal{D}$ on M making the the evaluation map smooth, it holds that $\mathcal{D} \subseteq \mathcal{D}_M$.

(1) *Covering:* Let $\gamma : U \to M$ be a constant parametrization onto the point $f \in C^\infty(X,Y)$, then for any plot $\alpha : V \to X$ the map

$$(u, v) \to \gamma \cdot \alpha(u, v) = f \circ \alpha(v)$$

is clearly smooth, hence $\gamma \in \mathcal{D}_M$.
Smoothness: Let $\gamma : U \to M$ be a parametrization, and assume that $\gamma \in \mathcal{D}_M$. Furthermore let $h : V \to U$ and $\alpha : W \to X$ be smooth maps, then the map

$$(v, w) \to (\gamma \circ h) \cdot \alpha(v, w) = \gamma \cdot \alpha(h(v), w)$$

is evidently smooth, hence $\gamma \circ h \in \mathcal{D}_M$.
Locality: Again let $\gamma : U \to M$ be a parametrization, and assume there

exist an open covering $\{U_i\}_{i\in\mathcal{I}}$ of U such that $\gamma|_{U_i} \in \mathcal{D}_M$. Then for any plot $\alpha : W \to X$ and each $i \in \mathcal{I}$ the map

$$U_i \times W \ni (u, w) \to \gamma \cdot \alpha(u, w)$$

is a smooth map into Y, i.e a plot for $\mathcal{D}_Y$. Since $\mathcal{D}_Y$ is a diffeology and $\{U_i \times W\}_{i\in\mathcal{I}}$ an open cover of $U \times W$ it followes by the smoothness of $\mathcal{D}_Y$ that $\gamma \cdot \alpha$ is smooth, hence $\gamma \in \mathcal{D}_M$.

(2) Let $\gamma : U \to M$ be a plot for the diffeology $\mathcal{D}$, and $\alpha : V \to X$ a plot for $\mathcal{D}_X$. Then the map

$$(u, v) \to \mathrm{eval}(\alpha(v), \gamma(u)) = \gamma(u)[\alpha(v)] = \gamma \cdot \alpha(u, v)$$

is smooth, hence $\gamma \in \mathcal{D}_M$. ▪

Definition 1.7.4 (Functional diffeology) Let $M \subseteq C^\infty(X, Y)$. The diffeology $\mathcal{D}_M$ of theorem 1.7.3 is called the *functional diffeology* on M.

We shall, unless otherwise stated, always assume that $C^\infty(X, Y)$ is equipped with the functional diffeology. And we shall by $\mathcal{D}_{C^\infty(X,Y)}$ denote the functional diffeology.

Functional diffeology and genrating families

Proposition 1.7.5 *Let Ω be a generating covering of X, and $M \subseteq C^\infty(X, Y)$. Then $\gamma : V \to M$ is a plot for the functional diffeology if and only if $\gamma \cdot \alpha$ is a plot for Y for all $\alpha \in \Omega$.*

Proof: Only one implication is not obvious. Assume that $\gamma \cdot \alpha \in \mathcal{D}_Y$ for all $\alpha \in \Omega$. And let $\beta : U \to X$ be a plot for $\mathcal{D}_X$, by theorem 1.2.4 $\beta \overset{\text{loc}}{\in} \mathcal{G}(\Omega)$. Hence there exist a open cover $\{U_i\}_{i\in\mathcal{I}}$ of U such that $\gamma \cdot \beta|_{V\times U_i} = \gamma \cdot \alpha_i$ for $\alpha_i \in \mathcal{G}(\Omega)$, that is $\gamma \cdot \beta \overset{\text{loc}}{\in} \mathcal{G}(\mathcal{D}_Y) = \mathcal{D}_Y$. Hence $\gamma \cdot \beta \in \mathcal{D}_Y$. ▪

1.7.1 Cartesian closure of the category of diffeological spaces

As shown below the category of diffeological spaces is cartesian closed i.e. for diffeological spaces X, Y and Z

$$C^\infty(X \times Y, Z) \simeq C^\infty(X, C^\infty(Y, Z)).$$

Definition 1.7.6 Let $\Phi : \mathrm{Maps}(X \times Y, Z) \to \mathrm{Maps}(X, \mathrm{Maps}(Y, Z))$ denote the map

$$\Phi(f)(x)(y) := f(x, y)$$

Remark 1.7.7 That is $\Phi(f)(x)$ is the map $y \to f(x, y)$.

Our next lemma shows that the category of sets is cartesian closed.

Lemma 1.7.8 *The inverse of the bijective map Φ is the map $\Phi^{-1} : \mathrm{Maps}(X, \mathrm{Maps}(Y, Z)) \to \mathrm{Maps}(X \times Y, Z)$ given by*

$$\Phi^{-1}(f)(x, y) := f(x)(y)$$

Proof: Let $f \in \text{Maps}(X \times Y, Z)$ then

$$\Phi^{-1} \circ \Phi(f)(x,y) = \Phi(f)(x)(y) = f(x,y).$$

Now if $f \in \text{Maps}(X, \text{Maps}(Y,Z))$ then

$$\Phi \circ \Phi^{-1}(f)(x)(y) = \Phi^{-1}(f)(x,y) = f(x)(y).$$

∎

Remark 1.7.9 Given a plot $\gamma : U \to C^\infty(X,Y)$, note that

$$\Phi^{-1}(\gamma)(u,x) = \gamma \cdot \text{id}_X(u,x) = \text{eval}(x, \gamma(u)).$$

We are now ready to prove that the cartesian closure property;

Theorem 1.7.10
Φ is a diffeomorphism between the spaces $C^\infty(X \times Y, Z)$ and $C^\infty(X, C^\infty(Y,Z))$.

Proof: The proof of this statement will basically be just unwinding the definitions. Our strategy will be to (1) show that Φ restrict to a bijection $C^\infty(X \times Y, Z) \to C^\infty(X, C^\infty(Y,Z)))$. Then show that $\Phi \circ \mathcal{D}_{C^\infty(X\times Y,Z)} = \mathcal{D}_{C^\infty(X,C^\infty(Y,Z))}$, this will then imply that

$$\overrightarrow{\Phi}\left(\mathcal{D}_{C^\infty(X\times Y,Z)}\right) = \mathcal{D}_{C^\infty(X,C^\infty(Y,Z))}.$$

By proposition 1.5.1 Φ is then a diffeomorphism as claimed.

In order to show the equality $\Phi \circ \mathcal{D}_{C^\infty(X\times Y,Z)} = \mathcal{D}_{C^\infty(X,C^\infty(Y,Z))}$ we will first (2) show that $\Phi \circ \mathcal{D}_{C^\infty(X\times Y,Z)} \subseteq \mathcal{D}_{C^\infty(X,C^\infty(Y,Z))}$ and then (3) the other inclusion.

(1) Since we know that Φ is injective we just need to show that it and its inverse, maps smooth maps to smooth maps. So let $f \in C^\infty(X \times Y, Z)$ we then wish to show that $\Phi(f) \in C^\infty(X, C^\infty(Y,Z))$. Let $\alpha : U \to X$ be a plot we should then check that the map $u \to \Phi(f)(\alpha(u))$ is a plot for functional diffeology on $C^\infty(Y,Z)$. So let $\beta : V \to Y$ be smooth then the map

$$(\Phi(f) \circ \alpha) \cdot \beta(u,v) = \Phi(f)(\alpha(u))(\beta(v)) = f(\alpha(u), \beta(v))$$

is obviously smooth, hence $\Phi(f)$ is smooth. Secondly let $f \in C^\infty(X, C^\infty(Y,Z))$, and let $\alpha : U \to X$ and $\beta : V \to Y$ be plots then the map

$$(u,v) \to \Phi^{-1}(f)(\alpha(u), \beta(v)) = f(\alpha(u))(\beta(v)) = (f \circ \alpha) \cdot \beta(u,v)$$

is clearly smooth.It followes that Φ^{-1} is smooth.

(2) Let $\gamma : U \to C^\infty(X \times Y, Z)$ be smooth. We wish to show that $\Phi \circ \gamma$ is a plot for $\mathcal{D}_{C^\infty(X,C^\infty(Y,Z))}$, that is for any plot $\alpha : V \to X$ the map $(\Phi \circ \gamma) \cdot \alpha : U \times V \to C^\infty(Y,Z)$ is smooth. So let $\beta : W \to Y$ be a plot then

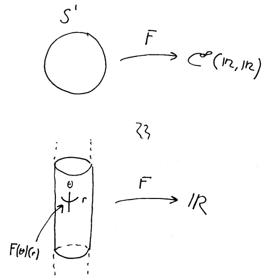
$$\begin{aligned}((\Phi \circ \gamma) \cdot \alpha) \cdot \beta(u,v,w) &= \Phi \circ \gamma(u)[\alpha(v)][\beta(w)] \\ &= \gamma(u)(\alpha(v), \beta(w)) \\ &= \gamma \cdot (\alpha \times \beta)(u,v,w)\end{aligned}$$

is smooth, hence $(\Phi \circ \gamma) \cdot \alpha$ is smooth.

(3) Let $\gamma : U \to C^\infty(X, C^\infty(Y, Z))$ be a plot, and let $\alpha : V \to X$ and $\beta : W \to Y$ be plots then the map

$$(u, v, w) \to \left(\Phi^{-1} \circ \gamma\right) \cdot (\alpha \times \beta)(u, v, w) = \gamma(u)[\alpha(v)][\beta(w)]$$

is smooth, hence $\Phi^{-1} \circ \gamma$ is a plot for $C^\infty(X \times Y, Z)$. ∎



See example 10.

Example 10 (A smooth map $S^1 \to C^\infty(\mathbb{R})$)
Let $F : S^1 \to C^\infty(\mathbb{R})$ be a smooth map, we may then consider F as a smooth map $S^1 \times \mathbb{R} \to \mathbb{R}$.

Example 11 (Differation is smooth)
Let $\xi_0 \in \mathbb{R}$ and consider the map $d_{\xi_0} : C^\infty(\mathbb{R}) \to \mathbb{R}$ given by

$$f \to \left.\frac{df(\xi)}{d\xi}\right|_{\xi_0}.$$

Let $\alpha : U \to C^\infty(\mathbb{R})$ be a plot, then

$$\begin{aligned} d_{\xi_0}(\alpha(u)) &= \left.\frac{d\alpha(u)(\xi)}{d\xi}\right|_{\xi_0} \\ &= \left.\frac{\partial \alpha \cdot \mathrm{id}_{\mathbb{R}}(u, \xi)}{\partial \xi}\right|_{(u,\xi_0)}. \end{aligned}$$

This implies that $u \to d_{\xi_0}(\alpha(u))$ is smooth, since the partial derivative of a smooth function is smooth. Consider now the map $\gamma : \mathbb{R} \to C^\infty(C^\infty(\mathbb{R}), \mathbb{R})$ given by $\gamma(t) := d_t$. Evidently

$$\gamma \cdot \alpha(t, u) = \left.\frac{\partial \alpha \cdot \mathrm{id}_{\mathbb{R}}(u, \xi)}{\partial \xi}\right|_{(u,t)},$$

hence $\gamma \cdot \alpha$ is smooth, i.e. γ is a plot. ∎

1.8 More examples

Example 12 (Diffeologies on the cross)
Consider the cross as in example 3, we shall here give some more examples of diffeologies on the cross. We will only concider diffeomorphisms such that the subspace diffeologies of the 4 subspaces (see the figure in example 3)

$$\mathcal{X}_a := (-\infty, 0)_v, \quad \mathcal{X}_b := (0, \infty)_v, \quad \mathcal{X}_c := (-\infty, 0)_h, \quad \mathcal{X}_d := (0, \infty)_h$$

are the canonical diffeology.

We shall by $\iota_a, \iota_b, \iota_c, \iota_d, \iota_h$ and ι_v denote the obvious inclusions. Furthermore we will for a function $f : \mathcal{X} \to \mathbb{R}$ denote by f_a the composition $f \circ \iota_a$ etc.

(a) *The weakest diffeology.* Consider the diffeology generated by the injections of the 4 subspaces given above, i.e. consider

$$\mathcal{D}_{\text{weak}} := \langle \iota_a, \iota_b, \iota_c, \iota_d \rangle.$$

This is the weakest diffeology such that the diffeologies of the subspaces $\mathcal{X}_a, \mathcal{X}_b, \mathcal{X}_c$ and $\mathcal{X}_d$ are the canonical diffeology, since if so then the 4 inclusions $\iota_a, \iota_b, \iota_c$ and ι_d must be smooth.

Now let $f : \mathcal{X} \to \mathbb{R}$ be $\mathcal{D}_{\text{weak}}$ smooth, then by lemma 1.2.6 the restriction of f to any of the 4 subspaces, from above, is smooth. Consider now any map $f : \mathcal{X} \to \mathbb{R}$ such that the restriction of f to any of the 4 subspaces is smooth, $f(0)$ may take any value. By lemma 1.2.6 f is $\mathcal{D}_{\text{weak}}$-smooth. Hence we conclude that

$$C^\infty(\mathcal{D}_{\text{weak}}) = \{f \in \text{Maps}(\mathcal{X}, \mathbb{R}) \mid f_a, f_b, f_c \text{ and } f_d \text{ are smooth}\}$$

Note that we use the notation $C^\infty(X)$ to denote the algebra $C^\infty(X, \mathbb{R})$.

(b) *The line diffeology.* Consider the diffeology

$$\mathcal{D} := \langle \iota_h, \iota_v \rangle$$

We claim that $\mathcal{D} = \mathcal{D}_{\text{line}}$ (see example 3), since $\iota_h, \iota_v \in \mathcal{D}_{\text{line}}$ the inclusion $\mathcal{D} \subseteq \mathcal{D}_{\text{line}}$ is evident. For the other inclusion, simply note that for a plot $\alpha \in \mathcal{D}_{\text{line}}$ there exist an open cover $\{U_i\}_{i\in\mathcal{I}}$ such that $\alpha|_{U_i} = \iota_h \circ h$ or $\alpha|_{U_i} = \iota_v \circ h$ for a usual smooth map h, hence $\alpha \overset{\text{loc}}{\in} \mathcal{D}$.

Clearly the injections ι_h and ι_v are $\mathcal{D}_{\text{line}}$-smooth. So if a function $f : \mathcal{X} \to \mathbb{R}$ is $\mathcal{D}_{\text{line}}$-smooth, then $f_v := f \circ \iota_v$ and $f_h := f \circ \iota_h$ are smooth also, hence

$$\lim_{x\to 0} f_v(x) = f(0) = \lim_{x\to 0} f_h(x)$$

in other words we have a kind of continuity of f at 0, this is true for f_v and f_h, but not for f'_v and f'_g (here f'_v etc denotes the first derivative of f_v). See also fig. 1.3.

Let us show by example this last claim, consider the map $f : \mathcal{X} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x & x \in \mathbb{R}_h \\ -x & x \in \mathbb{R}_v \end{cases}$$

this map is well defined (and $f(0) = 0$), it is clearly $\mathcal{D}_{\text{line}}$-smooth. Now $f_h(x) = x$ and $f_v(x) = -x$ hence

$$\lim_{x\to 0} f'_v(x) \neq \lim_{x\to 0} f'_h(x).$$

(c) *The subspace diffeology.*

Since $\mathcal{D}_{\text{line}} \subseteq \mathcal{D}_{\text{sub}}$ it follows that

$$C^\infty(\mathcal{D}_{\text{sub}}) \subseteq C^\infty(\mathcal{D}_{\text{line}})$$

infact we have equality. To see this let $f : \mathcal{X} \to \mathbb{R}$ be $\mathcal{D}_{\text{line}}$-smooth we then wish to show that it is $\mathcal{D}_{\text{sub}}$-smooth. In order to do this consider first the case where $f(0) \neq 0$ then let $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$\tilde{f}(x, y) := \frac{f_h(x) f_v(y)}{f(0)}$$

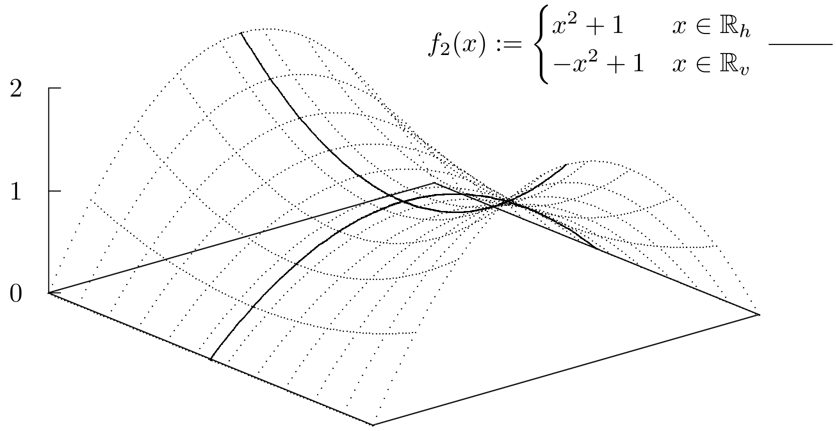


Figure 1.3: Plot of a smooth function f_2 (bold lines) on the cross with the line diffeology, and its extension $\tilde{f}_2$ (dotted lines) to all of $\mathbb{R}^2$, i.e. a smooth surface. See example 12 (c). For illustrative reasons we have illustrated a different function that the one used in example 12 (c), it is however clear that f_2 is not smooth "going around a corner", although it is C^1.

and note that

$$\tilde{f}(0,0) = f(0), \quad \tilde{f}(x,0) = f_h(x), \quad \tilde{f}(0,y) = f_v(y)$$

and since f_v and f_h are smooth $\tilde{f}$ is smooth, see fig. 1.3. Since any plot $\alpha \in \mathcal{D}_{\text{sub}}$ is in particular a smooth map into $\mathbb{R}^2$ it is clear that

$$f \circ \alpha = \tilde{f} \circ \alpha$$

is smooth, hence $f \in C^\infty(\mathcal{D}_{\text{sub}})$. For the case where $f(0) = 0$ simply consider the map $x \to f(x) + 1$ this is $\mathcal{D}_{\text{sub}}$-smooth, but then f is too.

(d) Let $\iota_\llcorner : \mathbb{R} \hookrightarrow X$ be given by

$$\iota_\llcorner := \begin{cases} \iota_a(x) & x \in (-\infty, 0) \\ 0 & x = 0 \\ \iota_d(x) & x \in (0, \infty) \end{cases}$$

And define

$$\mathcal{D}_c := \langle \iota_h, \iota_v, \iota_\llcorner \rangle$$

Then a smooth function $f : \mathcal{X} \to \mathbb{R}$ has in addition to the above the following properties

$$\lim_{x \to 0+} f_a^{(n)}(x) = \lim_{x \to 0+} (f \circ \iota_\llcorner)^{(n)}(x) = \lim_{x \to 0-} (f \circ \iota_\llcorner)^{(n)} = \lim_{x \to 0-} f_d^{(n)}(x)$$

By $\lim_{x\to 0-}$ we mean the limit approaching 0 from the right, i.e with x positive. And by $\lim_{x\to 0+}$ the limit approaching 0 from the left.

and likewise we may show that $\lim_{x\to 0-} f_b^{(n)}(x) = \lim_{x\to 0+} f_a^{(n)}(x)$ and $\lim_{x\to 0+} f_c^{(n)}(x) = \lim_{x\to 0-} f_d^{(n)}(x)$, hence we may talk of $f^{(n)}(0)$.

(e) Let $\iota_\neg : \mathbb{R} \hookrightarrow X$ be given by

$$\iota_\neg := \begin{cases} \iota_c(x) & x \in (-\infty, 0) \\ 0 & x = 0 \\ \iota_b(x) & x \in (0, \infty) \end{cases}$$

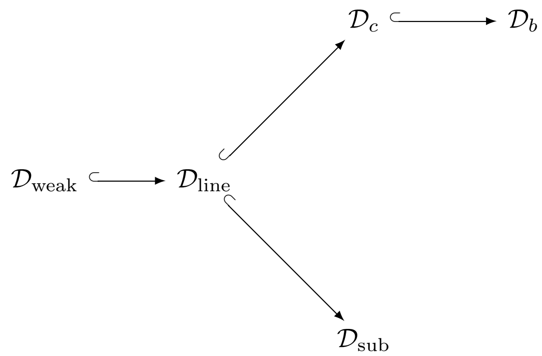
And define

$$\mathcal{D}_d := \langle \iota_h, \iota_v, \iota_\llcorner, \iota_\neg \rangle$$

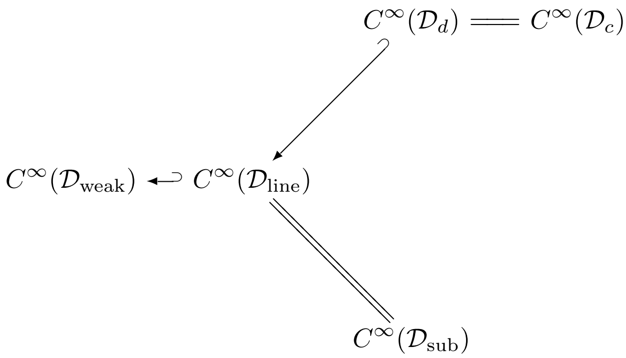
we note that the smooth functions $\mathcal{X} \to \mathbb{R}$ are exactly the same as in (c).

Now $\iota_\neg \notin \mathcal{D}_c$, since for any open neighbourhood $U \subseteq \mathbb{R}$ of 0, the image $\mathrm{Im}(\iota_\neg)$ is not a subset of any of the images $\mathrm{Im}(\iota_h)$, $\mathrm{Im}(\iota_v)$ or $\mathrm{Im}(\iota_\llcorner)$, hence $\iota_\neg|_U \notin \mathcal{G}(\{\iota_h, \iota_v, \iota_\llcorner\})$. ▪

To sum up we have the following relations between the diffeologies on the cross,



And between the algebras of functions



Example 13 (The cone)
Consider the following subspace of $\mathbb{R}^3$;

$$\Lambda := \left\{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2 \right\}.$$

We shall denote the subspace diffeology for the cone Λ by $\mathcal{D}_{\text{sub}}$. But the subspace diffeology is not the only diffeology on Λ we shall consider. Let $C \subseteq \mathbb{R}^3$ denote the cylinder of radius 1, with its canonical diffeology, i.e the subspace diffeology. Define a map $\pi : C \to \Lambda$ by

$$\pi(x, y, z) := (zx, zy, z).$$

π is easily seen to be a surjective map. We shall call the pushforward diffeology $\mathcal{D}_{\text{quo}} := \overrightarrow{\pi}(D_C)$ on Λ the *quotient cone diffeology*. Since π is a smooth map $(\mathbb{R}^3 \to \mathbb{R}^3)$ it follows that

$$\mathcal{D}_{\text{quo}} \subseteq \mathcal{D}_{\text{sub}}, \qquad \blacksquare$$

hence $C^\infty(\mathcal{D}_{\text{sub}}) \subseteq C^\infty(\mathcal{D}_{\text{quo}})$. Furthermore by lemma 1.3.14 a map $f : \Lambda \to \mathbb{R}$ is $\mathcal{D}_{\text{quo}}$-smooth if and only if $f \circ \pi : C \to \mathbb{R}$ is smooth.

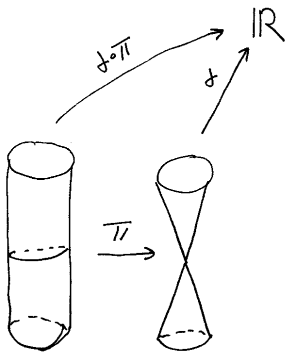


Figure 1.4: Smooth maps on the quotient cone.

The subspace cone is an embedding into $\mathbb{R}^2$. In order to justify this, we note that $\pi : C \to \Lambda$, as given above, is smooth, hence $\mathcal{D}$-continues (see section 1.6). Let U be a $\mathcal{D}$-neighbourhood of 0, and consider the $\mathcal{D}$-open set $\pi^{-1}(U) \subseteq C$. Notice that $C \hookrightarrow \mathbb{R}^3$ is an embedding and $\pi^{-1}(U)$ is open and containing the circle $\pi^{-1}(\{0\})$. Furthermore it is evident that $\pi|_{z \neq 0}$ is a diffeomorphism. It follows that there exist a neighbourhood $B \subseteq \mathbb{R}^3$ of 0 such that $B \cap U \subseteq U$, this implies that U is open in the subspace topology. (see also fig. 1.5)

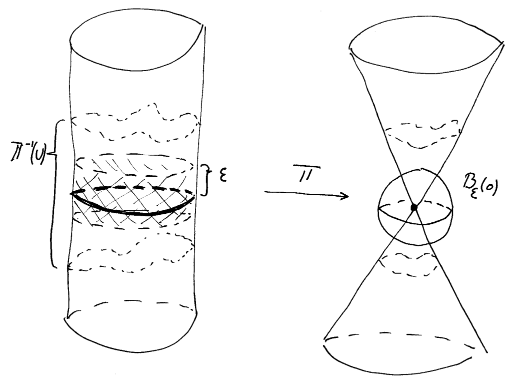


Figure 1.5: The subspace cone is an embedding into $\mathbb{R}^2$.

Example 14 (The canonical diffeology on Half spaces)
Let

$$\mathbb{H}^n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 \geq 0\}$$

We shall equip the half space $\mathbb{H}^n$ with the subspace diffeology, that is a parametrization $\alpha : U \to \mathbb{H}^n$ is a plot if it is smooth as a map into $\mathbb{R}^n$. To be precise, let $\iota : \mathbb{H}^n \hookrightarrow \mathbb{R}^n$ denote the canonical inclusion, the canonical diffeology on the half space $\mathbb{H}^n$ is then

$$\mathcal{D}_{\mathbb{H}^n} := \{\alpha \in \mathrm{Par}(\mathbb{H}^n) \mid \iota \circ \alpha \text{ is smooth}\}.$$

What about smooth functions on the half spaces. It can be shown that a $\mathcal{D}_{\mathbb{H}^n}$-smooth function $f : \mathbb{H}^n \to \mathbb{R}$ is smooth in the usual sense, that is it can be extended to a smooth function $\mathbb{R}^n \to \mathbb{R}$ (in the usual sense). This is not obvious, but it is however a direct consequence of the following old result of Whitney [1943],

Theorem 1.8.1
Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function even in the first coordinate, then there exist a smooth function $g : \mathbb{R}^n \to \mathbb{R}$ such that

$$f(x_1, \ldots, x_n) = g(x_1^2, \ldots, x_n).$$

To see how this imply the above claim, first note that the map $\alpha : \mathbb{R}^n \to \mathbb{H}^n$ given by

$$(x_1, \ldots, x_n) \xrightarrow{\alpha} (x_1^2, x_2, \ldots, x_n)$$

is a plot for $\mathcal{D}_{\mathbb{H}^n}$, i.e it is smooth in the usual sense. Hence if $f : \mathbb{H}^n \to \mathbb{R}$ is a $\mathcal{D}_{\mathbb{H}^n}$-smooth function then the function

$$\mathbb{R}^n \ni (x_1, \ldots, x_n) \to f(x_1^2, x_2, \ldots, x_n)$$

is smooth, and even in the first coordinate. By Whitney's theorem f may be extended to a smooth function on all of $\mathbb{R}^n$.

Roughly the same application of this theorem to diffeology is found in Iglesias-Zemmour [2007a], were in there also can be found a discussion about the diffeology of manifolds with boundary. ▪

Chapter 2

Diffeological vector spaces

In this chapter we are going to equip a vector space with a diffeology, in such a way that addition and scalar multiplication are smooth maps, the two together will then be called a diffeological vector space. We shall study smooth linear maps on diffeological vector spaces, and tensor product of diffeological vector spaces. We shall also in the last section take a brief look at diffeological algebras, as we study the algebra $C^\infty(X)$, which will be important in the following chapters.

Note that we shall only consider real vector spaces, although it should be straightforward to extend most of the results found in this section to vector spaces over arbitrary fields.

2.1 Diffeological Vector spaces

Definition 2.1.1 Let E be a real vector space and $\mathcal{D}_E$ a diffeology on E. The diffeological space $(E, \mathcal{D}_E)$ is said to be a *diffeological vector space* if vector addition and scalar multiplication are smooth mappings.

Remark 2.1.2 So $(E, \mathcal{D}_E)$ is a diffeological vectorspace if the map $(\sigma_1, \sigma_2) \to \sigma_1 + \sigma_2$ is an element of $C^\infty(E \times E, E)$, and the map $(\lambda, \sigma) \to \lambda\sigma$ an element of $C^\infty(\mathbb{R} \times E, E)$

A diffeology $\mathcal{D}_E$ on a vector space E is said to be a *vector space diffeology* if $(E, \mathcal{D}_E)$ is a diffeological vector space.

Example 15
$\mathbb{R}^n$ with the canonical diffeology is a diffeological vectorspace. ▪

Example 16
The functional space $C^\infty(X) := C^\infty(X, \mathbb{R})$ is a diffeological vector space when addition and scalar multiplication are defined pointwise. ▪

The proof for the following proposition is trivial, we shall therefore omit it.

Proposition 2.1.3 *A linear subspace of a diffeological vector space, is a diffeological vector space when equipped with the subspace diffeology.*

2.1.1 Linear maps between diffeological vector spaces

Definition 2.1.4 Let E and F be diffeological vector spaces over the field $\mathbb{R}$, then the set of *smooth linear maps* is

$$\mathcal{L}^\infty(E,F) := \{A : E \to F \mid A \text{ is linear and smooth}\}$$

Furthermore, as a diffeological space, $\mathcal{L}^\infty(E,F)$ is equipped with the functional diffeology.

Proposition 2.1.5 *$\mathcal{L}^\infty(E,F)$ is a diffeological vector space, with addition and scalar multiplication defined pointwise, hence for $A_1, A_2 \in \mathcal{L}^\infty(E,F)$*

$$(A_1 + A_2)(\sigma) := A_1(v) + A_2(v) \quad \text{and} \quad (kA_1)(\sigma) := kA_1(\sigma).$$

Proof: Let us first show that $\mathcal{L}^\infty(E,F)$ is a vector space, hence that given any two $A_1, A_2 \in \mathcal{L}^\infty(E,F)$ and $\lambda \in \mathbb{R}$ the linear maps $A_1 + A_2$ and λA_2 are smooth. First note that the map $\sigma \to (A_1(\sigma), A_2(\sigma)) \in F \times F$ is smooth (by proposition 1.4.2). Since the composition of smooth maps is smooth, the map

$$\sigma \longrightarrow (A_1(\sigma), A_2(\sigma)) \xrightarrow{+} A_1(\sigma) + A_2(\sigma) = (A_1 + A_2)(\sigma)$$

is smooth. Secondly note that the map $\sigma \to (\lambda, A_1(\sigma))$ is smooth, as scalar multiplication is smooth, it follows that the map $\sigma \to \lambda A_1(\sigma)$ is smooth. Hence $\mathcal{L}^\infty(E,F)$ is a vector space.

Next we wish to show that $\mathcal{L}^\infty(E,F)$ is a diffeological vector space, that is we must show that addition and scalar multiplication are smooth operations, hence that the maps

$$(A_1, A_2) \to A_1 + A_2 \quad \text{and} \quad A_1 \to \lambda A_1$$

are smooth. So let $\alpha_i : U_i \to \mathcal{L}^\infty(E,F)$, i=1,2 be plots and let $\beta : U_3 \to E$ be a plot, then

$$\begin{aligned}(u_1,u_2,u_3) \to (\alpha_1(u_1) + \alpha_2(u_2))[\beta(v)] = \alpha_1 \circ \pi_1(u_1,u_2,u_3)[\beta \circ \pi_3(u_1,u_2,u_3)] \\ + \alpha_2 \circ \pi_2(u_1,u_2,u_3))[\beta \circ \pi_3(u_1,u_2,u_3)]\end{aligned}$$

which is evidently a plot for F. Hence addition is smooth. The smoothness of the scalar multiplication follwoes by a similar argument. ▪

Example 17
The collection $\mathrm{Der}_x^\infty(C^\infty(X),\mathbb{R})$ of all smooth derivations, at $x \in X$, on the algebra $C^\infty(X)$ is a diffeological vector space. As it is a linear subspace of $\mathcal{L}^\infty(C^\infty(X),\mathbb{R})$. ▪

2.2 The weak diffeology for vector spaces

The weak diffeology for vector spaces as we introduce it below, is a generalization of *the fine diffeology* for vector spaces, as found in Iglesias-Zemmour [2007c].

Definition 2.2.1 Let E be a real vector space. And let $\mathcal{D}_E$ be any diffeology on E. Then the *weak vector space diffeology* on E generated by $\mathcal{D}_E$, is the weak diffeology generated by the collection of parametrizations of the form

$$U \ni u \to \sum_{i=1}^{n} \lambda_i(u)\gamma_i(u)$$

where $n \in \mathbb{N}$, $\lambda_i : U \to \mathbb{R}$ is smooth, and $\gamma_i : U \to E$ is a plot for $\mathcal{D}_E$.

We shall by $\sum \mathcal{D}_E$ denote the weak vector space diffeology on E generated by $\mathcal{D}_E$.

Theorem 2.2.2
The weak vector space diffeology on E generated by $\mathcal{D}_E$ is the weakest vector space diffeology on E containing D_E.

Proof: Let $\mathcal{D}_E$ be a diffeology on the vector space E. We must then, first of all, show that $\sum \mathcal{D}_E$ is a vector space diffeology. Let $\gamma : U \to E$ and $\gamma' : V \to E$ be two generating plots for $\sum \mathcal{D}_E$ hence we may write

$$\gamma(u) = \sum_{i=1}^{n} \lambda_i \gamma_i(u) \quad \text{and} \quad \gamma'(v) = \sum_{i=n+1}^{n+m} \lambda_i \gamma_i(v),$$

with $n, m \in \mathbb{N}$, $\lambda_i : U \to \mathbb{R}$ smooth, and $\gamma_i : U \to E$ plots for $\mathcal{D}_E$. Define the maps $\lambda_i(u, v) = \lambda_i(u)$ for $i = 1, \ldots, n$, $\lambda_i(u, v) = \lambda_i(v)$ for $i = n+1, \ldots, m$ and similar for γ_i, evidently all of these maps are smooth. Now

$$\gamma(u) + \gamma'(v) = \sum_{i=1}^{n+m} \lambda_i(u, v)\gamma_i(u, v)$$

hence $(u, v) \to \gamma(u) + \gamma(v)$ is smooth as a map into $(E, \sum \mathcal{D}_E)$. By lemmas 1.4.7 and 1.2.6 smoothness of $(u, v) \to \gamma(u) + \gamma(v)$ implies smoothness of addition on $(E, \sum \mathcal{D}_E)$. Smoothness of scalar multiplication follows by a similar argument.

Let us now show (i), hence let $\mathcal{D}$ be a vector space diffeology on E with $\mathcal{D}_E \subseteq \mathcal{D}$. Then evidently every generating plot for $\sum \mathcal{D}_E$ are plots for $\mathcal{D}$, hence $\sum \mathcal{D}_E \subseteq \mathcal{D}$. Claim (ii) and (iii) are just special cases of (i). ▪

Corollary *Theorem 2.2.2 implies that*

(i) If Ω is a collection of parametrizations on E. Then the weak vector space diffeology $\sum \langle \Omega \rangle$ is the weakest vector space diffeology, on E, such that the maps Ω are smooth.

(ii) The weak vector space diffeology $\sum \mathcal{D}_E^\circ$ is the weakest vector space diffeology on E.

We shall call the vector space diffeology $\sum \mathcal{D}_E^\circ$ the *weak vector space diffeology*. This diffeology, i.e the weak vector space diffeology, is called the fine diffeology for vector spaces in Iglesias-Zemmour [2007c].

Smooth linear map on weak diffeological vector spaces

Theorem 2.2.3
Let E be a vector spaces equipped with a diffeology D_E and let F be a diffeological vector space. Then a linear map $A : (E, \sum \mathcal{D}_E) \to F$ is smooth if and only if $A : (E, \mathcal{D}_E) \to F$ is smooth.

Proof: Assume that $A : (E, \mathcal{D}_E) \to F$ is smooth, and let $\gamma : U \to E$ be a generating plot for $\sum D_E$, hence

$$\gamma(u) = \sum_{i=1}^{n} \lambda_i \gamma_i(u),$$

with $n \in \mathbb{N}$, $\lambda_i : U \to \mathbb{R}$ smooth, and $\gamma_i : U \to E$ plots for $\mathcal{D}_E$. Then

$$A(\gamma(u)) = \sum_{i=1}^{n} \lambda_i A(\gamma_i(u))$$

hence $A : (E, \sum \mathcal{D}_E) \to F$ is smooth. The other implication is trivial. ▪

Corollary *Let E and F be diffeological vector space, if E is equipped with the weak diffeology then*

$$\mathcal{L}^\infty(E, F) = \{A : E \to F \mid A \text{ is linear}\} .$$

Example 18
The canonical diffeology of $\mathbb{R}^n$ is the weak vectorspace diffeology. ▪

Example 19 (Vector space genrated by a set)
Let X be a set, and let $\mathcal{D}_X$ be a diffeology on X. Let $E := \operatorname{span}\{X\}$, and note that we may consider $\mathcal{D}_X$ as a collection of parametrizations into E. Then consider the diffeology

$$\mathcal{D}_E := \sum \langle \mathcal{D}_X \rangle_E$$

on E. We may also concider $\sum \mathcal{D}_X$ as a diffeology on E, this make sense as in definition 2.2.1 we do not need that $\mathcal{D}_X$ is a diffeology. In fact $\sum \mathcal{D}_X$ is a vector space diffeology, this is seen by going through the first part of the proof of theorem 2.2.2, in particular we are using that the generating plots for $\sum \mathcal{D}_X$ make up a covering of E. Now trivially $\sum \mathcal{D}_X \subseteq \sum \langle \mathcal{D}_X \rangle_E$, it is also obvious that $\mathcal{D}_E^\circ \cup \mathcal{D}_X \subseteq \sum \mathcal{D}_X$. Lemma 1.2.3 then implies that $\langle \mathcal{D}_X \rangle_E \subseteq \sum \mathcal{D}_X$, as $\sum \langle \mathcal{D}_X \rangle_E$ is the weakest vector space diffeology containing $\langle \mathcal{D}_X \rangle_E$ it follows that

$$\sum \mathcal{D}_X = \sum \langle \mathcal{D}_X \rangle_E .$$

Let F be a diffeological vector space then by theorem 2.2.3 and lemma 1.2.6

$$\mathcal{L}^\infty(E, F) = \{A : E \to F \mid A \text{ is linear and } A \circ \mathcal{D}_X \subseteq \mathcal{D}_F\} .$$ ▪

Example 20 (Diffeological Dual)
Let E be a diffeological vector space over the field $\mathbb{K}$. The *diffeological dual vector space* of E is

$$E^* := \mathcal{L}^\infty(E, \mathbb{K}).$$

If E is equipped with the weak diffeology then, by theorem 2.2.3, the diffeological dual equals the algebraic dual. Consider for example the vector space $\mathbb{R}^\infty$ (consisting of infinite sequences where only finitely many elements are nonzero) equipped with the weak vector space diffeology. The diffeological dual is then $(\mathbb{R}^\infty)^* = \mathbb{R}^\mathbb{N}$ (the vector space consisting of *all* infinite sequences) equipped with the functional diffeology as a dual to $\mathbb{R}^\infty$. Let $\{e_i\}_{i \in \mathbb{N}}$ denote a basis for $\mathbb{R}^\infty$ then it follows that a parametrization $\alpha : U \to \mathcal{L}^\infty(\mathbb{R}^\infty, \mathbb{R})$ is a plot if and only if $u \to \alpha(u)[e_i]$ is smooth. Hence as a diffeological vector space

$$\mathbb{R}^\mathbb{N} = \prod_{\mathbb{N}} \mathbb{R}$$

(where addition and scalar multiplication are defined pointwise). ▪

2.3 Tensors and multilinear maps

We shall in this section introduce the diffeological tensor product.

Multilinear maps

Let $E_1, \dots, E_n$ and F be diffeological vector spaces. Recall that a map $A : E_1 \times \cdots \times E_n \to \Omega$ is said to be multilinear if it is separately linear in each n coordinates, i.e. if for each $i = 1, \dots, n$ and any $\lambda, \lambda' \in \mathbb{R}$

$$A(\sigma_1, \dots, \lambda\sigma_i + \lambda'\sigma_i', \dots, \sigma_n) = \lambda A(\sigma_1, \dots, \sigma_i, \dots, \sigma_n) + \lambda' A(\sigma_1, \dots, \sigma_i', \dots, \sigma_n).$$

Denote by

$$\mathcal{L}^\infty_{Mult}(E_1 \times \cdots \times E_n, F)$$

the collection of all multilinear and smooth maps $E_1 \times \cdots \times E_n \to F$. And by

$$\mathcal{L}^\infty_{Alt}(E_1 \times \cdots \times E_n, F)$$

the collection of all multilinear, alternating and smooth maps $E_1 \times \cdots \times E_n \to F$. We shall consider these two vector spaces as functional diffeological spaces, i.e. we equip them with the functional diffeology. Arguments similar to those found in section 2.1.1 will show that these spaces are in fact diffeological vector spaces.

Tensor product

The following result (theorem 2.3.1) may be found in Greub [1967]. Notice that there are no conditions of finite dimensionality.

Theorem 2.3.1 (Tensor product)
Let $E_1, \dots, E_n$ be vector spaces. Then there exist a unique (up to isomorphism) vector space, denoted $E_1 \otimes \cdots \otimes E_n$, and a multilinear map

$$\phi : E_1 \times \cdots \times E_n \to E_1 \otimes \cdots \otimes E_n \tag{2.1}$$

having the following universal property: Given any vector space F, for each multilinear map $A' : E_1 \times \cdots \times E_n \to F$ there exist a unique linear map $A : E_1 \otimes \cdots \otimes E_n \to F$ such that $A \circ \phi = A'$.

The construction in theorem 2.3.1 is called the tensor product. We shall sometimes denote it by $(E_1 \otimes \cdots \otimes E_n, \phi)$. We wish to equip the tensor product with a diffeology, the following tensor product diffeology is a natural choice;

Definition 2.3.2 Let $E_1, \dots, E_n$ be diffeological vector spaces. The *tensor product diffeology* on $E_1 \otimes \cdots \otimes E_n$ is the weak vector space diffeology $\sum \overrightarrow{\phi}(\mathcal{D}_{E_1 \times \cdots \times E_n})$. Where $\mathcal{D}_{E_1 \times \cdots \times E_n}$ is the product diffeology and ϕ the map 2.1.

The following remarks are direct consequence of the definition of the tensor product diffeology;

Remark 2.3.3 Let $\alpha : V \to E_1 \otimes \cdots \otimes E_n$ be a plot. Then for each $v_0 \in V$ there exist an open set $V_0 \subseteq V$ containing v_0 and such that

$$V_0 \ni v \to \alpha(v) = \sum_{i=1}^{k} \lambda_i(v)\phi[\beta_i(v)]$$

with $\lambda_i : V_0 \to \mathbb{R}$ smooth and $\beta_i : V_0 \to E_1 \times \cdots \times E_n$ plots for the product diffeology.

Remark 2.3.4 The tensor product ϕ (map 2.1) is smooth. The tensor product diffeology is in fact the weakest vector space diffeology on the tensor product such that ϕ is smooth.

Theorem 2.3.5
As diffeological vector spaces

$$\mathcal{L}^\infty(E_1 \otimes \cdots \otimes E_n, F) \simeq \mathcal{L}^\infty_{Mult}(E_1 \times \cdots \times E_n, F).$$

Proof: Define a map $\Phi : \mathcal{L}^\infty(E_1 \otimes \cdots \otimes E_n, F) \to \mathcal{L}^\infty_{Mult}(E_1 \times \cdots \times E_n, F)$ by

$$\Phi(A) := A \circ \phi, \tag{2.2}$$

where ϕ is the tensor product. First of all Φ is well defined, to see this let $A \in \mathcal{L}^\infty(E_1 \otimes \cdots \otimes E_n, F)$ and let $\alpha_i : U_i \to E_i$ be plots then

$$\Phi(A)(\alpha_1 \times \cdots \times \alpha_n(u_1, \ldots, u_n)) = A\left(\phi[\alpha_1 \times \cdots \times \alpha_n(u_1, \ldots, u_n)]\right)$$

hence $\Phi(A)$ is smooth.

We claim that Φ is a linear diffeomorphism. Φ is, by theorem 2.3.1, evidently linear and injective. To see that Φ is surjective let $A' \in \mathcal{L}^\infty_{Mult}(E_1 \times \cdots \times E_n, F)$, by theorem 2.3.1 there exist a unique linear map $A : E_1 \otimes \cdots \otimes E_n \to F$ such that $A \circ \phi = A'$, where ϕ is the map 2.1. This implies that $A(\phi[\mathcal{D}_{E_1 \times \cdots \times E_n}]) = A'(\mathcal{D}_{E_1 \times \cdots \times E_n})) \subseteq \mathcal{D}_F$, and by definition 1.3.4, lemma 1.2.6 and theorem 2.2.3 this implies that A is smooth. Hence Φ is surjective.

We must, in addition, show that the map defined by eq. (2.2) is a diffeomorphism. Let $\gamma : U \to \mathcal{L}^\infty(E_1 \otimes \cdots \otimes E_n, F)$ be a plot for the functional diffeology and let $\alpha : V \to E_1 \times \cdots \times E_n$ a plot for the product diffeology (note that $\phi \circ \alpha$ is a plot for the tensor product diffeology). Then the map

$$(u, v) \to \Phi(\gamma) \cdot \alpha(u, v) = \gamma(u) \circ \phi[\alpha(v)] = \gamma \cdot (\phi \circ \alpha)(u, v)$$

is seen to be smooth. That is Φ is smooth. Now let $\gamma' : U \to \mathcal{L}^\infty_{Mult}(E_1 \times \cdots \times E_n, F)$ be a plot for the functional diffeology. By theorem 2.3.1 there exist a unique map $\gamma : U \to \mathcal{L}^\infty(E_1 \otimes \cdots \otimes E_n, F)$ such that $\Phi(\gamma(u)) = \gamma(u) \circ \phi = \tilde{\gamma}(u)$. Let $\alpha : V \to E_1 \otimes \cdots \otimes E_n$ be a plot, by remark 2.3.3 we may assume that

$$\alpha(v) = \sum_{i=1}^{k} \lambda_i(v) \phi \circ \beta_i(v)$$

with $\lambda_i : V \to \mathbb{R}$ smooth and $\beta_i : V \to E_1 \times \cdots \times E_n$ plots. Hence the map

$$\begin{aligned}(u, v) \to \gamma(u)[\alpha(v)] &= \sum_{i=1}^{k} \lambda_i(v)(\gamma(u) \circ \phi)[\beta_i(v)] \\ &= \sum_{i=1}^{k} \lambda_i(v) \gamma' \cdot \beta_i(u, v)\end{aligned}$$

is smooth. It follows that γ is a plot for the functional diffeology, and by lemma 1.5.2 this implies that Φ is a diffeomorphism. ▪

2.4 The algebra of smooth real functions

Let X be a diffeological space, then the space $C^\infty(X) := C^\infty(X, \mathbb{R})$ (with the functional diffeology) is a diffeological vector space. As usual addition and scalar multiplication are defined pointwise. In addition pointwise multiplication is seen to be smooth. In short we may say that $C^\infty(X)$ is a diffeological algebra. It is not hard to verify that $C^\infty(X)$ is in fact a diffeological algebra, we shall therefore omit the proof.

Definition 2.4.1 Let $\varphi : X \to Y$ be a smooth map. Define $\hat{\varphi} : C^\infty(Y) \to C^\infty(X)$ by

$$\hat{\varphi}(f) := f \circ \varphi$$

Let us collect a few simple observations;

Proposition 2.4.2 *Let $\varphi : X \to Y$ be a smooth map. Then the map $\hat{\varphi}$ is a smooth algebra homomorphism.*

Proof: We shall show (1) that $\hat{\varphi}$ is smooth, (2) that it is an algebra homomorphism;

(1) Consider two plots $\gamma : U \to C^\infty(Y)$ and $\alpha : V \to X$. Then

$$\hat{\varphi}(\gamma(u))[\alpha(v)] = \gamma(u) \circ \varphi[\alpha(v)]$$

hence $(u, v) \to \hat{\varphi}(\gamma(u))[\alpha(v)]$ is smooth, as $\varphi \circ \alpha$ is a plot for Y.

(2) Consider two functions $f, g \in C^\infty(Y)$ and a constant $\lambda \in \mathbb{R}$, then

$$\hat{\varphi}(f + g) = (f + g) \circ \varphi = f \circ \varphi + g \circ \varphi = \hat{\varphi}(f) + \hat{\varphi}(g).$$

It is as well evident that $\hat{\varphi}(fg) = \hat{\varphi}(f)\hat{\varphi}(g)$ and $\hat{\varphi}(\lambda g) = \lambda\hat{\varphi}(g)$. ▪

Lemma 2.4.3 *Let $\varphi : X \to Y$ be a smooth map. Then*

(i) if φ is surjective then $\hat{\varphi}$ is injective.

(ii) if φ is a diffeomorphism then $\hat{\varphi}$ is an isomorphism.

Proof: (i) Assume that φ is surjective. Then for any two function $f, g \in C^\infty(Y)$ with $f \circ \varphi = g \circ \varphi$ surjectiveness of φ implies that $f = g$. Hence $\hat{\varphi}$ is injective.

(ii) Simply note that for $f \in C^\infty(X)$

$$\hat{\varphi} \circ \widehat{\varphi^{-1}}(f) = f \circ \varphi^{-1} \circ \varphi = f,$$

and $\widehat{\varphi^{-1}} \circ \hat{\varphi} = \mathrm{id}_{C^\infty(Y)}$. ▪

Determining $C^\infty(X)$ and Boman's theorem.

How can we determine which function $C^\infty(X)$ consist of? The following lemma can sometimes be helpful;

Lemma 2.4.4 *A map $f : X \to \mathbb{R}$ is smooth if and only if for each smooth curve $\alpha : \mathbb{R} \to X$ the map*

$$\xi \to f \circ \alpha(\xi)$$

is smooth.

In order to prove this we need to utilize Boman's theorem. This is not a trivial result and a prove may be found in Boman [1967].

Theorem 2.4.5 (Boman)
Let $f : \mathbb{R}^n \to \mathbb{R}$ be any function, then f is smooth if and only if $f \circ c$ is smooth for every smooth curve $c : \mathbb{R} \to \mathbb{R}^n$.

It is said that the theorem, given above, by J. Boman, inspired A. Frölicher in his work on, what later was to be known as, Frölicher spaces. Lemma 2.4.4 trivially follows by applying Boman's theorem. In fact lemma 2.4.4 holds if we replace $C^\infty(X)$ with $C^\infty(X, F)$ where F is any diffeological Frölicher space. An introduction to diffeological Frölicher spaces may be found in Vincent [2008].

Chapter 3

Diffeological Tangent spaces

We shall in this chapter construct the tangent space over a point x on a diffeological space X. Below is listed some natural requirement that a tangent space should fulfil. The constructed tangent space will fulfil all of these.

(a) To each point x of a diffeological space X is associated a vector space T_xX. And to each smooth map $\phi : X \to Y$ between diffeological spaces X and Y is associated a linear map $T_x\phi : \mathrm{T}_xX \to \mathrm{T}_{\phi(x)}Y$. This association should define a functor

$$\text{category of diffeological spaces} \to \text{category of vector spaces.}$$

We shall call this functor the tangent functor.

(b) The tangent functor should be an extension of the usual tangent functor on smooth manifolds.

(c) Each tangent space should be equipped with a natural diffeology, and for $U \in \mathrm{O}\mathbb{R}^\infty$ and each $u \in U$

$$\mathrm{T}_uU \simeq \mathbb{R}^{\dim(U)},$$

as diffeological spaces.

(d) T_xX should be a linear subspace of the vector space of smooth derivation at x on $C^\infty(X)$. And for each smooth map $\alpha : U \to C^\infty(X)$ and any tangent vector $\delta \in \mathrm{T}_xX$ the map

$$u \to \delta(\alpha(u))$$

should be smooth.

3.1 Plot derivations

Let, in the following, X be a diffeological space and x a point in X. As allready mentioned each plot $\alpha : \mathbb{R} \to X$ with $\alpha(0) = x$ induces a derivation, denoted $d\alpha$ on the algebra $C^\infty(X)$ of smooth functions on X. We shall in this section make this precise.

Definition 3.1.1 The set of *curves centered at* x is the set

$$\mathcal{P}_x(X) := \{\alpha \in C^\infty(\mathbb{R}, X) \mid \alpha(0) = x\}.$$

We shall equip $\mathcal{P}_x(X)$ with the functional diffeology. Hence we consider $\mathcal{P}_x(X)$ as a subspace of $C^\infty(\mathbb{R}, X)$. Note that by the cartesian closure property (see section 1.7) a plot $\gamma : U \to \mathcal{P}_x(X)$ for the space of curves centered at x, may be identified with a smooth map $\gamma : U \times \mathbb{R} \to X$ for which $\gamma(u, 0) = x$.

Definition 3.1.2 Let $r \in \mathbb{R}$ and $\alpha \in \mathcal{P}_x(X)$. Define the map $r * \alpha : \mathbb{R} \to X$ by

$$r * \alpha(u) := \alpha(ru)$$

Note that if α is a curve centered at x then $r * \alpha$ is too.

Lemma 3.1.3 *The map* $* : \mathbb{R} \times \mathcal{P}_x(X) \to \mathcal{P}_x(X)$ *is smooth.*

Proof: Consider a plot $\gamma : U \to \mathcal{P}_x(X)$, and a smooth parametrization $\lambda : V \to \mathbb{R}$. In order to prove the claim, we must show that the map

$$(u, v) \to \lambda(v) * \gamma(u)$$

is smooth, with $\lambda : V \to \mathbb{R}$ a smooth function. That is for any smooth function $h : W \to \mathbb{R}$, the map

$$(u, v, w) \to (\lambda(v) * \gamma(u)) \cdot h(w) = \gamma(u)[\lambda(v)h(w)]$$

is smooth, but it clearly is. ∎

We shall next define a *plot derivation* on X. Note that we use the notation

$$d_0(f) := \left.\frac{df(u)}{du}\right|_0,$$

with $f \in C^\infty(\mathbb{R}, \mathbb{R})$.

Definition 3.1.4 Let $\alpha \in \mathcal{P}_x(X)$ be a plot. The, by α, induced *plot derivation* is the map $d\alpha : C^\infty(X) \to \mathbb{R}$ given by

$$d\alpha(f) := d_0(f \circ \alpha).$$

Example 21
If $\alpha \in \mathcal{P}_x(X)$ is locally constant at x then $d\alpha = 0$. ∎

Lemma 3.1.5 *Let* $\alpha \in \mathcal{P}_x(X)$, *then* $d(r * \alpha) = rd\alpha$.

Proof: Let $f \in C^\infty(X, \mathbb{R})$ then

$$d(r * \alpha)(f) = \left.\frac{df \circ \alpha(ru)}{du}\right|_0 = rd\alpha(f).$$

∎

Theorem 3.1.6
Let $\alpha \in \mathcal{P}_x(X)$, *then the induced plot derivation* $d\alpha : C^\infty(X) \to \mathbb{R}$ *is a smooth derivation, at* x, *on the algebra* $C^\infty(X)$.

Proof: Let α be a curve on X, centered at $x \in X$. We shall then show (1) that $d\alpha : C^\infty(X) \to \mathbb{R}$ is smooth, and (2) that it is a derivation on $C^\infty(X)$.

(1) Let $\gamma : V \to C^\infty(X)$ be a plot, then the map $\gamma \cdot \alpha : V \times \mathbb{R} \to \mathbb{R}$ is smooth (see section 1.7). For each $v \in V$

$$d\alpha(\gamma(v)) = d_0(\gamma(v) \circ \alpha) = \left. \frac{\partial \gamma \cdot \alpha(v,u)}{\partial u} \right|_{(v,0)}$$

hence the map $v \to d\alpha(\gamma(v))$ is smooth, as $\gamma \cdot \alpha$ is smooth (and the partial derivative of a smooth map is smooth).

(2) $d\alpha$ is linear since for any two $f, g \in C^\infty(X)$

$$d_0([(f+g) \circ \alpha]) = d_0(f \circ \alpha) + d_0(g \circ \alpha),$$

and a derivation since in addition

$$d_0([(fg) \circ \alpha]) = d_0(f \circ \alpha) g(x) + f(x) d_0(g \circ \alpha)$$

■

3.2 The Tangent cone

The collection of plot derivation on X, makes up a cone in the vector space of derivations on X, called the tangent cone. We shall, in this section, equip the tangent cone with a natural diffeology, the tangent cone diffeology.

Definition 3.2.1 Let V be a real vector space, and A a subset of V then let

$$\mathbb{R} \cdot A := \{\lambda a \mid \lambda \in \mathbb{R} \quad \text{and} \quad a \in A\}$$

Definition 3.2.2 (Cone) Let V be a real vector space, a subset A of V is a (pointed) *cone* if

$$\mathbb{R} \cdot A \subseteq A$$

Definition 3.2.3 (Tangent cone) The *Tangent cone set* is the set

$$\mathrm{C}_x X := \{d\alpha \mid \alpha \in \mathcal{P}_x(X)\}.$$

Notice that we have the following inclusions

$$\mathrm{C}_x X \subseteq \mathrm{Der}^\infty(C^\infty(X), \mathbb{R}) \subseteq \mathcal{L}^\infty(C^\infty(X), \mathbb{R}) \subseteq C^\infty(C^\infty(X), \mathbb{R}).$$

Proposition 3.2.4 *The set $\mathrm{C}_x X$ is a pointed cone in the vectorspace of all derivations, at x, on $C^\infty(X)$.*

Proof: Evidently $\mathrm{C}_x X$ is a subset of the vectorspace of derivations, at x, on $C^\infty(X)$. And by lemma 3.1.5 $\mathrm{C}_x X$ is a cone. ■

Example 22 (Tangent cones on the cross)
Consider the cross as in examples 3 and 12. Let us go through the cases and find the tanget cone, at 0, in each case.

(a) *The weak diffeology.* Since the only curve centered at 0 is the constant it follows that

$$\mathrm{C}_0\mathcal{X} = \{0\}$$

(b) *The line diffeology.* Let α be a curve centered at 0 on $(\mathcal{X}, \mathcal{D}_{\text{line}})$. Then by theorem 1.2.4 α is locally constant at 0, or there exist a open neighbourhood $U \subseteq \mathrm{Dom}(\alpha)$ of 0 and a non constant smooth map $h : U \to \mathbb{R}$, such that $\alpha|_U = \iota_v \circ h$ or $\alpha|_U = \iota_h \circ h$. If α is locally constant at 0 then evidently $d\alpha = 0$, so assume that $\alpha|_U = \iota_v \circ h$. Clearly $h(0) = 0$, and a simple calculation will show that

$$d\alpha = h'(0)d\iota_v.$$

Hence in general $d\alpha$ must be zero or proportional to $d\iota_h$ or $d\iota_v$. To see that the two derivations $d\iota_h$ and $d\iota_v$ are not equal consider the map $f : \mathcal{X} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x & x \in \mathbb{R}_h \\ -x & x \in \mathbb{R}_v \end{cases}$$

this map is well defined (as $f(0) = 0$), and it is smooth since $f \circ i_h$ and $f \circ i_v$ are smooth. Now

$$d\iota_h(f) = 1 \quad \text{and} \quad d\iota_v(f) = -1.$$

We conclude that

$$\mathrm{C}_0\mathcal{X} = \mathbb{R} \cdot \{d\iota_h, d\iota_v\}.$$

(c) *The subspace diffeology.* The only curves which are not found in the line diffeology but are found in the subspace diffeology, are the singular curves (see example 3). If α is a singular curve then obviously $d\alpha = 0$, as $\alpha^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. Hence, bearing in mind that $C^\infty(\mathcal{D}_{\text{sub}}) = C^\infty(\mathcal{D}_{\text{line}})$, the tangent cone is the same as for the line diffeology.

(d) An argument similar to that in the first part of (b) shows that

$$\mathrm{C}_x\mathcal{X} \subseteq \mathbb{R} \cdot \{d\iota_h, d\iota_v, d\iota_\llcorner\}.$$

Now if $f : \mathcal{X} \to \mathbb{R}$ is $\mathcal{D}_c$ smooth, then

$$d\iota_h(f) = d_0(f \circ i_h) = f_h'(0)\,, \quad d\iota_v(f) = f_v'(0) \quad \text{and} \quad d\iota_\llcorner(f) = f_\llcorner'(0)$$

in example 12 it is shown that $f_h'(0) = f_v'(0) = f_\llcorner'(0)$, hence the tangent cone is a one dimensional vector space.

(e) A similar argument as for (c) shows that also in this case the tangent cone is a one dimensional vector space. ▪

Notice that, the above example in particular shows that the tangent cone is not always a vector space.

3.2.1 The tangent cone diffeology

We shall equip the tangent cone with a natural diffeology, the tangent cone diffeology, as defined in the following;

Definition 3.2.5 (Tangent cone diffeology) A parametrization $\gamma : U \to \mathrm{C}_x X$ is a plot for the *tangent cone diffeology*, if for each $u_0 \in U$ there exist an open neighbourhood U_0 of u_0 and a smooth parametrization $\gamma^\dagger : U_0 \to \mathcal{P}_x(X)$ such that

$$\gamma(u) = d[\gamma^\dagger(u)] \quad \text{for all } u \in U_0.$$

We shall call the map $\gamma^\dagger$, from the above definition, a to γ associated map. The collection of plots for the tangent cone diffeology will be denoted by $\mathcal{D}_{\mathrm{C}_x X}$, it is not hard to see that it is in fact a diffeology. Note in particular that the tangent cone diffeology is genrated by parametrizations of the form

$$u \to d[\gamma^\dagger(u)],$$

with $\gamma^\dagger : U \to \mathcal{P}_x(X)$ a plot, we shall call plots of this form for generating plots (although this is a bit misleading, as there may be other generating sets). It is important to note that the collection of generating plots is a smooth covering. By lemma 1.2.6 a map $A : \mathrm{C}_x X \to Y$ (with Y any diffeological space) is smooth if and only if

$$u \to A(d[\gamma^\dagger(u)])$$

is smooth for all $\gamma^\dagger : U \to \mathcal{P}_x(X)$ plots.

Lemma 3.2.6 *Let $x \in X$ then, the evaluation map* $\mathrm{eval} : C^\infty(X) \times \mathrm{C}_x X \to \mathbb{R}$ *is smooth.*

Proof: We must show that the diffeology of the cone space $\mathcal{D}_{\mathrm{C}_x X}$ is weaker than the functional diffeology on $\mathrm{C}_x X \subseteq C^\infty(C^\infty(X), \mathbb{R})$. Hence let $\gamma : U \to \mathrm{C}_x X$ be a generating plot for $\mathcal{D}_{\mathrm{C}_x X}$, and consider a smooth map $\phi : V \to C^\infty(X)$, then the map

$$\begin{aligned}\gamma \cdot \phi(u, v) &= d[\gamma^\dagger(u)](\phi(v)) \\ &= d_0(\phi(v) \circ \gamma^\dagger(u)) \\ &= \left.\frac{\partial[\phi \cdot (\gamma^\dagger \cdot \mathrm{id}_{\mathbb{R}})](v, u, \xi)}{\partial \xi}\right|_{(v,u,0)}\end{aligned}$$

is smooth, since the map $\phi \cdot (\gamma^\dagger \cdot \mathrm{id}_{\mathbb{R}}) : V \times U \times \mathbb{R} \to \mathbb{R}$ is smooth. Assume now that γ is a plot for $\mathcal{D}_{\mathrm{C}_x X}$ then $\gamma \cdot \phi$ is, by the above, smooth in a neighbourhood of every point of the domain. It follows that $\gamma \cdot \phi$ is smooth, that is γ is a plot for the functional diffeology. ∎

Example 23
Consider a plot $\gamma : U \to \mathrm{C}_0 \mathcal{X}$ for the line diffeology and write it as

$$\gamma(u) = \lambda_h(u) d\iota_h + \lambda_v(u) d\iota_v.$$

(Note that at each point u either $\lambda_h(u) = 0$, $\lambda_v(u) = 0$ or they are both zero). Consider the $\mathcal{D}_{\text{line}}$ smooth map $f : \mathcal{X} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x & x \in \mathbb{R}_h \\ 0 & x \in \mathbb{R}_v \end{cases}.$$

By lemma 3.2.6 the map $u \to \mathrm{eval}(f, \gamma(u)) = \gamma(u)(f) = \lambda_h(u)$ is smooth. It follows, by a similar argument, that $\lambda_v(u)$ is smooth.

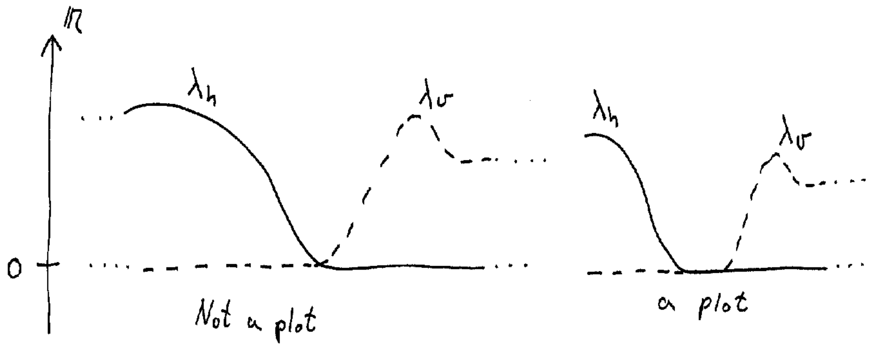


Figure 3.1: Plots for the tangent cone space at 0 on $\mathcal{X}$.

Let us make a more careful analysis of the tangent cone space at 0. Consider a plot $\gamma^\dagger : U \to \mathcal{P}_0(\mathcal{X})$, by the cartesian closure we may consider $\gamma^\dagger$ as a smooth map $U \times \mathbb{R} \to \mathcal{X}$ with $\gamma^\dagger(u)(0) = 0$. By restricting the domain we may assume that there exist a smooth map $h : U \times \mathbb{R} \to \mathbb{R}$ such that

$$\gamma^\dagger(u, r) = \iota_h \circ h(u, r) \quad \text{or} \quad \gamma^\dagger(v, r) = \iota_h \circ h(u, r).$$

It follows that

$$d[\gamma^\dagger(u)] = \left.\frac{\partial h(u, r)}{\partial r}\right|_{(u,0)} d\iota_h \quad \text{or} \quad d[\gamma^\dagger(u)] = \left.\frac{\partial h(u, r)}{\partial r}\right|_{(u,0)} d\iota_v.$$

■

Example 24
Consider the cone with the subspace diffeology (see example 13) , and the plots $\alpha_\theta(t) := (\cos(\theta)t, \sin(\theta)t, t)$. The map $\gamma^\dagger : \mathbb{R} \to C^\infty(\mathbb{R}, \Lambda)$ given by $\gamma^\dagger(\theta) := \alpha_\theta$ is smooth, and $\gamma^\dagger(\theta)(0) = 0$. Hence $\gamma(\theta) = d\alpha_\theta$ is a plot for the tangent cone space at 0.

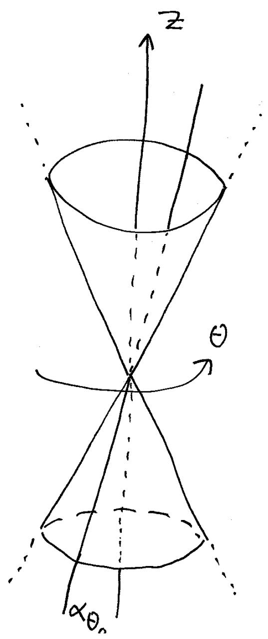


Figure 3.2: The plot α_θ.

Open question *Is γ a constant plot? (I think so, the problem is essentially to determine $C^\infty(\Lambda)$, and the behaviour of functions near the singularity).*

3.3 The Tangent Space

Definition 3.3.1 (Tangent space) The Tangent space at $x \in X$ is the vector space

$$\mathrm{T}_x X := \mathrm{span}\{\mathrm{C}_x X\}.$$

Notice that $\mathrm{C}_x X \subseteq \mathrm{T}_x X$ hence we may view the tangent cone diffeology as a collection of parametrizations on the tangent space. However $D_{\mathrm{C}_x X}$ is not necessarily a diffeology on the tangent space. The tangent space diffeology is defined as the weakest vectorspace diffeology containing $D_{\mathrm{C}_x X}$ see section 2.2 and example 19. In other words;

Definition 3.3.2 (Tangent space diffeology) *The tangent space diffeology* is the vector space diffeology

$$\mathcal{D}_{\mathrm{T}_x X} := \sum \mathcal{D}_{\mathrm{C}_x X}.$$

When we in the following talk of *the tangent space* we shall mean, the tangent space, as a diffeological space, where the diffeology is the tangent space diffeology as described above. Note that $\sum \mathcal{D}_{\mathrm{C}_x X} = \sum \langle \mathcal{D}_{\mathrm{C}_x X} \rangle_{\mathrm{T}_x X}$.

In order to clarify the constructions we shall, in the following proposition, sum up some important properties of the tangent space diffeology. The proof is essentially the discussion in example 19.

Proposition 3.3.3 (The tangent space diffeology) *Let X be a diffeological space. Then the following holds*

(i) A generating plot for $\mathcal{D}_{\mathrm{T}_x X}$ is of the form

$$u \to \sum_{i}^{n} \lambda_i(u) d[\gamma_i^\dagger(u)]$$

where $\lambda_i : U \to \mathbb{R}$ is smooth and $\gamma_i^\dagger : U \to C^\infty(\mathbb{R}, X)$ smooth with $\gamma_i^\dagger(u)(0) = x$ for all $u \in U$.

(ii) Let F be any diffeological vector space. Then a linear map $A : \mathrm{T}_x X \to F$ is smooth if and only if

$$u \to A(d[\gamma^\dagger(u)])$$

is smooth, for all plots $\gamma^\dagger : U \to C^\infty(\mathbb{R}, X)$ with $\gamma^\dagger(u)(0) = x$ for all $u \in U$.

(iii) $\mathcal{D}_{\mathrm{T}_x X}$ is the weakest vector space diffeology on $\mathrm{T}_x X$ such that the maps $u \to d[\gamma^\dagger(u)]$ are smooth.

Example 25 (The tangent space of a discrete diffeoloical space)
Consider any discrete diffeological space X, then at any point $x \in X$ by example 21 the tangent space is the zero space. ▪

Example 26 (The Tangent space of $\mathbb{R}^n$)
For each $0 < i \leq n$ define the following maps. Define $\rho_i : \mathbb{R} \to \mathbb{R}^n$ by

$$\rho_i(\xi) = (\underbrace{0, \ldots, 0}_{i-1 \text{ copies}}, \xi, 0, \ldots, 0).$$

And let $\xi_0 \in \mathbb{R}^n$ and define $\tau_i(\xi_0) : \mathbb{R} \to \mathbb{R}^n$ by

$$\tau_i(\xi_0)(\xi) := \rho_i(\xi) + \xi_0.$$

Then note that for $f \in C^\infty(\mathbb{R}^n, \mathbb{R})$, $\xi_0 \in \mathbb{R}^n$

$$d[\tau_i(\xi_0)](f) = d_0(f \circ \tau_i(\xi_0)) = \left. \frac{\partial f(\xi_1, \ldots, \xi_n)}{\partial \xi_i} \right|_{\xi_0}$$

so for any $\alpha \in \mathcal{P}_{\xi_0}(\mathbb{R}^n)$ (letting $\alpha(\xi) = (\alpha_1(\xi), \ldots, \alpha_n(\xi))$)

$$d\alpha(f) = d_0(f \circ \alpha) = \sum_{i=1}^{n} \left. \frac{d\alpha_i}{d\xi} \right|_0 d[\tau_i(\xi_0)](f). \tag{3.1}$$

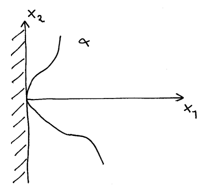
Hence the set of vectors $d[\tau_1(\xi_0)], \ldots, d[\tau_n(\xi_0)]$ spans the tangent space $\mathrm{T}_{\xi_0}\mathbb{R}^n$. It is not hard to see that these vectors are linearly independent, i.e they form a basis for the tangent space at ξ_0. Notice that $\mathrm{C}_{\xi_0}\mathbb{R}^n = \mathrm{T}_{\xi_0}\mathbb{R}^n$.

We have in fact that $T_a\mathbb{R}^n \simeq \mathbb{R}^n$, as the tangent space diffeology is the weak vector space diffeology. To see this, let $\gamma : U \to \mathrm{T}_{\xi_0}\mathbb{R}^n$ be a plot for the tangent cone diffeology, and let $\gamma^\dagger(u)(\xi) = (\gamma_1^\dagger(u)(\xi), \ldots, \gamma_n^\dagger(u)(\xi))$ be its associated map (see definition 3.2.5). Then by eq. (3.1)

$$\begin{aligned} \gamma(u) &= \sum_{i=1}^{n} d_0(\gamma_i^\dagger(u)) d[\tau_i(\xi_0)] \\ &= \sum_{i=1}^{n} \left. \frac{\partial \gamma_i^\dagger \cdot \mathrm{id}_{\mathbb{R}}(u,\xi)}{\partial \xi} \right|_{(u,0)} d[\tau_i(\xi_0)] \end{aligned}$$

hence γ is a plot for the weak vector space diffeology. It follows that the tangent space diffeology is the weak vector space diffeology. And by example 18 this implies that $T_a\mathbb{R}^n \simeq \mathbb{R}^n$. ▪

Example 27 (The Tangent space of half spaces)
Let $n \in \mathbb{N}$ and consider the half space $\mathbb{H}^n$ with the canonical diffeology, as defined in example 14. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a curve on $\mathbb{H}^n$ centered at 0 and let $f \in C^\infty(\mathbb{H}^n)$. As discussed, in example 14, we may extend f to a map $f' \in C^\infty(\mathbb{R}^n)$. Note that



A curve on $\mathbb{H}^2$.

$$d\alpha(f) = d_0(f \circ \alpha) = d_0(f' \circ \alpha) = \sum_{i=1}^{n} d_0(\alpha_i) \left. \frac{\partial f'}{\partial x_i} \right|_0 = \sum_{i=1}^{n} d_0(\alpha_i) d[\tau_i(0)]\,(f')\,.$$

Evidently $d_0(\alpha_1) = 0$, it follows that

$$d\alpha(f) = \sum_{i=2}^{n} \alpha_i'(0) d[\tau_i(0)](f).$$

The conclusion, the tangent space at an interior point has dimension n and at a boundary point it has, as expected, dimension $n-1$. ▪

3.3.1 The Cotangent space

Definition 3.3.4 The *dual tangent space* is the diffeological vector space

$$T_x^* X := \mathcal{L}^\infty(\mathrm{T}_x X, \mathbb{R})$$

Example 28 (The tangent space of the star)
Consider the set

$$X_* := \frac{\coprod_{n \in \mathbb{N}} \mathbb{R}_n}{\{0_n\}_{n \in \mathbb{N}}}$$

we shall call this set *The Star*. Consider the canonical injections $\iota_n : \mathbb{R} \to \mathbb{R}_n$, and define a diffeology for X_* by

$$\mathcal{D}_* := \sup\{\overrightarrow{i}_n(\mathcal{D}_{\mathbb{R}})\} = \left\langle \bigcup_{n \in \mathbb{N}} \iota_n \right\rangle$$

An argument similar to that given in example 22 will show that

$$\mathrm{C}_x X = \mathbb{R} \cdot \{d\iota_n \mid n \in \mathbb{N}\}$$

hence $\mathrm{T}_x X \simeq \mathbb{R}^\infty$, and $T_x^* X \simeq \mathbb{R}^{\mathbb{N}}$. ▪

3.4 The tangent map

Definition 3.4.1 Let $\varphi : X \to Y$ be a smooth map between diffeological spaces, and let $x \in X$. Define a map $T_x\varphi : \mathrm{C}_x X \to \mathrm{C}_{\varphi(x)} Y$ by

$$T_x\varphi(d\alpha) := d[\varphi \circ \alpha].$$

Note that for $f \in C^\infty(X)$

$$T_x\varphi(d\alpha)(f) = d\alpha(\hat{\varphi}(f)), \tag{3.2}$$

(see definition 2.4.1). This implies in particular that $T_x\varphi$ is well defined. We will show, below, that this map extends to a linear map between the tangent spaces, the tangent map.

Definition 3.4.2 Let A be a (pointed) cone of the real vectorspace V and B a (pointed) cone of the real vector space W. A map $S : A \to B$ is a cone map if for any $\lambda \in \mathbb{R}$ and any $a \in A$

$$S(\lambda a) = \lambda S(a)$$

Note that $T_x\varphi$ is a cone map as

$$T_x\varphi(\lambda d\alpha) = T_x\varphi\left(d[\lambda * \alpha]\right) = \lambda d[\varphi \circ \alpha].$$

Lemma 3.4.3 *Let A and B be (pointed) cones of the real vector spaces. A cone map $S : A \to B$ extends to a linear map (also called S) between the real vectorspaces* $\mathrm{span}(A)$ *and* $\mathrm{span}(B)$ *if and only if for any $a_1, \ldots, a_n \in A$*

$$a_1 + \cdots + a_n \in A \Rightarrow S(a_1 + \cdots + a_n) = S(a_1) + \cdots + S(a_n).$$

Proof: Define a map $S' : \mathrm{span}(A) \to \mathrm{span}(B)$ by

$$S'(a_1 + \cdots + a_n) := S(a_1) + \cdots + S(a_n)$$

where $a_1, \ldots, a_n \in A$. This map is properly defined since if $a_1 + \cdots + a_n = a_{n+1} + \cdots + a_m$ then by assumption

$$S(a_1) = S(a_{n+1}) + \cdots + S(a_m) - S(a_2) - \cdots - S(a_n)$$

hence

$$S'(a_1 + \cdots + a_n) = S'(a_{n+1} + \cdots + a_m).$$

The map S' as defined is clearly a linear extension of the cone map. ▪

Proposition 3.4.4 (Tangent map) *Let $\varphi : X \to Y$ be a smooth map. Then the map $T_x\varphi : \mathrm{C}_x X \to \mathrm{C}_{\varphi(x)} Y$ extends to a smooth linear map $T_x\varphi : \mathrm{T}_x X \to T_{f(x)} Y$.*

Proof: We must show (1) that the cone map $T_x\varphi$ extend to a linear map between the tangent spaces, and (2) that it is smooth.

(1) Let $\alpha_1, \dots, \alpha_n \in \mathcal{P}_x(X)$, and assume that there exist a $\alpha \in \mathcal{P}_x(X)$ such that
$$d\alpha = d\alpha_1 + \cdots + d\alpha_n.$$
It then follows, by using eq. (3.2), that for any $f \in C^\infty(Y)$
$$\begin{aligned} T_x\varphi(d\alpha)(f) &= (d\alpha_1 + \cdots + d\alpha_n)(\hat{\varphi}(f)) \\ &= T_x\varphi(d\alpha_1)(f) + \cdots + T_x\varphi(d\alpha_n)(f) \end{aligned}$$
This implies by lemma 3.4.3 that the cone map $T_x\varphi$ extends to a linear map $T_x\varphi : \mathrm{T}_xX \to T_{f(x)}Y$.

(2) Let $\gamma : U \to \mathrm{T}_xX$ be a generating plot for the tangent cone diffeology, and denote by $\gamma^\dagger$ its associated map. Then
$$T_x\varphi(\gamma(u)) = d[\varphi \circ \gamma^\dagger(u)]$$
evidently $u \to T_x\varphi(\gamma(u))$ is a plot for the tangent space $\mathrm{T}_{\varphi(x)}Y$. It follows that $T_x\varphi$ is smooth. ▪

Definition 3.4.5 The linear map $T_x\varphi : \mathrm{T}_xX \to T_{f(x)}Y$, as defined above, is called the *tangent map*.

Theorem 3.4.6 (Chain rule)
Let $\varphi : Y \to Z$ and $\psi : X \to Y$ be smooth maps, then
$$T_x(\varphi \circ \psi) = T_{\psi(x)}\varphi T_x\psi$$
Proof: Let $\alpha \in \mathcal{P}_x(X)$ then
$$\begin{aligned} T_{\psi(x)}\varphi(T_x\psi(d\alpha)) &= T_{\psi(x)}\varphi\left(d[\psi \circ \alpha]\right) \\ &= d[\varphi \circ \psi \circ \alpha] \\ &= T_x(\varphi \circ \psi)(d\alpha) \end{aligned}$$
▪

Example 29 (The tangent map between euclidian spaces)
Consider a smooth map $\varphi : \mathbb{R}^n \to \mathbb{R}^m$, let $\varphi_i = \pi_i \circ f$ that is
$$\varphi(\xi_1, \dots, \xi_n) = (\varphi_1(\xi_1, \dots, \xi_n), \dots, \varphi_m(\xi_1, \dots, \xi_n)).$$
Now for $\xi_0 \in \mathbb{R}^n$, and $f \in C^\infty(\mathbb{R}^m)$
$$\begin{aligned} T_{\xi_0}\varphi\left(d[\tau_i(\xi_0)]\right)(f) &= d[\varphi \circ \tau_i(\xi_0)](f) \\ &= d[\tau_i(\xi_0)](f \circ \varphi) \\ &= \left.\frac{\partial f \circ \varphi(\xi_1, \dots, \xi_n)}{\partial \xi_i}\right|_{\xi_0} \\ &= \sum_{j=1}^{m} \left.\frac{\partial \phi_j}{\partial \xi_i}\right|_{\xi_0} d[\tau_j(\phi(\xi_0))](f). \end{aligned}$$
Hence in the "tau" basis we have, as expected
$$T_{\xi_0}\varphi = \left(\left.\frac{\partial \phi_j}{\partial \xi_i}\right|_{\xi_0} \right)_{ij}$$
▪

3.5 Regular diffeological spaces

Let $\mathcal{A}$ be a $\mathcal{D}$-open subspace of a diffeological space X, x_0 a point in $\mathcal{A}$ and denote by $\iota_{\mathcal{A}}$ the canonical inclusion. We shall in this section study the following problem, when is $T_{x_0}\iota_{\mathcal{A}}$ an isomorphism, i.e. a linear isomorphism and a diffeomorphism. This is important when dealing with model space modelled on diffeological spaces, since we which to know if the tangent spaces are also modelled on the tangent spaces of the model spaces. We suggest the notion of regular diffeological spaces, it is simple and have the nice property that a space model on regular model space is it self regular. Furthermore for regular space $T_{x_0}\iota_{\mathcal{A}}$ is an isomorphism.

Examples of model space modelled on regular spaces are finite dimensional manifolds with or without boundary. This implies in particular that requirement (b) stated in the introduction of the chapter is fulfilled.

Definition 3.5.1 A diffeological space X is said to be *regular* if there for each $x \in X$ and each $\mathcal{D}$-open subset $x \in \mathcal{U} \subseteq X$ exist a $\mathcal{D}$-open subset $x \in \mathcal{V} \subseteq \mathcal{U}$, and a smooth map $X \to \mathbb{R}$ being 1 on $\mathcal{V}$ and zero outside $\mathcal{U}$.

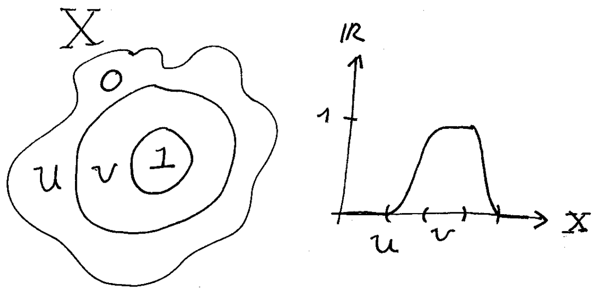


Figure 3.3: Regular diffeological space.

Open question *Which diffeological spaces are regular? In particular are there non regular diffeological spaces? (I think so).*

Example 30
$\mathbb{R}^n$, with the canonical diffeology, is regular. ▪

Lemma 3.5.2 *Let $U \subseteq \mathbb{R}^n$ be open and let $b : \mathbb{R}^n \to \mathbb{R}$ be any smooth function being zero outside U. And let $f : U \to \mathbb{R}$ be any smooth function, then the function*

$$\mathbb{R}^n \ni \xi \to \begin{cases} b(\xi)f(\xi) & \xi \in U \\ 0 & \xi \in \mathbb{R}^n - U \end{cases}$$

is smooth.

Proof: By Boman's theorem (see theorem 2.4.5) it is enough to prove the claim for $n = 1$. Let ξ_0 be a point on the boundary of $\overline{U}$, and choose a sequence $\xi_i \in U$ converging to ξ_0, evidently $b^{(k)}(\xi_i) \to 0$ for all $k \in \mathbb{N}$. By repeated use of Leibniz product rule it then follows that

$$(b \cdot f)^{(k)}(\xi_i) \to 0.$$

This implies the claim. ▪

Theorem 3.5.3
Let X be a regular diffeological space , $\mathcal{A}$ any $\mathcal{D}$-open subspace of X and let $x_0 \in \mathcal{A}$ Then $T_{x_0}\iota_{\mathcal{A}}$ is a isomorphism (where $\iota_{\mathcal{A}}$ is the canonical inclusion). In particular $\mathrm{T}_{x_0}\mathcal{A} \simeq \mathrm{T}_{x_0}X$.

Proof: Let $\iota_{\mathcal{A}}$ be the canonical inclusion $\mathcal{A} \hookrightarrow X$. We shall then show that $T_{x_0}\iota_{\mathcal{A}}$ is a diffeomorphism, we shall do this in three steps (1) surjectiveness, (2) injectiveness and finally (3) that $T_{x_0}\iota_{\mathcal{A}}$ is a diffeomorphism. (Note that the critical point, where we need that the space is regular, is to show injectiveness).

(1) Let $d\alpha \in \mathrm{C}_{x_0}X$, that is $\alpha : \mathbb{R} \to X$ is a plot with $\alpha(0) = x$. Let $U := \alpha^{-1}(\mathcal{A})$ as $\mathcal{A}$ is $\mathcal{D}$-open $U \subseteq \mathbb{R}$ is open. Hence there exist a $\varepsilon > 0$ such that $B_\varepsilon(0) \subseteq U$, and a smooth map $h : \mathbb{R} \to B_\varepsilon(0)$ with $d_0 h = 1$. Now let $\beta := \alpha \circ h$, evidently $\beta : \mathbb{R} \to A$ is a plot. And for $f \in C^\infty(X)$

$$\begin{aligned} T_{x_0}\iota_{\mathcal{A}}(d\beta)(f) &= d_0(f \circ \iota_{\mathcal{A}} \circ \alpha \circ h) \\ &= d_0 h\, d_0(f \circ \alpha) \\ &= d\alpha(f). \end{aligned}$$

Hence $T_{x_0}\iota_{\mathcal{A}}$ is surjective.

(2) Let $d\alpha, d\beta \in \mathrm{C}_{x_0}A$ and assume that $d\alpha \neq d\beta$, i.e. there exist a smooth function $f \in C^\infty(A)$ such that $d\alpha(f) \neq d\beta(f)$. By regularity there exist a bump function $\chi : X \to \mathbb{R}$ 1 in a neighbourhood of x_0 and zero outside $\mathcal{A}$. Let

$$g(x) := \begin{cases} \chi(x)f(x) & x \in \mathcal{A} \\ 0 & x \in X - \mathcal{A} \end{cases}$$

then since $\mathcal{A}$ is $\mathcal{D}$-open and by lemma 3.5.2 g is smooth, i.e $g \in C^\infty(X)$. Furthermore it is evident that $g(x) = f(x)$ in a neighbourhood of $x_0 = \alpha(0)$. Hence

$$\begin{aligned} T_x\iota_{\mathcal{A}}(d\alpha)(g) &= d_0(g \circ \iota_{\mathcal{A}} \circ \alpha) \\ &= d_0(f \circ \alpha) \\ &= d\alpha(f), \end{aligned}$$

it follows that $T_{x_0}\iota_{\mathcal{A}}$ is injective.

(3) By proposition 3.4.4 $T_{x_0}\iota_{\mathcal{A}}$ is smooth. We wish to apply lemma 1.5.2, so consider a genrating plot $\gamma : U \to \mathrm{C}_X x_0$, this is consider a plot $\gamma^\dagger_U \times \mathbb{R} \to X$ with $\gamma^\dagger(u)(0) = x$. Note that $V := (\gamma^\dagger)^{-1}(\mathcal{A}) \subseteq U \times \mathbb{R}$ is open, and that $U \times \{0\} \subseteq V$. Let $u_0 \in U$ then there exist a $\varepsilon > 0$ such that $B_\varepsilon(u_0) \times (-\varepsilon, \varepsilon) \subseteq W$. And a smooth map $h : \mathbb{R} \to (-\varepsilon, \varepsilon)$ with $d_0 h = 1$. Let $\tilde{\gamma}^\dagger : B_\varepsilon(u_0) \times \mathbb{R} \to A$ be given by

$$\tilde{\gamma}^\dagger(u, \xi) := \gamma^\dagger(u, h(\xi))$$

notice that $\tilde{\gamma}^\dagger$ is indeed a well defined plot, as $(u, h(\xi)) \in W$. Furthermore for $f \in C^\infty(X)$

$$\begin{aligned} u \to T_{x_0}\iota_{\mathcal{A}}(d[\tilde{\gamma}^\dagger(u)])(f) &= d_0(f\iota_{\mathcal{A}} \circ \gamma^\dagger(u) \circ h(\xi)) \\ &= d_0 h\, d_0(f \circ \gamma^\dagger(u)) \\ &= d[\gamma^\dagger(u)](f). \end{aligned}$$

Hence $\gamma \overset{\text{loc}}{\in} T_{x_0}\iota_{\mathcal{A}}(\mathcal{D}_{\mathrm{C}_{x_0}\mathcal{A}})$, it follows that $T_{x_0}\iota_{\mathcal{A}}$ is a diffeomorphism. ▪

Lemma 3.5.4 *Let Y be a subspace of a regular diffeological space X. If the inclusion $Y \hookrightarrow X$ is an embedding then Y is regular.*

Proof: Let $x_0 \in Y$ and let $\mathcal{U} \subseteq Y$ be $\mathcal{D}$-open with $x_0 \in \mathcal{U}$. Then there exist a $\mathcal{D}$-open $\mathcal{U}' \subseteq X$ such that $\mathcal{U} = \mathcal{U}' \cap Y$. By regularity of X there exist a $\mathcal{D}$-open $\mathcal{V}' \subseteq \mathcal{U}'$ with $x_0 \in \mathcal{V}'$, and a smooth map $\chi : X \to \mathbb{R}$ being 1 on $\mathcal{V}'$ and zero outside $\mathcal{U}'$. It follows that $\chi|_Y : Y \to \mathbb{R}$ is smooth and 1 on $\mathcal{V} := \mathcal{V} \cap Y \subseteq \mathcal{U}$, and zero outside $\mathcal{U}$. ▪

Example 31
The half spaces, the subspace cross and the subspace cone are regular. ▪

Theorem 3.5.5
If a diffeological space X is locally diffeomorphic to a regular space at every point then X is itself regular.

Proof: Let $x_0 \in X$ and let $\mathcal{U} \subseteq X$ be $\mathcal{D}$ open with $x_0 \in \mathcal{U}$. By assumption there exist a local diffeomorphism $\phi : X \to Y$ at x_0, i.e. there exist a $\mathcal{D}$-open subset $\mathcal{A}$ of X such that $\phi(\mathcal{A})$ is $\mathcal{D}$-open and $\phi|_{\mathcal{A}} : \mathcal{A} \to \phi(\mathcal{A})$ is a diffeomorphism. Consider the $\mathcal{D}$-open subset $\mathcal{U}' := \phi(\mathcal{U} \cap \mathcal{A})$ of Y, by regularity of Y there exist a $\mathcal{D}$-open subset $\mathcal{V}' \subseteq \mathcal{U}'$ with $x_0 \in \mathcal{V}'$, and a smooth map $\chi_Y : Y \to \mathbb{R}$ being 1 on $\mathcal{V}'$ and zero outside $\mathcal{U}'$. See fig. 3.4. Consider then a plot $\beta : U \to Y$ with $U \subseteq \mathbb{R}$.Then if ξ_0 is a point on the boundary of $\overline{\beta^{-1}(\phi(\mathcal{A}))}$ and $\xi_n \in \beta^{-1}(\phi(A))$ a sequence converging to ξ_0. Then evidently for any $k \in \mathbb{N}$

$$(\chi_Y \circ \beta)^{(k)}(\xi_n) \to (\chi_Y \circ \beta)^{(k)}(\xi_0) = 0,$$

(where k denotes the k'th derivative).

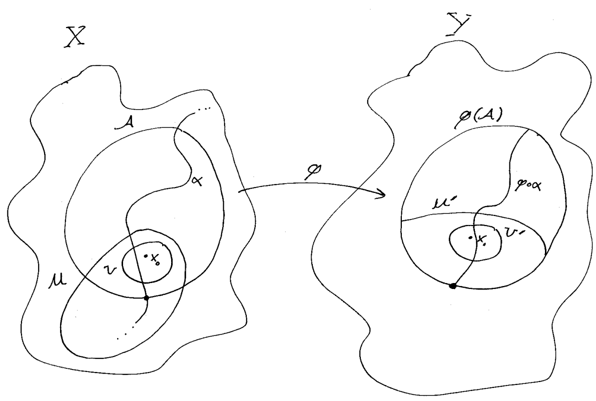


Figure 3.4: The sets $\mathcal{A}, \mathcal{U}, \mathcal{V}$ and $\phi(A), \mathcal{U}', V'$. And the plot α and $\phi \circ \alpha|_{\alpha^{-1}(\mathcal{A})}$.

Now let $\alpha : \mathbb{R} \to X$ be a plot, and let ξ_0 be a point on the boundary of $\overline{\alpha^{-1}(\mathcal{A})}$ and $\xi_n \in \alpha^{\mathcal{A})}$ a sequence converging to ξ_0. Then by using the above observation we conclude that

$$(\chi_Y \circ \phi \circ \alpha)^{(k)}(\xi_n) \to 0,$$

as $\phi \circ \alpha|_{\mathcal{A}}$ is a plot for Y.

It follows that the map $\chi_X : X \to \mathbb{R}$ give by

$$\chi(x) := \begin{cases} \chi_Y \circ \phi(x) & x \in \mathcal{A} \\ 0 & x \in X - A \end{cases}$$

is smooth. In order to see this let $\alpha : \mathbb{R} \to X$ be a plot, then

$$\chi \circ \alpha(\xi) = \begin{cases} \chi_Y \circ \phi \circ \alpha(\xi) & \xi \in \alpha^{-1}(\mathcal{A}) \\ 0 & \xi \in \alpha^{-}(X - A) \end{cases}$$

evidently this map is smooth everywhere except on the boundary of $\overline{\alpha^{-1}(\mathcal{A})}$. By the above it follows that all of its derivatives are continuous also on the boundary of $\overline{\alpha^{-1}(\mathcal{A})}$, this implies smoothness. Hence χ is smooth by lemma 2.4.4. ▪

Lemma 3.5.6 *Let X and Y be diffeological spaces. Then*

(i) If $\varphi : X \to Y$ is a diffeomorphism then $\mathrm{T}_x X \simeq \mathrm{T}_{\varphi(x)} Y$.

(ii) If X and Y are regular spaces and $\varphi : X \to Y$ is a local *diffeomorphism at $x \in X$ then $\mathrm{T}_x X \simeq \mathrm{T}_{\varphi(x)} Y$.*

Proof: Assume first that $\varphi : X \to Y$ is a diffeomorphism then note that by the chain rule (theorem 3.4.6)

$$T_{\varphi(x)}\varphi^{-1} T_x \varphi = T_x(\varphi^{-1} \circ \varphi) = T_x \mathrm{id}_X = \mathrm{id}_{\mathrm{T}_x X},$$

and $T_x \varphi T_{\varphi(x)} \varphi^{-1} = \mathrm{id}_{\mathrm{T}_{\varphi(x)} Y}$. Hence, in this case, $\mathrm{T}_x X \simeq \mathrm{T}_{\varphi(x)} Y$. Consider now the case where φ is a local diffeomorphism at $x \in X$. Then there exist a $\mathcal{D}$-open neighborhood A of x such that $\varphi(A)$ is $\mathcal{D}$-open and $\varphi : A \to \varphi(A)$ is a diffeomorphism. And

$$\begin{aligned} \mathrm{T}_x X &\simeq \mathrm{T}_x A && \text{since } X \text{ is regular} \\ &\simeq \mathrm{T}_{\varphi(x)} \varphi(A) && \text{by the case above} \\ &\simeq \mathrm{T}_{\varphi(x)} Y. \end{aligned}$$

▪

3.5.1 Examples of model spaces modelled on regular spaces

Let $\mathcal{Q}_i$ be a collection of regular diffeological spaces, and let for each $i \in \mathcal{I}$ q_i be a point of Q_i. We shall say that a diffeological space $\mathcal{M}$ is modelled on the collection $\{(Q_i, q_i)\}_{i \in \mathcal{I}}$ if for each point $m \in \mathcal{M}$ there exist a local diffeomorphism into one of the $Q_i's$ and such that $\phi(m) = q_i$.

Example 32
Finite dimensional mainfolds are modelled on $(\mathbb{R}^n, 0)$. ▪

Example 33
Finite dimensional manifolds with boundary are modelled on $\{(\mathbb{R}^n, 0), (\mathbb{H}^n, 0)\}$. ▪

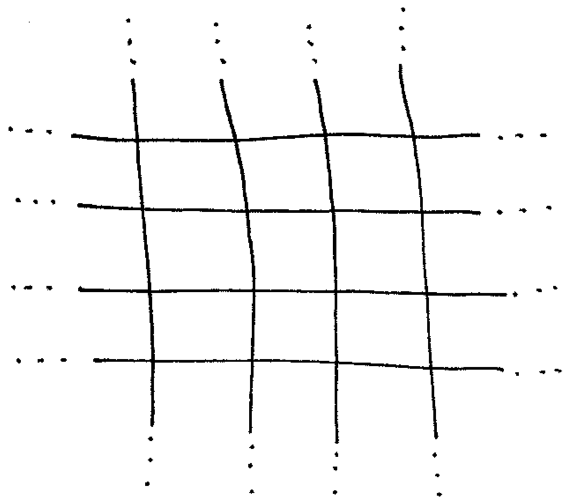


Figure 3.5: A space model on $\{(\mathcal{X},0),(\mathbb{R},0)\}$.

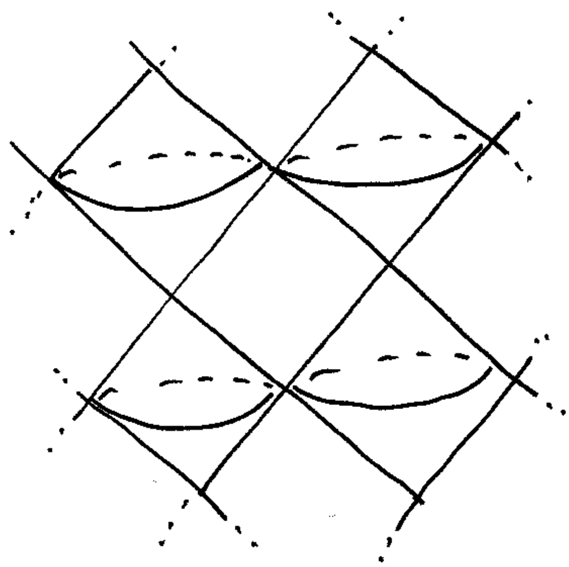


Figure 3.6: A space model on $\{(\Lambda,0),(\mathbb{R}^2,0)\}$.

Chapter 4

Diffeological bundles

The fibres of a tangent bundle should be the tangent spaces. Hence, by considering for example the cross example 22, it is clear that we can not, in general, assumed that the tangent bundles are locally trivial. So we need a notion of diffeological bundles which can handle this. The approach we shall take is very general, it is however worth noting that all of the constructions we define carries over to there natural counterpart for trivial bundles. For example we will define the product of two bundles over the same base base, as discussed later in this chapter. If the two bundles are locally trivial with fibres respectively F_1 and F_2 then there product is locally trivial with fibres $F_1 \times F_2$.

4.1 Bundles

Definition 4.1.1 A *bundle* over the space B, is a space E together with a subduction $\pi : E \to B$.

Usually we shall denote a bundle by $E \xrightarrow{\pi} B$, the space B is called the *base space*, E the *total space* and π the *bundle projection*. If the base space and the projection are understood from the context we shall sometimes denote a bundle just by the total space E.

Definition 4.1.2 Let $E \xrightarrow{\pi} B$ and $E' \xrightarrow{\pi'} B'$ be bundles. A *bundle morphism* from E to E', is a pair of smooth maps $A : E \to E'$ and $a : B \to B'$, such that

$$\begin{array}{ccc} E & \xrightarrow{A} & E' \\ \pi \downarrow & & \downarrow \pi' \\ B & \xrightarrow{a} & B' \end{array}$$

commutes.

Note that we have a natural composition of bundle morphisms. In fact the collection of diffeological bundles together with bundle morphisms makes up a category. We shall sometimes denote a bundle morphism by (A, a).

Definition 4.1.3 Let $E \xrightarrow{\pi} B$ be a bundle. The fiber over $b \in B$ is the supspace $E_b := \pi^{-1}(b)$ of E.

Note that if E is a bundle over B, then the fiber over $b \in B$ is denote by E_b. Furthermore we shall by ι_b denote the canonical inclusion $E_b \hookrightarrow E$, by the above definition this is an induction. By lemma 1.4.9, the canonical injection $\coprod_{b\in B} E_b \hookrightarrow E$ is a smooth bijection.

Definition 4.1.4 A smooth *section* of a bundle $E \xrightarrow{\pi} B$ is a smooth map $s : B \to E$ such that $\pi \circ s = \mathrm{id}_B$.

Given a diffeological bundle $E \to B$, we shall by $\Gamma(E)$ denote the set of smooth sections of E. $\Gamma(E)$ becomes a diffeological space as a subspace of $C^\infty(B, E)$, i.e. $\Gamma(E)$ is given the functional diffeology.

Example 34
Consider the cone Λ with the subspace diffeology (see example 13). Define a projection $\pi : \Lambda \to \mathbb{R}$ by setting $\pi(x, y, z) := z$. Clearly π is smooth, hence $\overrightarrow{\pi}(\mathcal{D}_\Lambda) \subseteq \mathcal{D}_\mathbb{R}$. Consider now a plot for $\mathbb{R}$, i.e. a smooth map $\alpha : U \to \mathbb{R}$, then define a map $\beta : U \to \mathbb{R}^3$ by setting $\beta(u) := (\alpha(u), 0, \alpha(u))$. Clearly β is smooth, and its range is contained in the cone, hence it is a plot for $\mathcal{D}_\Lambda$. It follows that $\mathcal{D}_\mathbb{R} \subseteq \overrightarrow{\pi}(\mathcal{D}_\Lambda)$, as $\pi \circ \beta = \alpha$. Hence π is a subduction. And the conclusion is that we may view the cone as a bundle over $\mathbb{R}$.

It is not hard to see that the fiber over a non zero point is diffeomorphic to S^1. And that the fiber over zero is simply a point. Examples of sections on the subspace cone are

$$s_\theta := (\cos(\theta)\xi, \sin(\theta)\xi, \xi),$$ ∎

for $\theta \in \mathbb{R}$. See fig. 4.1.

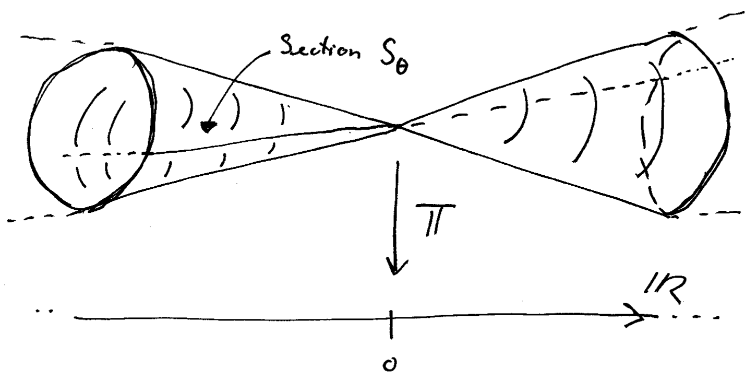


Figure 4.1: The subspace cone as a bundle over $\mathbb{R}$, see example 34.

4.1.1 Trivial bundles

Definition 4.1.5 Let B be a diffeological space, and F a diffeological vector space. A *trivial bundle* over B with fiber F, is simply a bundle isomorphic (in the category of bundles) to the bundle $B \times F \xrightarrow{\mathrm{pr}_B} B$.

We shall sometimes denote the trivial bundle $B \times F \xrightarrow{\mathrm{pr}_B} B$ by F_B.

Remark 4.1.6 Note that we may assume that the trivializing isomorphism is of the form (Φ, id_B). As given any trivializing isomorphism (Φ, ϕ) we may construct the following bundle isomorphism

$$\begin{array}{ccccc} E & \xrightarrow{\Phi} & B \times F & \xrightarrow{(\phi^{-1}(b), v)} & B \times F \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \mathrm{pr}_B} & & \downarrow{\scriptstyle \mathrm{pr}_B} \\ B & \xrightarrow{\phi} & B & \xrightarrow{\phi^{-1}} & B \end{array}$$

Example 35
Let F be a diffeological space. And denote by F_B the trivial bundle $B \times F \to B$. Then as diffeological spaces

$$\Gamma(F_B) \simeq C^\infty(B, F).$$

■

4.1.2 Subbundles

Definition 4.1.7 Let $E \xrightarrow{\pi} B$ and $E' \xrightarrow{\pi} B'$ be diffeological vector bundles. We shall say that E is a *subbundle* of E' if

(i) E is a subset of E' and the canonical injection $E \xhookrightarrow{\iota_T} E'$ is an induction.

(ii) B is a subset of B' and the canonical injection $B \xhookrightarrow{\iota_B} B'$ is an induction.

(iii) (ι_T, ι_B) is a bundle morphism.

Trivializing subbundles

Consider two diffeological bundles $E \xrightarrow{\pi} B$ and $E' \xrightarrow{\pi} B'$. And assume that E is a subbundle of E', assume furthermore that E' is trivial. That is we have a vector bundle isomorphism $(\Phi : E' \to B' \times F, \mathrm{id}_{B'})$ trivializing E'. Consider then the composition

$$\begin{array}{ccccc} E & \xhookrightarrow{\iota_T} & E' & \xrightarrow{\Phi} & B' \times F \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \mathrm{pr}_{B'}} \\ B & \xhookrightarrow{\iota_B} & B' & \xrightarrow{\mathrm{id}} & B' \end{array}$$

And note that if for any two $b_1, b_2 \in B$ it holds that

$$\mathrm{pr}_F \circ \Phi \circ \iota_T(E_{b_1}) = \mathrm{pr}_F \circ \Phi \circ \iota_T(E_{b_2})$$

then E is trivial with fiber $\mathrm{pr}_F \circ \Phi \circ \iota_T(E_{b_1})$.

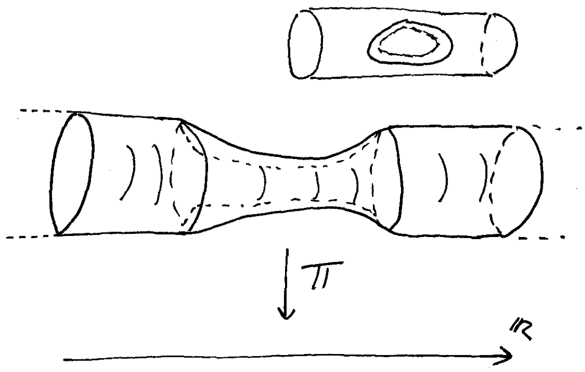


Figure 4.2: An example of a non trivial subbundle of the cylinder.

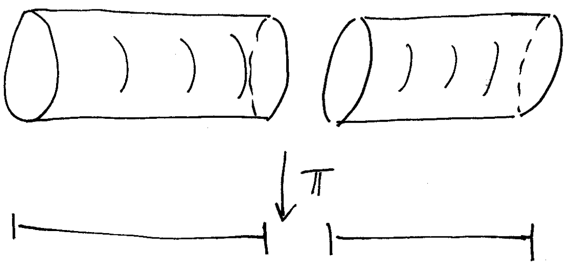


Figure 4.3: An example of a trivial subbundle of the cylinder.

4.2 Pre-bundles

Definition 4.2.1 Given a collection of sets $\{E_b\}_{b\in B}$ index by a diffeological space B. Consider the disjoint union $E := \cup_{b\in B} E_b$, and the map $\pi : E \to B$ mapping E_b to b. The triple $E \xrightarrow{\pi} B$ is said to be a *pre-bundle* with fibers $\{E_b\}_{b\in B}$. A *bundle diffeology* for E is any diffeology $\mathcal{D}_E$ on E such that $\overrightarrow{\pi}(\mathcal{D}_E) = \mathcal{D}_B$.

In other words a bundle diffeology on a pre-bundle is any diffeology that will make the pre-bundle a diffeological bundle. Often we shall, when constructing a bundle, specify a pre-bundle along with diffeologies for the fibers. It is then important to check that a given bundle diffeology induces the correct diffeology on the fibers. Notice that a pre-bundle may become a bundle in many ways, even if we have specified the diffeologies of the fibers. That is, there are many bundle diffeologies for a pre-bundle which will induce the same diffeologies on the fibers. We may consider the bundle

$$\coprod_{b\in B} E_b \xrightarrow{\pi} B$$

This is the "smallest" bundle having as its fibers the diffeological spaces $\{E_b\}_{b \in B}$, i.e. the coproduct diffeology is the weakest bundle diffeology inducing specific diffeologies on the fibers. We may think of the fibers in this bundle as separated, the bundle diffeology then determines how the fibers are to be glued together.

4.2.1 Fiberwize defined maps

Definition 4.2.2 Let $E \xrightarrow{\pi} B$ and $E' \xrightarrow{\pi'} B$ be pre-bundles. Assume given a collection of maps $\{\varphi_b : E_b \to E'_b\}_{b \in B}$. We shall then say that the map $\varphi : E \to E'$ defined by

$$\varphi(\sigma) := \hat{\iota}_b \circ \varphi_{\pi(\sigma)}(\sigma)$$

is fiberwise defined by $\{\varphi_b\}$.

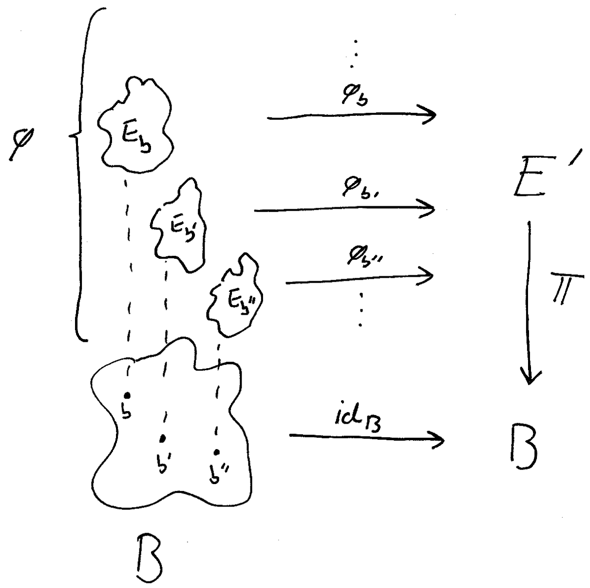


Figure 4.4: Fiberwise defined maps.

Assuming that E and E' are bundles, we note that if a fiberwise defined map is smooth then it is a bundle morphism. And if it is a diffeomorphism then it is a bundle isomorphism.

Another trivial but usefull observation is that, if $\phi : E \to E'$ is fiberwise defined then $\pi' \circ \phi = \pi$. Furthermore the push forward of a fiberwise defined map is a bundle diffeology as we show below.

Lemma 4.2.3 *Let $E \xrightarrow{\pi} B$ be a bundle and $E' \xrightarrow{\pi'} B$ a pre-bundle. If $\varphi : E \to E'$ is fiberwise defined then $\overrightarrow{\varphi}(\mathcal{D}_E)$ is a bundle diffeology for E'. And the fiber over $b \in B$ of the bundle $(E', \overrightarrow{\varphi}(\mathcal{D}_E))$ is the diffeological space $(E'_b, \overrightarrow{\varphi_b}(\mathcal{D}_{E_b}))$, where $\mathcal{D}_{E_b}$ is the diffeology of the fiber over b of the bundle E.*

Proof: The first part of the claim follows by the observation that

$$\overrightarrow{\pi'} \circ \overrightarrow{\varphi}(\mathcal{D}_E) = \overrightarrow{\pi' \circ \varphi}(\mathcal{D}_E) = \overrightarrow{\pi}(\mathcal{D}_E).$$

For the second claim note that

$$\overrightarrow{\phi_b}(\mathcal{D}_{E_b}) = \langle \phi_b \circ \mathcal{D}_{E_b} \rangle = \langle \phi \circ \mathcal{D}_{E_b} \rangle \subseteq \overrightarrow{\phi}(\mathcal{D}_E)$$

hence $\overrightarrow{\phi_b}(\mathcal{D}_{E_b}) \subseteq \left\{ \alpha \in \mathrm{Par}(E'_b) \mid \alpha \in \overrightarrow{\phi}(\mathcal{D}_E) \right\}$. For the other inclusion let $\alpha : U \to E'_b$ be a plot for $\overrightarrow{\phi}(\mathcal{D}_E)$. Then note that given any $u_0 \in U$, by the definition of the pushforward (definition 1.3.4), there exist an open $U_0 \subseteq U$ with $u_0 \in U_0$ and such that $\alpha|_{U_0}$ is constant or $\alpha|_{U_0} = \phi \circ \beta$ for a plot $\beta \in \mathcal{D}_E$. In any case it is evident that

$$\pi \circ \beta(u) = \pi' \circ \phi \circ \beta(u) = \pi' \circ \alpha|_{U_0}(u) = b$$

for all $u \in U_0$, hence $\beta \in \mathcal{D}_{E_b}$. It follows that $\alpha|_{U_0} \in \overrightarrow{\phi_b}(\mathcal{D}_{E_b})$, and by locality this implies that $\alpha \in \overrightarrow{\phi_b}(\mathcal{D}_{E_b})$. ▪

4.3 Product bundles

Let $n \in \mathbb{N}$ and let $E_1 \xrightarrow{\pi_1} B, \dots, E_n \xrightarrow{\pi_n} B$ be a collection of bundles over the same base space B. Consider then the pre-bundle

$$E_1 \times \cdots \times E_n := \bigcup_{b \in B} (E_1)_b \times \cdots \times (E_n)_b.$$

Definition 4.3.1 The *product bundle diffeology* on the pre-bundle $E_1 \times \cdots \times E_n$ is the strong diffeology induced by the fiberwise defined projections $\mathcal{P}_i : E_1 \times \cdots \times E_n \to E_i$. The resulting bundle $E_1 \times \cdots \times E_n$ is called the *product bundle*.

Note that by lemma 1.3.15 the product bundle diffeology is the collection

$$\{\gamma \in \mathrm{Par}(E_1 \times \cdots \times E_n) \mid \mathcal{P}_i \circ \gamma \text{ is smooth for all } i = 1, \dots, n\}.$$

It is a bundle diffeology, as we shall see in the proposition given below. If $\sigma_i \in E_i$ with $\pi(\sigma_i) = b$ for $i = 1, \dots, n$, then the element $(\sigma_1, \dots, \sigma_n)$ is a well defined element in the fiber over b of the product bundle. Note that each element σ in the product bundle is of this form, namely $(\mathcal{P}_1(\sigma), \dots, \mathcal{P}_n(\sigma))$. Given n plots $\alpha_i : U \to E_i$ with $\pi_1 \circ \alpha_1 = \cdots = \pi_n \circ \alpha_n$, we define a map $\alpha_1 \times \cdots \times \alpha_n : U \to E_1 \times \cdots \times E_n$ by setting

$$\alpha_1 \times \cdots \times \alpha_n(u) = (\alpha_1(u), \dots, \alpha_n(u)),$$

note that this is a plot for the product bundle diffeology. And that given any plot $\alpha : U \to E_1 \times \cdots \times E_n$

$$\alpha(u) = \alpha_1 \times \cdots \times \alpha_n(u)$$

where $\alpha_i := \mathcal{P}_i \circ \alpha$ are plots. Futher more a map $\phi : E_1 \times E_2 \to E_3$ between bundles is smooth if and only if

$$u \to \phi(\alpha_1 \times \alpha_2(u))$$

is smooth for all pairs of plots $\alpha_i : U \to E_i$ $(i = 1, 2)$ with $\pi \circ \alpha_1 = \pi \circ \alpha_2$.

Proposition 4.3.2 *The product bundle diffeology is a bundle diffeology. And the fiber over $b \in B$ is the diffeological product $(E_1)_b \times \cdots \times (E_n)_b$.*

Proof: Let us first show that the product diffeology is in fact a bundle diffeology. That is, we must show that π is a subduction. Let $\alpha : U \to E_1 \times \cdots \times E_n$ be a plot for the product bundle diffeology, then $\mathcal{P}_1 \circ \alpha$ is a plot for E_1, hence

$$\pi \circ \alpha = \pi \circ \mathcal{P}_1 \circ \alpha \in \overrightarrow{\pi}(\mathcal{D}_{E_1}) = \mathcal{D}_B.$$

That is π is smooth. Let $\beta : U \to B$ be a plot for the base space, as π is a subduction we may, by restricting the domain, assume that there exist plots $\alpha_i : U \to E_i$ such that $\pi \circ \alpha_i = \beta$ (see remark 1.3.11). Let

$$\alpha(u) := \alpha_1 \times \cdots \times \alpha_n(u)$$

evidently α is a plot for the product bundle diffeology with $\pi \circ \alpha = \beta$. This implies that π is a subduction.

Secondly note that the subspace diffeology of the fiber over $b \in B$ is simply

$$\begin{aligned}&\{\alpha \in \mathrm{Par}((E_1)_b \times \cdots \times (E_n)_b) \mid \mathcal{P}_i \circ \alpha \text{ is smooth for all } i = 1, \ldots, n\}\\&\quad = \{\alpha \in \mathrm{Par}((E_1)_b \times \cdots \times (E_n)_b) \mid \mathrm{pr}_i \circ \alpha \text{ is smooth for all } i = 1, \ldots, n\}\end{aligned}$$

hence it is the diffeological product, as claimed. ▪

Theorem 4.3.3
If $E_1 \xrightarrow{\pi} B, \ldots, E_n \xrightarrow{\pi} B$ are trivial with fibers respectively $F_1, \ldots, F_n$. Then the product bundle $E_1 \times \cdots \times E_n \xrightarrow{\pi} B$ is trivial with fiber $F_1 \times \cdots \times F_n$.

Proof: By assumption we have, for each $i = 1, \ldots, n$, bundle isomorphisms

$$(\Phi_i : E_i \to B \times F_i, \mathrm{id}_B)$$

Define a map $\Psi : E_1 \times \cdots \times E_n \to B \times F_1 \times \cdots \times F_n$ by

$$\Psi(\sigma) := \left(\pi(\sigma), \mathrm{pr}_{F_1} \circ \Phi_1(\mathcal{P}_1(\sigma)), \ldots, \mathrm{pr}_{F_n} \circ \Phi_n(\mathcal{P}_n(\sigma))\right), \tag{4.1}$$

we claim that (Ψ, id_B) is a bundle isomorphism. It is clearly a bijection and if we can show that it is a diffeomorphism then it easily follows that it is a bundle isomorphism. It is smooth since for any plot $\alpha : U \to E_1 \times \cdots \times E_n$ the map

$$u \to \Psi(\alpha(u)) = \left(\pi \circ \alpha(u), (\mathrm{pr}_{F_1} \circ \Phi_1(\mathcal{P}_1 \circ \alpha(u)), \ldots, \mathrm{pr}_{F_n} \circ \Phi_n(\mathcal{P}_n \circ \alpha(u))\right)$$

is the composition of smooth maps, hence it is smooth. Next note that (for $\varsigma_i \in F_i$)

$$\Psi^{-1}(b, (\varsigma_1, \ldots, \varsigma_n)) = (\Phi_1^{-1}(b, \varsigma_1), \ldots, \Phi_n^{-1}(b, \varsigma_n))$$

is also clearly smooth. ▪

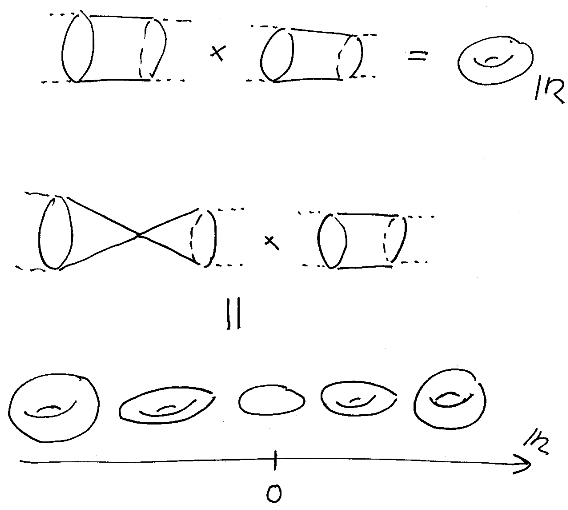


Figure 4.5: Examples of product bundles.

Chapter 5

Diffeological vector bundles

This chapter is a continuation of the previous chapter. We shall study diffeological vector bundles, we shall mainly be concerned with what we shall call regular vector bundles (not to be confused with regular spaces). In section 5.2 we define the tensor product of vector bundles, which will be a bundle with fibres tensor products. In section 5.3 we define the dual of a vector bundle, this will be a bundle with fibres the dual spaces. By the statements proved in these sections it will be evident, that for locally trivial bundles the tensor product bundle is locally trivial with fibres the tensor products. Furthermore any locally trivial bundle with finite dimensional fibres is self dual.

Finally in section 5.4 we discuss the dual of the tensor product bundle, as this will be helpful in the following chapter.

5.1 Vector bundles

Definition 5.1.1 A *vector bundle* is a diffeological bundle, where each fiber has the structure of a diffeological vector space.

Definition 5.1.2 Let $E \xrightarrow{\pi} B$ and $E' \xrightarrow{\pi'} B'$ be vector bundles. A *vector bundle morphism* from E to E' is a bundle morphism (A, a) from E to E', such that for each $b \in B$ the map

$$A|_{E_b} : E_b \to E'_{a(b)}$$

is linear.

The composition of two vector bundle morphism is again a vector bundle morphism. It is therefor clear that the collection of vector bundles together with vector bundles morphisms form a subcategory of the category of bundles. Note, in particular, that a trivialization of a vector bundle must be fiberwise linear. Furthermore if E is a vector subbundle of E' then the canonical injection must be fiberwise linear.

A *vector pre-bundle* is a pre-bundle where each fiber carries the structure of a vector space. Note that since a vector space structure on a set is not unique, a triple $E \xrightarrow{\pi} B$ may be a vector pre-bundle in many ways. The point is that when we say that a triple $E \xrightarrow{\pi} B$ is a vector pre-bundle we are specifying the vector space structure on each fiber. A vector bundle diffeology for a vector pre-bundle is then a diffeology on E such that each fiber becomes a diffeological vector bundle.

Regular vector bundles

Let E be a vector bundle over B. The fiberwise addition on E is a map

$$E \times E \to E$$

(where $E \times E$ is the product bundle) defined as addition on each fiber. Let $\mathbb{R}_B$ denote the trivial bundle $B \times \mathbb{R} \to B$. Then fiberwise scalar multiplication is a map

$$\mathbb{R}_B \times E \to E.$$

defined as scalar multiplication on each fiber.

Definition 5.1.3 A vector bundle is said to be regular if fiberwise addition and scalar multiplication is smooth.

Example 36
For examples of non regular diffeological vector bundles, see figs. 5.1 and 5.2. And for an example of a regular but non locally trivial vector bundle see fig. 5.3.∎

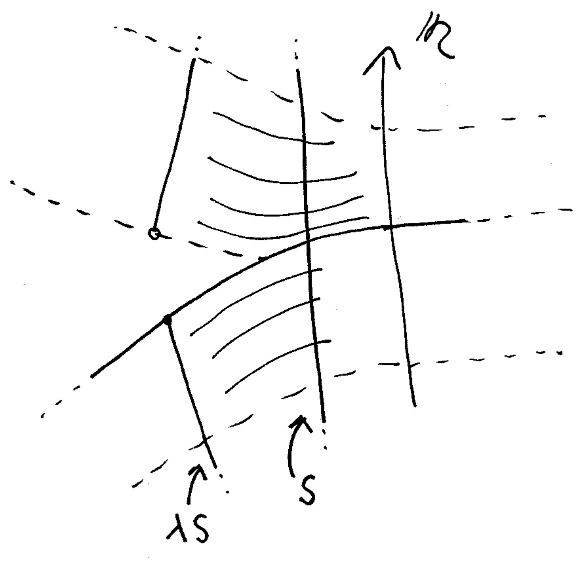


Figure 5.1: A non regular vector bundle over ℝ, with fibers ℝ.

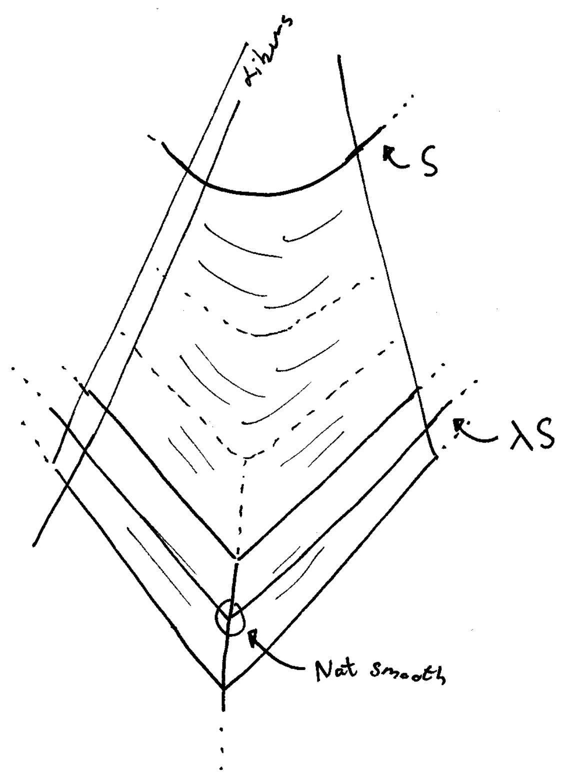


Figure 5.2: A connected non regular vector bundle over $\mathbb{R}$, with fibers $\mathbb{R}$.

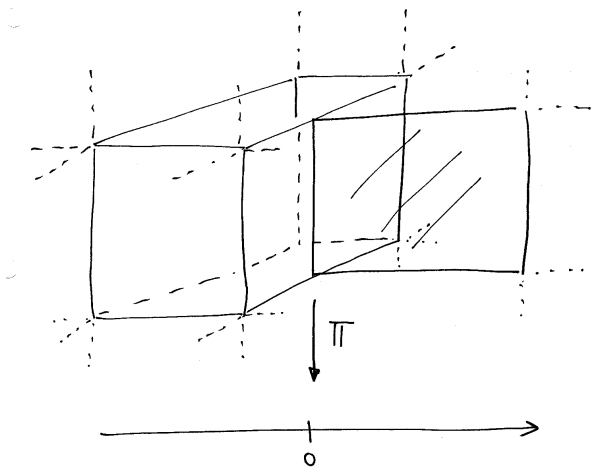


Figure 5.3: A regular vector bundle with two types of fibres $\mathbb{R}^2$ and $\mathbb{R}$.

Regular vector bundles have the following desirable property;

Lemma 5.1.4 *If E is a regular diffeological vector bundle over B then $\Gamma(E)$ is a $C^\infty(B)$ module.*

Proof: Let $s_1, s_2 \in \Gamma(E)$ and $f \in C^\infty(B)$. If we let

$$(s_1 + s_2)(b) := s_1(b) + s_2(b) \quad \text{and} \quad fs_1(b) := f(b)s_1(b)$$

then as fiberwise addition and scalar multiplication are smooth it follows that $s_1 + s_2$ and fs_1 are sections. ▪

5.1.1 *The Weak vector bundle diffeology*

Definition 5.1.5 Let $E \xrightarrow{\pi} B$ be a vector pre-bundle and let $\mathcal{D}_E$ be any bundle diffeology on E. Then the *weak vector bundle diffeology* on E generated by $\mathcal{D}_E$ is the weak diffeology generated by parametrizations of the form

$$u \to \sum_{i=1}^{n} \lambda_i(u)\alpha_i(u)$$

with $\lambda_i : U \to \mathbb{R}$ smooth and $\alpha_i : U \to E$ plots for $\mathcal{D}_E$ with $\pi \circ \alpha_1 = \cdots = \pi \circ \alpha_n$.

We shall use the notation $\sum \mathcal{D}_E$ to indicate the weak vector bundle diffeology generated by $\mathcal{D}_E$. Note the notational similarity with the weak diffeology for vector spaces. Hopefully, this will not lead to confusion.

Theorem 5.1.6
Let $E \xrightarrow{\pi} B$ be a vector pre-bundle, and let $\mathcal{D}_E$ be any bundle diffeology on E. Then the weak vector bundle diffeology on E generated by $\mathcal{D}_E$ is the weakest regular vector bundle diffeology on E containing $\mathcal{D}_E$. And for each $b \in B$, the fiber over b is the diffeological vector space $(E_b, \sum \mathcal{D}_{E_b})$.

Remark 5.1.7 Note that $\mathcal{D}_{E_b} = \{\alpha \in \text{Par}(E_b) \mid \alpha \in \mathcal{D}_E\}$ is the diffeology of the fiber over $b \in B$ of the bundle $(E, \mathcal{D}_E)$.

Proof (Proof of theorem 5.1.6): Let $\gamma : U \to E$ be a plot for the weak vector bundle diffeology, then there exist a plot $\hat{\gamma} : U \to E$ for $\mathcal{D}_E$ such that $\pi \circ \gamma = \pi \circ \hat{\gamma}$ hence

$$\overrightarrow{\pi}\left(\sum \mathcal{D}_E\right) = \overrightarrow{\pi}(\mathcal{D}_E).$$

That is $\sum \mathcal{D}_E$ is a bundle diffeology. Let $b \in B$, $\mathcal{D}_{E_b} = \{\alpha \in \text{Par}(E_b) \mid \alpha \in \mathcal{D}_E\}$ and let $\mathcal{D}_b$ denote the subspace diffeology of the fiber over b of the weak vector bundle diffeology. We which to show that $\sum \mathcal{D}_{E_b} = \mathcal{D}_b$ (note that here $\sum \mathcal{D}_{E_b}$ denote the weak vector space diffeology, see definition 2.2.1). Let $\gamma : U \to E_b$ be a plot for $\mathcal{D}_b$, hence

$$\gamma(u) = \sum_{i=1}^{n} \lambda_i(u)\gamma_i(u)$$

where λ_i is smooth, $\pi \circ \gamma_i(u) = b$ for each $u \in U$ and $\gamma_i \in \mathcal{D}_E$. But this implies that $\gamma_i \in \mathcal{D}_{E_b}$ hence $\gamma \in \sum \mathcal{D}_{E_b}$. The other inclusion is, also, obvious.

Let $\mathcal{D}$ be another regular vector bundle diffeology on E containing $\mathcal{D}_E$. Since $\mathcal{D}$ is regular, each genrating plot for $\sum \mathcal{D}_E$ must be smooth, hence $\sum \mathcal{D}_E \subseteq \mathcal{D}$. It follows that the weak vector bundle diffeology $\sum \mathcal{D}_E$ is the weakest regular vector bundle diffeology on E containing $\mathcal{D}_E$. ▪

Lemma 5.1.8 *Let $E \xrightarrow{\pi} B$ be a vector pre-bundle and let $\mathcal{D}_E$ be any bundle diffeology on E. Then a fiberwise linear diffeomorphism*

$$\Phi : (E, \mathcal{D}_E) \to (F, \mathcal{D}_F) \times B$$

becomes a trivialization of vector bundles $(E, \sum \mathcal{D}_E) \to (F, \sum \mathcal{D}_F) \times B$.

Proof: We must show that

$$\Phi : \left(E, \sum \mathcal{D}_E\right) \to \left(F, \sum \mathcal{D}_F\right) \times B$$

is a diffeomorphism. Consider n smooth functions and n plots for $\mathcal{D}_E$, say $\lambda_i : U \to \mathbb{R}$ and $\alpha_i : U \to E$, with $\pi \circ \alpha_1 = \cdots = \pi \circ \alpha_n$. Then

$$\mathrm{pr}_F \circ \Phi \left(\sum_{i=1}^{n} \lambda_i(u) \alpha_i(u) \right) = \sum_{i=1}^{n} \lambda_i(u) \mathrm{pr}_F \circ \Phi(\alpha_i(u))$$

hence Φ is smooth, as $u \to \mathrm{pr}_F \circ \Phi(\alpha_i(u))$ is a plot for $\mathcal{D}_F$. Next, consider a generating plot for $(F, \sum \mathcal{D}_F) \times B$, that is (by lemma 1.4.7) $n \in \mathbb{N}$ smooth functions and n plots for $\mathcal{D}_F$, say $\lambda_i : U \to \mathbb{R}$ and $\alpha_i : U \to F$, and in addition also a plot $\beta : V \to B$. Then let

$$\gamma_i := \Phi^{-1} \circ \alpha_i \times \beta$$

which is a plot for $\mathcal{D}_E$, and

$$(u, v) \to \Phi \left(\sum_{i=1}^{n} \lambda_i(u, v) \gamma_i(u) \right) = \left(\sum_{i=1}^{n} \lambda_i(u) \alpha_i(u), \beta(v) \right)$$

which is our generating plot. It follows, by lemma 1.5.2, that Φ is a diffeomorphism. ▪

5.2 Tensor product bundles

Let $n \in \mathbb{N}$ and let $E_1 \xrightarrow{\pi} B, \ldots, E_n \xrightarrow{\pi} B$ be a collection of vector bundles over the same base space B. Consider then the vector pre-bundle

$$E_1 \otimes \cdots \otimes E_n := \bigcup_{b \in B} ((E_1)_b \otimes \cdots \otimes (E_n)_b, \phi_b)$$

Definition 5.2.1 The *tensor product bundle diffeology* on the vector pre-bundle $E_1 \otimes \cdots \otimes E_n$ is the weak vector bundle diffeology

$$\sum \overrightarrow{\phi}(\mathcal{D}_{E_1 \times \cdots \times E_n}).$$

Where $\mathcal{D}_{E_1 \times \cdots \times E_n}$ is the product bundle diffeology and $\phi : E_1 \times \cdots \times E_n \to E_1 \otimes \cdots \otimes E_n$ is the fiberwise defined tensor product map 2.1. The resulting vector bundle $E_1 \otimes \cdots \otimes E_n$ is called the *tensor product bundle.*

Proposition 5.2.2 *The tensor product bundle is a vector bundle. And the fiber over $b \in B$ of the tensor product bundle $E_1 \otimes \cdots \otimes E_n$ is the diffeological tensor product $((E_1)_b \otimes \cdots \otimes (E_n)_b, \phi_b)$.*

Proof: By the first parts of lemma 4.2.3 and theorem 5.1.6 it follows that the tensor product bundle diffeology is a vector bundle diffeology. Hence the tensor product bundle is a vector bundle. For the second part of the proposition note that by theorem 5.1.6 and lemma 4.2.3 the diffeology of the fiber over $b \in B$ of the tensor bundle is

$$\sum \overrightarrow{\phi_b}(\mathcal{D}_{(E_1)_b \times \cdots \times (E_n)_b})$$

but this is the diffeological tensor product $((E_1)_b \otimes \cdots \otimes (E_n)_b, \phi_b)$, as defined in section 2.3. ▪

Theorem 5.2.3
If the vector bundles $E_1 \xrightarrow{\pi_1} B, \ldots, E_n \xrightarrow{\pi_n} B$ *are trivial with fibers respectively* $F_1, \ldots, F_n$*. Then the tensor product bundle* $E_1 \otimes \cdots \otimes E_n \xrightarrow{\pi} B$ *is trivial with fiber* $F_1 \otimes \cdots \otimes F_n$*.*

Proof: By assumption and theorem 4.3.3 the map 4.1 (denoted Ψ) is a trivialization. Let $\phi' : F_1 \times \cdots \times F_n \to F_1 \otimes \cdots \otimes F_n$ denote the tensor product, and let $\Psi' := \phi' \circ \mathrm{pr}_{F_1 \times \cdots \times F_n} \circ \Psi$. Then for $(\sigma_1, \ldots, \sigma_n) \in (E_1)_b \times \cdots \times (E_n)_b$ it is evident that

$$\Psi'(\sigma_1, \ldots, \sigma_n) = \phi'(\mathrm{pr}_{F_1} \circ \Phi_1(\sigma_1) \ldots, \mathrm{pr}_{F_n} \circ \Phi_n(\sigma_n)),$$

where $\Phi_i : E_i \to B \times F_i$ are the fiberwise linear trivializations. It follows that Φ' is multilinear on each fiber. By theorem 2.3.1 there exist for each $b \in B$ a unique linear map

$$\Phi_b : (E_1)_b \otimes \cdots \otimes (E_n)_b \to F_1 \otimes \cdots \otimes F_n \times B$$

such that $\Phi_b \circ \phi_b = \Psi'|_{(E_1)_b \times \cdots \times (E_n)_b}$, where $\phi_b : (E_1)_b \times \cdots \times (E_n)_b \to (E_1)_b \otimes \cdots \otimes (E_n)_b$ is the tensor product. Let now $\Phi : E_1 \otimes \cdots \otimes E_n \to B \times F_1 \otimes \cdots \otimes F_n$ be fiberwise defined by the collection $\{\Phi_b\}$. Hence

$$\mathrm{pr}_{F_1 \otimes \cdots \otimes F_n} \circ \Phi \circ \phi = \Psi',$$

where $\phi : E_1 \times \cdots \times E_n \to E_1 \otimes \cdots \otimes E_n$ is the fiberwise defined tensor product. We shall show below (1) that Φ is bijective and (2) that it is a diffeomorphism between the diffeological spaces

$$\left(E_1 \otimes \cdots \otimes E_n, \overrightarrow{\phi}(\mathcal{D}_{E_1 \times \cdots \times E_n})\right) \to \left(F_1 \otimes \cdots \otimes F_n \times B, \left\langle \overrightarrow{\phi'}(\mathcal{D}_{F_1 \times \cdots \times F_n}) \times \mathcal{D}_B \right\rangle\right).$$

This will by lemma 5.1.8 implie that

$$\Phi : E_1 \otimes \cdots \otimes E_n \to F_1 \otimes \cdots \otimes F_n \times B,$$

is a trivialization.

(1) Φ_b is surjective as it is linear, and given $\rho_1 \otimes \cdots \otimes \rho_n \in F_1 \otimes \cdots \otimes F_n$ there exist (by using the bijectiveness of $\Phi_1, \ldots, \Phi_n$) $(\sigma_1, \ldots, \sigma_n) \in (E_1)_b \times \cdots \times (E_n)_b$ such that

$$\Psi'(\sigma_1, \ldots, \sigma_n) = \rho_1 \otimes \cdots \otimes \rho_n.$$

Furthermore Φ_b is injective as it is evident that $\ker \Phi_b = \{0\}$. It follows that Φ is bijective.

(2) First note that $\mathrm{pr}_{F_1\otimes\cdots\otimes F_n} \circ \Phi \circ \phi = \Psi'$ is smooth, by lemma 1.3.14 (iii) this implies that $\mathrm{pr}_{F_1\otimes\cdots\otimes F_n} \circ \Phi$ is smooth. Furthermore it is clear that $\mathrm{pr}_B \circ \Phi = \pi$ is smooth. By proposition 1.4.2 the two above observations implies that Φ is smooth. Consider now plots $\alpha : U \to F_1 \times \cdots \times F_n$ and $\beta : V \to B$, then define a plot for $\overrightarrow{\phi}(\mathcal{D}_{E_1\times\cdots\times E_n})$ by

$$\gamma := \phi \circ \Psi^{-1} \circ \alpha \times \beta.$$

Now

$$\begin{aligned} \mathrm{pr}_{F_1\otimes\cdots\otimes F_n} \circ \Phi \circ \gamma &= \mathrm{pr}_{F_1\otimes\cdots\otimes F_n} \circ \Phi \circ \phi \circ \Psi^{-1} \circ \alpha \times \beta \\ &= \phi' \circ \mathrm{pr}_{F_1\times\cdots\times F_n} \circ \Psi \circ \Psi^{-1} \circ \alpha \times \beta \\ &= \phi' \circ \alpha \circ \mathrm{pr}_U \end{aligned}$$

and

$$\mathrm{pr}_B \circ \Phi \circ \gamma = \pi \circ \gamma = \beta \circ \mathrm{pr}_V$$

hence $\Phi \circ \gamma = (\phi' \circ \alpha) \times \beta$. Since any plot for $\langle \mathcal{D}_{F_1\times\cdots\times F_n}) \times \mathcal{D}_B\rangle$ is locally of the form $(\phi' \circ \alpha) \times \beta$ or constant, it follows by lemma 1.5.2 that Φ is a diffeomorphism. ▪

5.3 Dual bundles

Consider two pre-bundles, over B, say $E \xrightarrow{\pi} B$ and $E' \xrightarrow{\pi'} B$. For any two parametrizations $\gamma : U \to E$ and $\alpha : V \to E'$ let

$$\Delta_{\gamma,\alpha} := \{(u,v) \in U \times V \mid \pi \circ \gamma(u) = \pi' \circ \alpha(v)\}.$$

Let $E \xrightarrow{\pi} B$ be a vector bundle. And consider the vector pre-bundle

$$E^* := \bigcup_{b\in B} \mathcal{L}^\infty(E_b, \mathbb{R}).$$

Definition 5.3.1 The *dual bundle diffeology* for the vector pre-bundle E^* is the collection of parametrizations $\gamma : U \to E^*$ such that

(i) $\pi \circ \gamma$ is smooth.

(ii) For all plots $\alpha : V \to E$ and any smooth map $c = (c_1, c_2) : W \to U \times V$ with $\mathrm{Im}(c) \subseteq \Delta_{\gamma,\alpha}$ the map

$$w \to \gamma(c_1(w))[\alpha(c_2(w))]$$

is smooth.

The dual bundle diffeology is, as we shall see, a regular bundle diffeology. The resulting vector bundle E^* is called the *dual bundle*.

Example 37 (Zero maps)
Let $0_b : E_b \to \mathbb{R}$ denote the zero map. And let $\alpha : U \to B$ be a plot for the base space B. Then define a map $0_\alpha : U \to E^*$ by

$$0_\alpha(u) := 0_{\alpha(u)}.$$

0_α is clearly smooth (that is a plot for the dual bundle diffeology). ▪

Proposition 5.3.2 *Definition 5.3.1 defines a regular vector bundle diffeology on E^*. And the fiber over $b \in B$ is the diffeological vector space $\mathcal{L}^\infty(E_b, \mathbb{R})$.*

Proof: We must show (1) that definition 5.3.1 indeed defines a diffeology, (2) that π is a subduction and (3) we need to show that the subspace diffeology for the fibers $\mathcal{L}^\infty(E_b, \mathbb{R})$ are in fact the functional diffeology. And (4) that fiberwise addition and scalar multiplication is smooth.

(1) *Covering.:* It is not hard to see that every constant parametrization is a plot.

Smoothness.: Consider a plot γ for the dual diffeology and let $\gamma' = \gamma \circ h$ for a smooth map $h : U \to \text{Dom}(\gamma)$. The task is then to show that γ' is a plot for the dual diffeology. So let $\alpha : V \to E$ be a plot and consider a smooth map $c = (c_1, c_2) : W \to U \times V$ with $\text{Im}(c) \subseteq \Delta_{\gamma',\alpha}$. Let $c' := (h \circ c_1, c_2)$ then $c' : W \to \text{Dom}(\gamma) \times V$ is smooth and for each $w \in W$

$$\pi \circ \gamma(h \circ c_1(w)) = \pi \circ \gamma'(c_1(w)) = \pi \circ \alpha(c_2(w))$$

that is $\text{Im}(c') \subseteq \Delta_{\gamma,\alpha}$. Hence

$$w \to \gamma'(c_1(w))[\alpha(c_2(w))] = \gamma(c_1'(w))[\alpha(c_2'(w))]$$

is smooth.

Locality.: Assume that $\gamma : U \to E^*$ locally is a plot for the dual diffeology (i.e. locally belongs to the diffeology). And consider, as above, a plot $\alpha : V \to E$ and a smooth map $c : W \to U \times V$ with $\text{Im}(c) \subseteq \Delta_{\gamma,\alpha}$. We then wish to show that the map

$$w \to \gamma(c_1(w))[\alpha(c_2(w))]$$

is smooth. We only need to show that that this map is smooth in a neighbourhood of every point of W, so let $w_0 \in W$ and set $(u_0, v_0) := c(w_0)$. By assumption there exist a open set $U_0 \subset U$ with $u_0 \in U_0$ and such that $\gamma' := \gamma|_{U_0}$ is a plot for the dual diffeology. If we now let $W_0 := c^{-1}(U_0 \times V)$ and $c' := c|_{W_0}$ then $\text{Im}(c') \subseteq \Delta_{\gamma',\alpha}$, hence

$$w \to \gamma(c_1(w))[\alpha(c_2(w))] = \gamma'(c_1'(w))[\alpha(c_2')]$$

is smooth.

(2) By definition π is smooth, hence by lemma 1.3.6 $\overrightarrow{\pi}(\mathcal{D}_{E^*}) \subseteq \mathcal{D}_B$. Let $\beta : U \to B$ be a plot for B then the map $0_\beta : U \to E^*$ is a plot (see example 37) and $\pi \circ 0_\beta = \beta$. It follows that $\mathcal{D}_B \subseteq \langle \pi \circ \mathcal{D}_{E^*} \rangle = \overrightarrow{\pi}(\mathcal{D}_{E^*})$. Since π is also surjective it is seen to be a subduction.

(3) At each point $b \in B$ we must show that the subspace diffeology of the fiber is the functional diffeology. So let $\gamma : U \to E^*$ be a plot with range contained in the fiber over b, that is in the vector space $\mathcal{L}^\infty(E_b, \mathbb{R})$. Let $\alpha : V \to E_b$ be a plot, then

$$(u, v) \to \gamma(u)[\alpha(v)] = \gamma \cdot \alpha(u, v)$$

is smooth, i.e γ is a plot for the functional diffeology on the fiber $\mathcal{L}^\infty(E_b,\mathbb{R})$. On the other hand, let $\gamma : U \to \mathcal{L}^\infty(E_b,\mathbb{R})$ be a plot for the functional diffeology on the fiber. Let $\alpha : V \to E$ be any plot and consider a smooth map $c = (c_1, c_2) : W \to U \times V$ with $\mathrm{Im}(c) \subseteq \Delta_{\gamma,\alpha}$. Then

$$w \to \gamma(c_1(w))[\alpha(c_2(w))] = \gamma \cdot \alpha \circ c(w)$$

is smooth, as it is the composition of smooth maps.

(4) Let $\gamma : U \to E^* \times E^*$ be a plot, and let $\gamma_i = \mathcal{P}_i \circ \gamma$ for $i = 1, 2$. We must show that

$$u \to \gamma_1(u) + \gamma_2(u)$$

is a plot for E^*. So let $\alpha : V \to E$ be a plot and consider a smooth map $c = (c_1, c_2) : W \to U \times V$ with $\mathrm{Im}(c) \subseteq \Delta_{\gamma,\alpha}$. Then

$$\begin{aligned} w \to (\gamma_1(c_1(w)) + \gamma_2(c_2(w)))\,[\alpha(c_2(w))] \\ = \gamma_1(c_1(w))[\alpha(c_2(w))] + \gamma_2(c_1(w))[\alpha(c_2(w))] \end{aligned}$$

which is smooth, since the γ_i's are plots for E^*. ▪

Lemma 5.3.3 *The diffeology defined in definition 5.3.1 is the strongest diffeology on E^* such that (i) and (ii) below holds for all plots $\gamma : U \to E^*$.*

(i) $\pi \circ \gamma$ is smooth.

(ii) For any plot $\alpha : U \to E$ with $\pi \circ \gamma = \pi \circ \alpha$ the map $u \to \gamma(u)[\alpha(u)]$ is smooth.

Proof: Let $\gamma : U \to E^*$ be a plot for a diffeology $\mathcal{D}'$ satisfying (i) and (ii). We must then show that γ is a plot for the dual bundle diffeology, as defined in definition 5.3.1. Let $\alpha : V \to E$ be a plot and $c = (c_1, c_2) : W \to U \times V$ a smooth map with $\mathrm{Im}(c) \subseteq \Delta_{\gamma,\alpha}$, then the map

$$W \ni w \to \gamma \circ c_1(w)[\alpha \circ c_2(w)]$$

is smooth, by (ii) and the smoothness of $\mathcal{D}'$ and $\mathcal{D}_E$. ▪

The above lemma implies the following important proposition.

Proposition 5.3.4 *For any smooth section s of E^* and any smooth section t of E the map*

$$B \ni b \to s(b)[t(b)]$$

is smooth.

Theorem 5.3.5
If $E \xrightarrow{\pi} B$ is trivial with fiber F, then $E^* \xrightarrow{\pi^*} B$ is trivial with fiber $\mathcal{L}^\infty(F,\mathbb{R})$.

Proof: Since E is trivial we have a vector bundle isomorphism

$$(\Phi : B \times F \to E, \mathrm{id}_B).$$

Let $\phi_b : F \to E_b$ denote the induced linear diffeomorphism on the fibers, i.e $\phi_b := \Phi|_{\{b\}\times F}$ consider as a map onto E_b. Define $\Psi : E^* \to B \times \mathcal{L}^\infty(F, \mathbb{R})$ by setting

$$\Psi(\sigma) := (\pi(\sigma), \sigma \circ \phi_{\pi(\sigma)}),$$

we claim that (Ψ, id_B) is a vector bundle isomorphism. The only thing which is not clear is that Ψ is diffeomorphism, hence that it and its inverse are smooth. We shall show (1) that Ψ is smooth and (2) that its inverse is smooth.

(1) Let $\gamma : U \to E^*$ be a plot for the dual diffeology, we then wish to show that the map

$$u \to \Psi(\gamma(u)) = \left(\pi^* \circ \gamma(u), \gamma(u) \circ \phi_{\pi^*\circ\gamma(u)}\right)$$

is smooth. Hence we must show that given any plot $\alpha : V \to F$ the map

$$(u, v) \to \gamma(u) \left[\phi_{\pi\circ\gamma(u)}(\alpha(v))\right] \tag{5.1}$$

is smooth. Now $\pi \circ \gamma(u) = \pi \circ \Phi(\pi \circ \gamma(u), \alpha(v))$ for all $(u, v) \in U \times V$, and the map

$$(u, v) \to \Phi(\pi \circ \gamma(u), \alpha(v)) = \phi_{\pi\circ\gamma(u)}(\alpha(v))$$

is smooth. Hence by lemma 5.3.3 the map 5.1 is smooth.

(2) Our first observation is that inverse $\Psi^{-1} : B \times \mathcal{L}^\infty(F, \mathbb{R}) \to E^*$, of Ψ, is given by

$$\Psi^{-1}(b, \sigma) = \iota_b \left[\sigma \circ \phi_b^{-1}\right]$$

where ι_b is the canonical inclusion $\mathcal{L}^\infty(E_b, \mathbb{R}) \hookrightarrow E^*$. In order to verify this observation note that

$$\Psi \circ \Psi^{-1}(b, \sigma) = \Psi\left(\iota_b \left[\sigma \circ \phi_b^{-1}\right]\right) = (b, \sigma \circ \phi_b^{-1} \circ \phi_b) = (b, \sigma)$$

and

$$\Psi^{-1} \circ \Psi(\sigma) = \Psi^{-1}(\pi(\sigma), \sigma \circ \phi_{\pi(\sigma)}) = \iota_{\pi(\sigma)}(\sigma) = \sigma.$$

The task is to show that Ψ^{-1} is smooth. Consider a plot for $B \times \mathcal{L}^\infty(F, \mathbb{R})$ say $\alpha = (\alpha_1, \alpha_2) : U \to B \times \mathcal{L}^\infty(F, \mathbb{R})$. Then define a map $\gamma : U \to E^*$ by

$$\gamma(u) := \Psi^{-1}(\alpha(u)) = i_{\alpha_1(u)} \left[\alpha_2(u) \circ \phi^{-1}_{\alpha_1(u)}\right].$$

If we can show that γ is a plot for the dual bundle diffeology we are done. That is we must show (i) and (ii) of definition 5.3.1. (i) is easy as $\pi^* \circ \gamma = \alpha_1$ which is smooth. For (ii) consider a plot $\beta : V \to E$ and a smooth map $c : W \to U \times V$ with $\mathrm{Im}(c) \subseteq \Delta_{\gamma,\beta}$ then

$$\begin{aligned}\gamma(c_1(w))[\beta(c_2(w))] &= \alpha_2(c_1(w)) \circ \phi^{-1}_{\alpha_1(c_1(w))}[\beta(c_2(w))] \\ &= \alpha_2(c_1(w)) \left[\mathrm{pr}_F \circ \Phi^{-1}(\beta(c_2(w)))\right]\end{aligned}$$

which is smooth. ▪

Example 38
Let B be a diffeological space, then evidently $(\mathbb{R}^n)^*_{\mathbb{B}} \simeq (\mathbb{R}^n)_{\mathbb{B}}$. ▪

5.4 The Dual of the tensor product bundle

Consider n vector bundles, say $E_1, \dots, E_n$ over B. Then let

$$T := (E_1 \otimes \cdots \otimes E_n)^*,$$

and consider a plot $\gamma : U \to T$. By propositions 5.2.2 and 5.3.2 and theorem 2.3.5 the fiber over b of T is

$$\mathcal{L}^\infty((E_1)_b \otimes \cdots \otimes (E_n)_b, \mathbb{R}) \simeq \mathcal{L}^\infty_{Mult}((E_1)_b \times \cdots \times (E_n)_b, \mathbb{R}),$$

hence we may, for each $u \in U$ consider $\gamma(u)$ as a multilinear map $(E_1)_{\pi\circ\gamma(u)} \times \cdots \times (E_n)_{\pi\circ\gamma(u)} \to \mathbb{R}$. To be precise we see, by looking at the proofs of section 2.3, that the map defined by

$$\tilde{\gamma}(u) := \gamma(u) \circ \phi_{\pi\circ\gamma(u)}$$

is precisely this, to γ associated, multilinear map. Where $\phi_{\pi\circ\gamma(u)}$ is the tensor product map for the tensor product of the fibers over $\pi \circ \gamma(u)$. We shall however in general not use the notation $\tilde{\gamma}$, but also denote the to γ associated multilinear map by γ.

Theorem 5.4.1
A map $\gamma : U \to T$ is smooth if and only if

(i) *$\pi \circ \tilde{\gamma}$ is smooth.*

(ii) *Given any n plots $\alpha_i : V \to E_i$ with $\pi \circ \alpha_1 = \cdots = \pi \circ \alpha_n$, and any smooth map $c : W \to U \times V$ with $\mathrm{Im}(c) \subseteq \Delta_{\gamma,\alpha_1}$ the map*

$$w \to \tilde{\gamma}(c_1(w))[\alpha_1(c_2(u)), \dots, \alpha_n(c_2(w))]$$

is smooth.

Proof: Note that

$$\begin{aligned}\tilde{\gamma}(c_1(w))[\alpha_1(c_2(w)), \dots, \alpha_n(c_2(w))] &= \gamma \circ \phi_{\pi\circ\gamma(c_1(w))}[\alpha_1(c_2(w)), \dots, \alpha_n(c_2(w))] \\ &= \gamma(c_1(w))[\phi \circ \alpha_1 \times \cdots \times \alpha_n(c_2(w))]\end{aligned}$$

as the fiberwise tensor product $\phi : E_1 \times \cdots \times E_n \to E_1 \otimes \cdots \otimes E_n$ is smooth, one implication follows (by lemma 5.3.3). For the other implication assume that (i) and (ii) holds. And let $\beta : V \to E_1 \otimes \cdots \otimes E_n$ be a generating plot, hence

$$\beta(v) = \sum_{i=1}^{n} \lambda_i(v)\phi \circ \beta_i(v)$$

with $\lambda_i : V \to \mathbb{R}$ smooth and $\beta_i : V \to E_1 \times \cdots \times E_n$ plots for the product diffeology. Let $c : W \to U \times V$ be any smooth map with $\mathrm{Im}(c) \subseteq \Delta_{\gamma,\alpha_1}$ then

$$\begin{aligned}\gamma(c_1(w))[\beta(c_2(w))] &= \sum_{i=1}^{n} \lambda_i(c_2(w))\gamma(c_1(w))[\phi \circ \beta_i(c_2(w))] \\ &= \sum_{i=1}^{n} \lambda_i(c_2(w))\tilde{\gamma}(c_1(w))[\beta_i(c_2(w))]\end{aligned}$$

by the assumptions (and the discussion in section 4.3) it follows that γ is smooth. ▪

Corollary *Let* $\alpha_i : U \to E_i$ *for* $i = 1, \ldots, n$ *be plots, with* $\pi \circ \alpha_i = \pi \circ \gamma$*. Then the function*

$$u \to \gamma(u)[\alpha_1(u), \ldots, \alpha_n(u)]$$

is smooth.

Chapter 6

Differential forms in diffeology

We shall in this chapter reach the main objective of this thesis, by constructing a tensor bundle and proving that it is sensible, as discussed in the introduction. In sections 6.1 and 6.2 we construct the tangent bundle TX, of a diffeological space X. And in section 6.3 we define the k-tensor bundle as

$$T^k(X) := \Big(\underbrace{TX \otimes \cdots \otimes TX}_{\text{k copies}}\Big)^* .$$

In this last section we also defines differential forms and we shall see that each differential form naturally induces a $\mathcal{D}$-form.

Having in mind the results on diffeological bundles from the previous chapters, as well as the requirements for the tensor bundle, as discussed in the introduction, we arrive at the following requirements for the tangent bundle:

(a) The fibre over $x \in X$ should be the tangent space $\mathrm{T}_x X$.

(b) For $U \in \mathrm{O}\mathbb{R}^\infty$ the tangent bundle TU should be trivial with fibre $\mathbb{R}^{\dim(U)}$.

(c) The set of smooth sections $\Gamma(TX)$ on TX should, in a natural way, be a $C^\infty(X)$ module.

(d) Given a smooth map $\phi : X \to Y$, the fiberwise defined tangent map $T\phi : TX \to TY$ should be smooth.

(e) Given any smooth vector field $V \in \Gamma(TX)$ and a function $f \in C^\infty(X)$ the map

$$x \to V(x)(f)$$

should be smooth.

As we shall see, the constructed tangent bundle will fulfil all of these requirements, and this will imply that the tensor bundle fulfil the requirements listed in the introduction.

6.1 The tangent cone bundle

Consider the pre-bundle

$$CX := \bigcup_{x\in X} \mathrm{C}_x X.$$

We shall equip this pre-bundle with a bundle diffeology.

Definition 6.1.1 A parametrization $\gamma : U \to \mathrm{C}_x X$ is said to be a *homogeneous plot* if for each $u_0 \in U$ there exist an open neighbourhood U_0 of u_0 and a smooth parametrization $\gamma^\dagger : U_0 \to C^\infty(\mathbb{R}, X)$ such that

$$\gamma(u) = d[\gamma^\dagger(u)] \quad \text{for all } u \in U_0.$$

Note that if $\gamma : U \to CX$ is a homogeneous plot then $\pi \circ \gamma(u) = \gamma^\dagger(u)(0)$. We shall by $\mathcal{D}_{CX}$ denote the collection of all homogeneous plots. Note that, as a direct consequence of the definition, this collection is a diffeology on CX. In fact it is a bundle diffeology, as we shall see.

Remark 6.1.2 A remark on notation, if γ is a homogeneous plot we shall by $\gamma^\dagger$, in the following, always mean the to γ associated map, as defined in definition 6.1.1.

Example 39 (The zero section)
Let X be a diffeological space. Denote by 0_x the zero element in $\mathrm{C}_x X \subseteq CX$. Given any plot $\alpha : U \to X$ for $\mathcal{D}_X$ define a map $0_\alpha : U \to CX$ by

$$0_\alpha(u) := 0_{\alpha(u)}.$$

We claim that this map is a homogeneous plot. To see this consider the smooth map $\gamma^\dagger U \to (\mathbb{R}, X)$ defined by

$$\gamma^\dagger(u)(\xi) := \alpha(u),$$

that is $\gamma^\dagger(u)$ is the constant plot onto $\alpha(u)$. Clearly $d[\gamma^\dagger(u)] = 0_{\alpha(u)} = 0_\alpha(u)$, i.e. 0_α is a homogeneous plot.

Furthermore the map $0 : X \to CX$ defined by $0(x) := 0_x$ is a smooth section of CX. To see this simply note that, for any plot $\alpha : U \to X$ the conposition $u \to 0(\alpha(u)) = 0_{\alpha(u)}$ is, as we saw above, a homogeneous plot. We shall call this section the *zero section*. ▪

Lemma 6.1.3 *The collection of homogeneous plots is a bundle diffeology on the pre-bundle CX. And the fiber over $x \in X$ is the tangent cone space at x.*

Proof: In order to show that $CX \xrightarrow{\pi} X$ is a diffeological bundle, with fibers the tangent cone spaces, we must show (1) that π is a subduction and (2) that the subspace diffeology for $\mathrm{C}_x X$, as a subspace of CX, is the tangent cone space diffeology.

(1) Let $\gamma : U \to TX$ be a homogeneous plot. Then $\pi \circ \gamma = \gamma^\dagger(u)(0)$ is clearly smooth. It follows that $\pi : CX \to X$ is smooth, hence by lemma 1.3.6 $\overrightarrow{\pi}(\mathcal{D}_{CX}) \subseteq \mathcal{D}_X$. For the other inclusion let $\alpha : U \to X$ be a plot for D_X, and consider the homogeneous plot $0_\alpha : U \to CX$ as describe above in example 39. As $\pi \circ 0_\alpha = \alpha$ it follows, by definition 1.3.4, that $\mathcal{D}_X \subseteq \langle \pi \circ \mathcal{D}_{CX} \rangle = \overrightarrow{\pi}(\mathcal{D}_{CX})$. Since π is also surjective it is a subduction.

(2) Let $x \in X$ and consider the subspace $\mathrm{C}_x X$ of CX. Note the following direct consequence of the definitions;

- · a plot for the tangent cone diffeology is a homogeneous plot,
- · a homogeneous plot with its image contained in $\mathrm{C}_x X$, is simply a plot for the tangent cone diffeology at x.

It follows that the diffeology of the fiber is the tangent cone diffeology. ▪

Lemma 6.1.4 *Let $\gamma : U \to TX$ be a generating plot, then for any smooth map $\phi : V \to C^\infty(X)$ the map $\gamma \cdot \phi$ is smooth.*

Proof: Note that the map $(u, \xi) \to \gamma^\dagger(u)(\xi)$ is smooth, hence the map $(u, v, \xi) \to [\phi(v) \circ \gamma^\dagger(u)](\xi)$ is smooth. This implies that

$$\gamma \cdot \phi(u, v) = \gamma(u)[\phi(v)] = d[\gamma^\dagger(u)](\phi(v)) = \left.\frac{\partial[\phi(v) \circ \gamma^\dagger(u)](\xi)}{\partial \xi}\right|_{(u,v,\xi=0)}$$

is smooth. ▪

6.2 The Tangent bundle

Consider the pre-bundle

$$TX := \bigcup_{x \in X} \mathrm{T}_x X,$$

as usual the *base point projection* is the map $\pi : TX \to X$ defined by sending $\mathrm{T}_x X \subseteq TX$ to $x \in X$.We shall equip this pre-bundle with a bundle diffeology, thereby making it into our tangent bundle.

Definition 6.2.1 (Tangent bundle) The *tangent bundle diffeology* $\mathcal{D}_{TX}$ on TX is the weak vector bundle diffeology generated by the homogeneous plots, that is

$$\mathcal{D}_{TX} := \sum \langle \mathcal{D}_{CX} \rangle_{TX}.$$

The *tangent bundle* is the diffeological bundle $(TX, \mathcal{D}_{TX})$.

Evidently $\mathcal{D}_{TX}$ is a bundle diffeology on TX. We shall see below (proposition 6.2.2) that the fibers of the tangent bundle are in fact the tangent spaces.

Proposition 6.2.2 *Let X be a diffeological space.*

(i) A generating plot for $\mathcal{D}_{\mathrm{T}_x X}$ is of the form

$$u \to \sum_i^n \lambda_i(u) d[\gamma_i^\dagger(u)] \tag{6.1}$$

where $n \in \mathbb{N}$, $\lambda_i : U \to \mathbb{R}$ smooth and $\gamma_i^\dagger : U \to C^\infty(\mathbb{R}, X)$ smooth with $\gamma_1^\dagger(u)(0) = \cdots = \gamma_n^\dagger(u)(0)$ for all $u \in U$. Furthermore the collections of generating plots is a smooth covering.

(ii) *Let $E \to B$ be any diffeological vector bundle. Then A fiberwise linear bundle map $(A, a) : TX \to E$ is smooth if and only if*

$$u \to A(d[\gamma^\dagger(u)]) \quad \text{and} \quad a : X \to B$$

are smooth, for all plots $\gamma^\dagger : U \to C^\infty(\mathbb{R}, X)$.

(iii) *$\mathcal{D}_{TX}$ is the weakest regular vector space diffeology on TX such that the maps $u \to d[\gamma^\dagger(u)]$ are smooth.*

(iiii) *The fiber over $x \in X$ of TX is the tangent space $T_x X$.*

Proof: (i) The collection of parametrization of the form given in eq. (6.1) is clearly a subcollection of $\mathcal{D}_{TX}$ and a smooth covering. Hence if we can show that $\mathcal{D}_{TX}$ locally belongs to this collection, it follows that it generates $\mathcal{D}_{TX}$. Let $\gamma : U \to \mathcal{D}_{TX}$ be a plot, by restricting the domain, we may assume that

$$\gamma(u) = \sum_{i=1}^{n} \lambda_i(u)\gamma_i(u)$$

where $n \in \mathbb{N}$, $\lambda_i : U \to \mathbb{R}$ smooth and $\gamma_i : U \to TX$ plots for $\langle \mathcal{D}_{CX} \rangle_{TX}$ with $\pi \circ \gamma_1 = \cdots = \pi \circ \gamma_n$. Let $u_0 \in U$ then (since $\mathcal{D}_{CX}$ is a smooth collection and by theorem 1.2.4) there exist a neighbourhood $U_i \subseteq U$ of u_0 such that $\gamma_i|_{U_i}$ is constant, or there exist a plot $\gamma_i^\dagger : U_i \to C^\infty(\mathbb{R}, X)$ with $\gamma_i^\dagger(u)(0) = \pi \circ \gamma(u)$ and such that

$$\gamma_i|_{U_i}(u) = d[\gamma_i^\dagger(u)].$$

If $\gamma_i|_{U_i}$ is constant then

$$\gamma_i|_{U_i} = \sum_{j=1}^{k} \lambda_j d[\alpha_j]$$

with $\lambda_j \in \mathbb{R}$ and $\alpha_j \in \mathcal{P}_{\pi\circ\gamma(u_0)}(X)$. Now $U' = \cap_{i=1}^{n} U_i$ is a non empty neighbourhood of u_0, by the above arguments we conclude that $\gamma|_{U'}(u)$ equals a linear expansion of the form eq. (6.1).

(ii) Having in mind lemma 1.2.6 and example 19 this is a obvious consequence of (i).

(iii) By theorem 5.1.6 $\mathcal{D}_{TX}$ is the weakest regular vector bundle diffeology containing $\langle \mathcal{D}_{CX} \rangle$. As plots of the for $u \to d[\gamma^\dagger(u)]$ generates $\mathcal{D}_{CX}$ and therefor also $\langle \mathcal{D}_{CX} \rangle$ it follows that $\mathcal{D}_{TX}$ is the weakest regular vector bundle diffeology containing these plots.

(iiii) Let $\gamma : U \to TX$ be a plot and assume that $\mathrm{Im}(\gamma) \subseteq \mathrm{T}_x X$. Using (i) we may then, by restricting the domain, assume that

$$u \to \sum_{i}^{n} \lambda_i(u) d[\gamma_i^\dagger(u)]$$

where $n \in \mathbb{N}$, $\lambda_i : U \to \mathbb{R}$ smooth and $\gamma_i^\dagger : U \to C^\infty(\mathbb{R}, X)$ smooth with $\gamma_i^\dagger(u)(0) = x$ for all $u \in U$. It follows by locality of $\mathcal{D}_{\mathrm{T}_x X}$ that $\gamma \in \mathcal{D}_{\mathrm{T}_x X}$. The other inclusion is trivial. ▪

Example 40 (The tangent bundle of euclidian spaces)
Let $U \subseteq \mathbb{R}^n$ be open then TU is trivial. Let $u \in U$ and consider a tangent vector $d\alpha \in T_uU$ (i.e $\alpha \in \mathcal{P}_u(\mathbb{R}^n)$) then we have the equality (see example 26)

$$d\alpha = \sum_{i=1}^{n} \frac{d\alpha_i}{d\xi}\bigg|_0 d\tau_i(u),$$

where $\alpha_i = \mathrm{pr}_i \circ \alpha$. We claim that the map $\Psi : TU \to U \times \mathbb{R}^n$ given by

$$\Psi(d\alpha) := \left(u, \frac{d\alpha_1}{d\xi}\bigg|_0, \dots, \frac{d\alpha_n}{d\xi}\bigg|_0 \right)$$

is a vector bundle isomorphism. Given that it is a diffeomorphism, it is not hard to see that it is a isomorphism of vector bundles. Hence the task is to show that Ψ is a diffeomorphism. It is clearly a bijection, let $\gamma^\dagger : V \to C^\infty(\mathbb{R}, \mathbb{R}^n)$ be a plot, and let $\gamma(v) := d\gamma^\dagger(v)$ then

$$v \to \Psi(\gamma(v)) = \left(\gamma^\dagger(v)(0), \frac{d\gamma_1^\dagger(v)}{d\xi}\bigg|_0, \dots, \frac{d\gamma_n^\dagger(v)}{d\xi}\bigg|_0 \right)$$

which is seen to be smooth, as

$$v \to \frac{d\gamma_i^\dagger(v)}{d\xi}\bigg|_0 = \frac{\partial\gamma_i^\dagger \cdot \mathrm{id}_{\mathbb{R}}(v, \xi)}{\partial\xi}\bigg|_{\xi=0}$$

is smooth. Hence Ψ is smooth.

Now let $\alpha : V \to U \times \mathbb{R}^n$ be smooth, let us write $\alpha(v) = (\alpha_0(v), \alpha_1(v), \dots, \alpha_n(v))$ with $\alpha_0 : V \to U$ and $\alpha_i : V \to \mathbb{R}$ for $i = 1, \dots, n$. Then define a map $\gamma^\dagger : V \to C^\infty(\mathbb{R}, \mathbb{R}^n)$ by

$$\gamma^\dagger(v)(\xi) := (\alpha_0(v), \alpha_1(v)\xi, \dots, \alpha_n(v)\xi)\,.$$

Clearly $\gamma^\dagger$ is a plot, as it is smooth considered as a map $V \times \mathbb{R} \to \mathbb{R}^n$, furthermore $\Psi(d\gamma^\dagger) = \alpha$. This implies, by lemma 1.5.2, that Ψ is a diffeomorphism. ▪

Vector fields

Smooth sections on TX, i.e elements of $\Gamma(TX)$, are called (smooth) *vector fields*. By lemma 5.1.4 it follows that $\Gamma(TX)$ is $C^\infty(X)$ module.

Example 41
Consider the subspace cone. Note that the map $\gamma^\dagger : \mathbb{R}^2 \to C^\infty(\mathbb{R}, \Lambda)$ given by

$$\gamma^\dagger(\theta, \xi)(t) = (\cos(\theta)(\xi - t),\, \sin(\theta)(\xi - t),\, \xi - t)$$ ▪

is a smooth, as it is smooth as a map $\mathbb{R}^3 \to \Lambda$. If we use cylindrical coordinates (θ, ξ) to denote points on the cone, then $\gamma^\dagger(\theta, z)(0) = (\theta, z) \in \Lambda$. Hence we may define a vector field by setting $V(\theta, z) := d[\gamma^\dagger(\theta, z)]$ (see fig. 6.1).

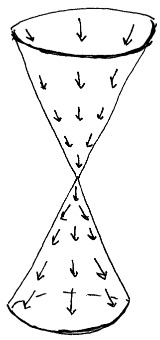


Figure 6.1: The vector field V, see example 41.

The Tangent map

Definition 6.2.3 Let $\varphi : X \to Y$ be smooth. The tangent map $T\varphi : TX \to TY$ is the map defined by the rule

$$T\varphi(v) := T_{\pi(v)}\varphi(v).$$

Lemma 6.2.4 *Let $\varphi : X \to Y$ be smooth. Then the tangnet map $T\varphi$ is smooth.*

Proof: Let $\gamma : U \to TX$ be a homogeneous plot. Then

$$\begin{aligned} T\varphi(\gamma(u)) &= T_{\pi\circ\gamma(u)}\varphi(\gamma(u)) \\ &= T_{\pi\circ\gamma(u)}\varphi(d[\gamma^\dagger(u)]) \\ &= d[\varphi\circ\gamma^\dagger(u)]. \end{aligned}$$

Hence $u \to T\varphi(\gamma(u))$ is a homogeneous plot for TY, as $\varphi\circ\gamma^\dagger \in C^\infty(U, C^\infty(\mathbb{R}, Y))$ is smooth. Proposition 6.2.2 (ii) then implies smoothness of $T\varphi$ ▪

Remark 6.2.5 Note that the tangent map is a vector bundle morphism.

6.3 Differential forms

Tensor fields

Definition 6.3.1 (Tensor bundle) Let X be a diffeological space. For any $k \in \mathbb{N}$ let

$$T^k(X) := \left(\underbrace{TX \otimes \cdots \otimes TX}_{\text{k copies}}\right)^*$$

i.e. the dual bundle of the tensor product bundle $TX \otimes \cdots \otimes TX$. Furthermore it will be convenient to make the convention that $T^0(X) := \mathbb{R}_X$ (i.e. the trivial bundle over X with fiber $\mathbb{R}$).

Note that the fibers of the tensor bundle may be consider as multilinear maps. To be precise the fiber over $x \in X$ is

$$\mathcal{L}^\infty(T_xX \otimes \cdots \otimes T_xX, \mathbb{R}) \simeq \mathcal{L}^\infty_{Mult}(T_xX \times \cdots \times T_xX, \mathbb{R}).$$

See also section 5.4. It is also worth noticing that theorem 5.4.1 implies that, given a plot $\gamma : U \to T^k(X)$ and k smooth vector fields on X, say $V_1, \dots, V_k$, the map

$$u \to \gamma(u)[V_1(\pi \circ \gamma(u)), \dots, V_k(\pi \circ \gamma(u))]$$

is smooth. Smooth sections of $T^k(X)$ are called *k-tensor fields* on X. Consider a k-tensor field $\mathcal{T}$ on X, and k smooth vector fields $V_1, \dots, V_k$ on X. By what we just noted above, it follows that the map

$$X \ni x \to \mathcal{T}(x)[V_1(x), \dots, V_k(x)]$$

is smooth.

Example 42
Let $U \subseteq \mathbb{R}^n$ be open, then as shown in example 40 $TU \simeq \mathbb{R}^n \times U$. Now

$$\begin{aligned} T^k(U) &\simeq (U \times \mathbb{R}^n \otimes \cdots \otimes \mathbb{R}^n)^* && \text{by theorem 5.2.3} \\ &\simeq U \times \mathcal{L}^\infty(\mathbb{R}^n \otimes \cdots \otimes \mathbb{R}^n, \mathbb{R}) && \text{by theorem 5.3.5} \\ &\simeq U \times \mathcal{L}^\infty_{Mult}(\mathbb{R}^n, \mathbb{R}) && \text{by theorem 2.3.5} \end{aligned}$$

hence a smooth section of $T^k(U)$ is a smooth map $U \to \mathcal{L}^\infty_{Mult}(\mathbb{R}^n, \mathbb{R})$. By using the cartesian closer property of the functional diffeology, we conclude that the set of tensor fields on $T^k(U)$ consists of all smooth functions

$$\phi : U \times \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{\text{k copies}} \to \mathbb{R}$$

with $\phi(u)$ multilinear for each $u \in U$. ▪

The pullback of tensor fields

Lemma 6.3.2 *Let $\varphi : X \to Y$ be a smooth map and let $\mathcal{T}$ be any k-tensor field on Y, then the map $\varphi^*(\mathcal{T}) : X \to T^k(X)$ defined as*

$$\varphi^*(\mathcal{T})(x)[\sigma_1, \dots, \sigma_k] := \mathcal{T}(\varphi(x))(T_x\varphi(\sigma_1), \dots, T_x\varphi(\sigma_k))$$

is a k-tensor field on X.

Proof: By linearity and smoothness of the tangent maps $T_x\varphi$ it follows that $\varphi^*(\mathcal{T})$ defines a section of the bundle $T^k(X)$. We still need to show that $\varphi^*(\mathcal{T})$ is smooth, so let $\alpha : U \to X$ be a plot for X, and let $\gamma(u) := \varphi^*(\mathcal{T})(\alpha(u))$. We claim that γ is a plot for $T^k(X)$. To see this, consider k plots $\beta_i : V \to TX$ with $\pi \circ \beta_i = \cdots = \pi \circ \beta_k$ and a smooth map $c : W \to U \times V$ with $\mathrm{Im}(c) \subseteq \Delta_{\gamma,\beta_1}$ then

$$\begin{aligned} &\gamma(c_1(w))[\beta_1(c_2(w)), \dots, \beta_k(c_2(w))] \\ &\qquad = \mathcal{T}(\varphi \circ \alpha(c_1(w)))[T\varphi(\beta_1(c_2(w)), \dots, T\varphi(\beta_1(c_2(w))]. \end{aligned}$$

By lemma 6.2.4 $w \to T\varphi(\beta_1(c_2(w))$ is smooth, and obviously $w \to \varphi \circ \alpha(c_1(w)$ is smooth. This implies (by theorem 5.4.1) that $w \to \gamma(c_1(w))[\beta_1(c_2(w)), \dots, \beta_k(c_2(w))]$ is smooth. Hence (i) and (ii) of theorem 5.4.1 holds, it follows that γ is a plot for $T^k(X)$. ▪

Definition 6.3.3 The map $\varphi^* : \Gamma(T^k(Y)) \to \Gamma(T^k(X))$, as defined in lemma 6.3.2, is called the pullback along φ.

Note that if $\varphi : U \to V$, with $U, V \in \mathrm{O}\mathbb{R}^\infty$, is smooth then the pullback of φ is the usual one.

Lemma 6.3.4 *Let $\phi : X \to Y$ and $\psi : Y \to Z$ be smooth maps. Then*

$$(\psi \circ \phi)^* = \phi^* \circ \psi^*.$$

Proof: A straightforward calculation will show this. ▪

Differential forms

Definition 6.3.5 Let X be a diffeological space. For any $k \in \mathbb{N}$ define the pre-bundle

$$\Lambda^k(X) := \bigcup_{x \in X} \mathcal{L}^\infty_{Alt}(\underbrace{\mathrm{T}_x X \times \cdots \times \mathrm{T}_X X}_{\text{k copies}}, \mathbb{R}).$$

We equip $\Lambda^k(X)$ with a bundle diffeology, namely the subspace diffeology $\Lambda^k(X) \subseteq T^k(X)$. This is indeed a bundle diffeology, as π is easily seen to be a subduction. Furthermore it is clear that the fiber over $x \in X$ of $\Lambda^k(X)$ is simply $\mathcal{L}^\infty_{Alt}(\mathrm{T}_x X \times \cdots \times \mathrm{T}_X X, \mathbb{R})$. Note also that we use the convention that $\Lambda^0(X) := \mathbb{R}_X$. A diffeological *k-form* is a smooth section of the bundle $\Lambda^k(X)$. Note that the pullback along a smooth map takes a k-form to a k-form.

Example 43
By examples 42 and 35 (and section 4.1.2) we see that for $U \subseteq \mathbb{R}^n$ open,

$$\Gamma(\Lambda^k(U)) = C^\infty(U, \mathcal{L}^\infty_{Alt}(\mathbb{R}^n \times \cdots \times \mathbb{R}^n, \mathbb{R})).$$

In other words the set of k-forms on U consists of all smooth functions

$$\phi : U \times \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{\text{k copies}} \to \mathbb{R}$$

with $\phi(u)$ multilinear and alternating for each $u \in U$. ▪

Differential forms and $\mathcal{D}$-forms

$\mathcal{D}$-forms are defined as follows;

Definition 6.3.6 ($\mathcal{D}$-form) A k-$\mathcal{D}$-form $\tilde{\omega}$ on a diffeological space X is a mapping of the form

$$D_X \to \bigcup_{U \in \mathrm{O}\mathbb{R}^\infty} C^\infty(U, \mathcal{L}^\infty_{Alt}(\underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{\text{k copies}}, \mathbb{R})), \qquad (6.2)$$

where $n := \dim U$. Furthermore ω must fulfill the following

(i) For any n-plot α

$$\tilde{\omega}(\alpha) \in C^\infty(\mathrm{Dom}(\alpha), \mathcal{L}^\infty_{Alt}(\mathbb{R}^n \times \cdots \times \mathbb{R}^n, \mathbb{R}))$$

(ii) For any smooth $h : U \to \mathrm{Dom}(\alpha)$

$$\tilde{\omega}(\alpha \circ h) = h^*(\tilde{\omega}(\alpha))$$

Let ω be differential k-form on a diffeological space X, i.e. a smooth section of $\Lambda^k(X)$. Then define a mapping of the form given in eq. (6.2) by

$$\alpha \to \alpha^*(\omega).$$

Evidently this defines a $\mathcal{D}$-form. In fact we see that given any tensor bundle fulfilling requirement (b) and (c) from the introduction this will define a $\mathcal{D}$-form.

We shall end this thesis with an open question, for which it would be nice to have an answer, or a partial answer.

Open question *Let X be a diffeological space. And consider the mapping*

$$\Gamma(\Lambda^k(X)) \to \mathcal{D}\text{-}\mathit{forms}$$

as defined above. For which spaces X is this mapping subjective respectively injective?

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